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THE FUNDAMENTAL SOLUTION TO $\Box_b$ ON QUADRIC MANIFOLDS WITH NONZERO EIGENVALUES

ALBERT BOGGESS AND ANDREW RAICH

ABSTRACT. This paper is part of a continuing examination into the geometric and analytic properties of the Kohn Laplacian and its inverse on general quadric submanifolds of $\mathbb{C}^n \times \mathbb{C}^m$. The goal of this article is explore the complex Green operator in the case that the eigenvalues of the directional Levi forms are nonvanishing. We (1) investigate the geometric conditions on M which the eigenvalue condition forces, (2) establish optimal pointwise upper bounds on complex Green operator and its derivatives, (3) explore the L^p and L^p-Sobolev mapping properties of the associated kernels, and (4) provide examples.

1. Introduction

In this paper, we investigate the complex Green operator N on quadric submanifolds $M \subset \mathbb{C}^n \times \mathbb{C}^m$ for which all the eigenvalues of the directional Levi forms are nonzero. The complex Green operator is the (relative) inverse to the Kohn Laplacian $\Box_b$. By definition, a *quadric submanifold* is defined as

$$M = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m : \operatorname{Im} w = \phi(z,z)\} \tag{1}$$

where $\phi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^m$ is a sesquilinear vector-valued quadratic form. The *Levi form in the direction of* $\nu \in S^{m-1}$, the unit sphere in $\mathbb{R}^m$, is defined as $\phi_\nu(z,z) = \phi(z,z)\cdot\nu$. The *Kohn Laplacian* is defined as $\Box_b = \bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b$ where $\bar\partial_b$ is the usual tangential Cauchy-Riemann operator and $\bar\partial_b^*$ is its L^2 adjoint. The (relative) inverse to $\Box_b$ on (p,q)-forms, when it exists, is called the complex Green operator and denoted by $N_{p,q}$. The existence of the complex Green operator produces the L^2-minimizing solution operator to the $\bar\partial_b$-equation, $\bar\partial_b^* N_{p,q}$, in a canonical fashion. For background on the $\bar\partial_b$ and $\Box_b$-operators, please see [Bog91, CS01, BS17].

In this paper, our main interest is the class of quadrics with codimension $m \geq 2$ where the matrix associated to the scalar Levi form, $\phi_\nu(z,z)$ has only nonzero eigenvalues for each $\nu \in S^{m-1}$. We show that the nonvanishing eigenvalue condition forces n to be even (so replace n with $2n$) with exactly half of the eigenvalues to be positive and half negative. For $0 \leq q \leq 2n$, we establish sharp upper bounds on the size of $N_{0,q}$ and its derivatives in terms of the control geometry on M that are analogous to the classical estimates on N for the Heisenberg group or the finite type hypersurface type case (that is, $m = 1$) in $\mathbb{C}^2$ [NRSW89, Chr91a, Chr91b, FK88].

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This allows us to invoke the theory of homogeneous groups to prove L^p and L^p-Sobolev mapping properties for appropriate derivatives of N. When $q = n$, $\Box_b$ is not solvable by [PR03], but we can still estimate the canonical relative fundamental solution for $\Box_b$ given by $\int_0^\infty e^{-s\Box_b}(I - S_n)\,ds$ where S_n is the orthogonal projection onto $\ker \Box_b$. We also provide several examples, illustrating our estimates.

More generally, when the eigenvalues are not bounded away from zero, the control distance fails to govern estimates on $N_{0,q}$. This failure is apparent in some general hypersurface type CR manifolds as well as some simple higher codimension quadrics [Mac88, NS06, BR21]. In higher codimension, the correct geometry is far from understood as the singularities of N occur both on and off of the diagonal.

For a bit more background and history, the tangential Cauchy-Riemann operator, or $\bar\partial_b$, and the associated Kohn Laplacian $\Box_b$ are arguably the most important operators in several complex variables because they are intrinsically intertwined with the complex geometry, topology, and analysis of CR manifolds. Solving the $\Box_b$-equation is often a product of hard analysis and sophisticated functional analysis, and the solution produced by these techniques may have excellent function theoretic properties but is not constructive (e.g., [Sha85, Koh86, HR11, HR15, CR21]). Often, this approach is not (yet) sufficient to produce the estimates we seek on the solution in the higher codimension setting. Hence we restrict to the class of quadrics, which have a Lie group structure which helps provide a more explicit formula for the solution that is suitable to estimate.

In our opinion, one of the most beautiful results is the computation of $N_{p,q}$ on the Heisenberg group by Folland and Stein [FS74]. The problem, though, is that their technique does not easily generalize, especially to higher codimension. Consequently, one of main approaches to the $\Box_b$-problem on these manifolds is through the $\Box_b$-heat equation. The first results in this direction were for the sub-Laplacian on the Heisenberg group by Hulanicki [Hul76] and Gaveau [Gav77]. More results followed for $\Box_b$ on quadrics of increasing generality [BR09, YZ08, CCT06, BGG96, BGG00, Eld09] culminating (so far) with our paper [BR11] where we compute the $\Box_b$-heat kernel on a general quadric. Virtually all of these results rely on the fact that we can identify M with its tangent space at the orgin, $\mathbb{C}^n \times \mathbb{R}^m$, and push the problem forward onto $\mathbb{C}^n \times \mathbb{R}^m$. The problem with these papers (ours included) is that if we put coordinates (z, t) on M, the solution is only given up to a partial Fourier transform in t. Given that [FS74] is the gold standard (for us), we are taking the formula from [BR11] and trying to undo the Fourier transform and integrate out the time variable. This allows us to recover to both the projection onto $\ker \Box_b$ as well as $N_{0,q}$. In the earlier parts of the series, [BR13, BR22, BR20, BR21], we started with the formula for $\Box_b$-heat kernel and generated an integral formula for both the diagonal part of the complex Green operator as well as the projection onto $\ker \Box_b$. We also categorized the class of quadrics of codimension 2 in $\mathbb{C}^4$ into three $\Box_b$-invariant groups and computed 0th order asymptotics for the kernels for each of these groups. We noticed that in one case, where the directional Leviform has nonvanishing eigenvalues, the complex Green operator was both solvable and hypoelliptic. Additionally, the estimates were particularly good, allowing us to prove continuity results in L^p-Sobolev spaces, $1 < p < \infty$. In many respects, the current paper is a generalization of this case.

In addition to our series of papers, Mendoza proves the following: Let M be a CR manifold of CR codimension > 1 whose Levi form is everywhere nondegenerate.

Then $\Box_b$ computed with respect to any Hermitian metric is hypoelliptic in all degrees except those corresponding to the number of positive or negative eigenvalues of the Levi form [Men]. Additionally, in the special case that $\phi(z,z)$ is a sum of squares, Nagel, Ricci, and Stein [NRS01] proved pointwise upper bounds on both the complex Green operator and the projections onto $\ker\Box_b$, and they established the L^p theory in addition.

The outline of the paper is as follows. In the next section, we state our main results, primarily Theorem 2.1. We continue in Section 3 where we define our notation and explore the geometric consequences of our hypotheses. The proof of Theorem 2.1 for $q \neq n$ is spread over Sections 4–11. In Section 12, we discuss the adjustments to adapt the argument for the $q = n$ case. We conclude the paper with several new examples in Section 13.

2. Main results

Define the projection $\pi : \mathbb{C}^{2n} \times \mathbb{C}^m \to \mathbb{C}^{2n} \times \mathbb{R}^m$ by $\pi(z, t+is) = (z,t)$. For each quadric $M \subset \mathbb{C}^{2n} \times \mathbb{C}^m$, the projection π induces a CR structure and Lie group structure on $\mathbb{C}^{2n} \times \mathbb{R}^m$, and we call this Lie group G (or G_M). The projection is therefore a CR isomorphism and we use the same notation for objects on M and their pushfowards/pullbacks on G.

We introduce only the notation necessary to state the main results. Define the *norm function* $\rho : \mathbb{C}^{2n} \times \mathbb{R}^m \to [0,\infty)$ by

$$\rho(z,t) = \max\{|z|, |t|^{1/2}\} \approx |z| + |t|^{1/2}.$$

For a multiindex $I = (I^1, I^2) \in \mathbb{N}_0^{4n+m}$, the multiindex $I^1 \in \mathbb{N}_0^{4n}$ records the differentiation in the z and $\bar{z}$-variables, and $I^2 \in \mathbb{N}_0^m$ records the t-derivatives. Given such a multiindex I, define the *weighted order* of I by $\langle I \rangle = |I^1| + 2|I^2|$ and the *order* of I by $|I| = |I_1| + |I_2|$.

Theorem 2.1. *Let $M \subset \mathbb{C}^{2n} \times \mathbb{C}^m$ be a quadric submanifold defined by* (1) *with associated projection G, and assume that eigenvalues of the directional Levi forms are nonzero. Let $0 \leq q \leq 2n$ and $N = N_{0,q}$. For any multiindex $I \in \mathbb{N}_0^{4n+m}$, there exists a constant $C_I > 0$ so that*

$$|D^I N(z,t)| \leq \frac{C_I}{\rho(z,t)^{2(2n+m-1)+\langle I \rangle}}.$$

Remark 2.2.

(1) The homogeneous dimension of M is $2(2n+m)$, and we are inverting an order two operator (with respect to ρ). This explains the power of ρ in the denominator of Theorem 2.1.

(2) The case $q = n$ is special because $\ker\Box_b \neq 0$. The relative fundamental solution that we estimate is $\int_0^\infty e^{-s\Box_b}(I - S_n)\, ds$ where $S_n : L^2_{0,n}(M) \to \ker\Box_b \cap L^2_{0,n}(M)$ is the orthogonal projection.

Let $W^{k,p}(M)$ denote the Sobolev space of forms on M with z, $\bar{z}$ and t derivatives of order k are in $L^p(M)$.

Theorem 2.3. *Let $M \subset \mathbb{C}^{2n} \times \mathbb{C}^m$ be a quadric submanifold defined by* (1) *with associated projection G, and assume that eigenvalues of the directional Levi forms are nonzero. Let $0 \leq q \leq 2n$ and $N = N_{0,q}$. Given a multiindex $I \in \mathbb{N}_0^{4n+m}$ so that $\langle I \rangle = 2$, the operator $D^I N_{0,q}$ is exactly regular on $W^{k,p}(M)$ for all $k \geq 0$ and all*

$1 < p < \infty$. *In other words,* $D^I N_{0,q}$ *extends to a bounded operator on* $W^{k,p}(M)$. *In particular,* $D^I N_{0,q}$ *is a hypoelliptic operator.*

Proof. The proof follows easily following the approach of [BR20, Section 3]. Identifying M with $\mathbb{C}^{2n} \times \mathbb{R}^m$, we can view M as a homogeneous group with norm function $\rho(z,t)$. From Theorem 2.1, it follows that the integration kernel of $D^I N_{0,q}$ and its derivatives have the appropriate pointwise decay. A second consequence of Theorem 2.1 is that $D^I N_{0,q}$ is a tempered distribution, and combining this fact with the natural dilation structure and that $D^I N_{0,q}$ is a convolution operator shows that $D^I N_{0,q}$ is uniformly bounded on normalized bump functions. This is exactly what is required to establish the L^p boundedness, $1 < p < \infty$. From the fact that $D^I N_{0,q}$ is a convolution operator, boundedness on $W^{k,p}(\mathbb{C}^n \times \mathbb{R}^m)$ follows immediately. □

3. Notation and hypotheses

Suppose that M is the quadric submanifold

$$M = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m : \operatorname{Im} w = \phi(z,z)\}.$$

Recall that for $\nu \in S^{m-1}$, $\phi_\nu(z,z) = \phi(z,z) \cdot \nu = z^* A_\nu z$ where A_ν is a Hermitian symmetric matrix.

Proposition 3.1. *If* $m \geq 2$ *and if the eigenvalues of* A_ν *are all nonzero for each* $\nu \in S^{m-1}$, *then* n *must be even. Furthermore for each* $\nu \in S^{m-1}$, *half of the eigenvalues of* A_ν *are positive and half of the eigenvalues are negative, counting multiplicity.*

Proof. Note that if λ is an eigenvalue for A_ν, then $-\lambda$ is an eigenvalue for $A_{-\nu}$. If n is odd, then $\det A_{-\nu} = -\det A_\nu$. If $m \geq 2$, this change of sign in the determinant means that $\det A_{\nu'} = 0$ for some other $\nu' \in S^{m-1}$. Therefore, the assumption that all of eigenvalues are nonzero for each $\nu \in S^{m-1}$ implies that n must be even.

Also note that all the eigenvalues of A_ν are real. Let $p_\nu(\lambda) = \det(A_\nu - \lambda I)$ be the characteristic polynomial for A_ν. Let P_ν be the set of the positive roots of p_ν. We are assuming that P_ν is bounded away from zero for all $\nu \in S^{m-1}$. Let K be a compact disc in the open right half plane which contains P_ν in its interior for all $\nu \in S^{m-1}$. The number of roots in P_ν is given by the Argument Principle:

$$\text{Number of positive roots of } p_\nu = \frac{1}{2\pi i} \oint_{\partial K} \frac{p'_\nu(\lambda)\, d\lambda}{p_\nu(\lambda)}.$$

This is clearly a continuous integer-valued function of $\nu \in S^{m-1}$ which is a connected set for $m \geq 2$. Therefore, the number of positive roots of p_ν is constant for all $\nu \in S^{m-1}$. Since n is even and $A_{-\nu} = -A_\nu$, we see that $p_{-\nu}(-\lambda) = p_\nu(\lambda)$. Therefore if the number of positive roots of p_ν is k, then the number of negative roots of $p_{-\nu}(\cdot)$ is also k, which in turn implies that the number of positive roots of $p_{-\nu}$ is $n-k$. Since the number of positive roots is constant in ν, we conclude that $k = n-k$, and hence $k = n/2$. □

3.1. **The complex Green operator.** As a consequence of the above discussion, we assume the following:

- For each $\nu \in S^{m-1}$, there are n positive eigenvalues μ_j^ν for j in some index set P^ν of cardinality n from the set $\{1, 2, \dots, 2n\}$ and n negative eigenvalues μ_k^ν for $k \in (P^\nu)^c$, the complement of P^ν.

Remark 3.2. Given that our eigenvalues stay bounded away from 0 independently of $\nu \in S^{m-1}$, we may arrange the indices so that $P^\nu = P$ is independent of ν.

Denote the set of increasing q-tuples by $\mathcal{I}_q = \{K = (k_1, \dots, k_q) \in \mathbb{N}^q : 1 \le k_1 < k_2 < \cdots < k_q \le 2n\}$. To write the fundamental solution for $\Box_b$ [BR22] applied to a $(0, q)$-form of the form $f_K \, d\bar{z}^K$ for a fixed $K \in \mathcal{I}_q$, we need to establish some notation. Fix $\lambda \in \mathbb{R}^m \setminus \{0\}$ and set $\nu = \frac{\lambda}{|\lambda|} \in S^{m-1}$. We write $z \in \mathbb{C}^n$ in terms of the unit eigenvectors of ϕ_λ which means that $z_j^\lambda = z_j^\nu$ is given by

$$z^\nu = Z(\nu, z) = U(\nu)^* \cdot z$$

where $U(\nu)$ is the matrix whose columns are the eigenvectors, v_k^ν, $1 \le k \le 2n$ of the directional Levi form ϕ^ν, and $\cdot$ represents matrix multiplication with z written as a column vector. Note that the corresponding orthonormal basis of $(0, 1)$-covectors for this basis is

$$d\bar{Z}_j(\nu, z), \quad 1 \le j \le 2n, \quad \text{where} \quad d\bar{Z}(\nu, z) = U(\nu)^T \cdot d\bar{z}$$

where $d\bar{z}$ is written as a column vector of $(0, 1)$-forms and the superscript T stands for transpose. Note that $z^\nu = Z(\nu, z)$ depends smoothly on $z \in \mathbb{C}^n$ but only *locally integrable* as a function of $\nu \in S^{m-1}$ [Rai11].

For each $K \in \mathcal{I}_q$, we will need to express $d\bar{z}^K$, in terms of $d\bar{Z}(\nu, z)^L$ for $L \in \mathcal{I}_q$. We have

$$d\bar{z}^K = \sum_{L \in \mathcal{I}_q} \det(\bar{U}(\nu)_{K,L}) \, d\bar{Z}(\nu, z)^L \tag{2}$$

where $\bar{U}(\nu)_{K,L}$ is the $q \times q$ minor $\bar{U}(\nu)$ comprised of elements in the rows K and columns L. Note that if $q = 2n$, then the above sum only has one term and $\det(\bar{U}(\nu)_{K,K}) = 1$. In addition, $\mathcal{I}_0 = \emptyset$, so the sum (2) does not appear.

Until Section 12, we work under the assumption that $0 \le q \le 2n$ is fixed and $q \ne n$. Since the $|\mu_j^\nu|$ are bounded uniformly away from zero for ν in the unit sphere, it is straightforward to show that the integrability hypothesis on the associated heat kernel in [BR22, Theorem 2.3] is satisfied. Therefore, the fundamental solution to $\Box_b$ on $(0, q)$-forms spanned by $d\bar{z}^K$ is given by convolution with the kernel

(3)

$$\begin{aligned} &N_K(z, t) \\ &= K_{n,m} \sum_{L \in \mathcal{I}_q} \int_{\nu \in S^{m-1}} \det(\bar{U}(\nu)_{K,L}) \, d\bar{Z}(\nu, z)^L \\ &\times \int_{r=0}^1 \Bigg(\prod_{\substack{j \in L^c \cap P \\ j \in L \cap P^c}} \frac{r^{|\mu_j^\nu|} |\mu_j^\nu|}{(1 - r^{|\mu_j^\nu|})} \prod_{\substack{k \in L \cap P \\ k \in L^c \cap P^c}} \frac{|\mu_k^\nu|}{(1 - r^{|\mu_k^\nu|})} \Bigg) \frac{1}{(A(r, \nu, z) - i\nu \cdot t)^{2n+m-1}} \frac{dr \, d\nu}{r} \end{aligned}$$

where $d\nu$ is surface measure on the unit sphere S^{m-1}, the dimensional constant

$$K_{n,m} = \frac{4^{2n} (2n + m - 2)!}{2(2\pi)^{m+2n}}, \tag{4}$$

and

$$A(r, \nu, z) = \sum_{j=1}^{2n} |\mu_j^\nu| \left(\frac{1 + r^{|\mu_j^\nu|}}{1 - r^{|\mu_j^\nu|}} \right) |z_j^\nu|^2.$$

Taking derivatives in z_k or t_ℓ is relatively straight forward because z only appears in $A(r,\nu,z)$ and t only appears in the $\nu\cdot t$ term. In particular, we compute that for $1\le k\le 2n$,

$$\frac{\partial}{\partial z_k}A(r,\nu,z)=\sum_{j=1}^{2n}|\mu_j^\nu|\left(\frac{1+r^{|\mu_j^\nu|}}{1-r^{|\mu_j^\nu|}}\right)U(\nu)^*_{j,k}\cdot\overline{Z_j(\nu,z)}. \tag{5}$$

Similarly, $\frac{\partial^2 A(r,\nu,z)}{\partial z_{k_1}\partial z_{k_2}}=0$ as are all third (and higher) order derivatives. Also,

$$\frac{\partial}{\partial z_{k_1}\partial \bar z_{k_2}}A(r,\nu,z)=\sum_{j=1}^{2n}|\mu_j^\nu|\left(\frac{1+r^{|\mu_j^\nu|}}{1-r^{|\mu_j^\nu|}}\right)U(\nu)^*_{j,k_1}\cdot\overline{U(\nu)^*_{j,k_2}}. \tag{6}$$

A key fact which will be used later is the following: If $P(u)$ is a polynomial in $u\in\mathbb{C}$, then

$$\sum_{j=1}^{2n}P(\mu_j^\nu)|Z_j(\nu,z)|^2=z^*\cdot U(\nu)\cdot P(D_\nu)\cdot U(\nu)^*\cdot z=z^*\cdot P(A_\nu)\cdot z \tag{7}$$

where D_ν is the diagonal matrix with the eigenvalues of A_ν as its diagonal entries. The importance of this equation is as follows. The right side is a quadratic expression in z and $\bar z$ with coefficients that are polynomials in the coordinates of ν (since A_ν depends linearly on ν).

3.2. **Derivative notation.** We define a multiindex $I=(I^1,I^2)\in\mathbb{N}_0^{4n+m}$ where $I^1\in\mathbb{N}_0^{4n}$ is multiindex that records the z and $\bar z$-derivatives and $I^2\in\mathbb{N}_0^m$ records the t-derivatives. Recall that the weighted order of I is $\langle I\rangle=|I^1|+2|I^2|$ and the order of I is $|I|=|I_1|+|I_2|$. Each derivative in a t-variable introduces a component of ν into the numerator and increases the power of $(A(r,\nu,z)-i\nu\cdot t)$ in the denominator by 1. A derivative in a z-variable is more complicated to write down – either the power of $(A(r,\nu,z)-i\nu\cdot t)$ increases by one in the denominator and a component of $\nabla_z A(r,\nu,z)$ is introduced in the numerator or the denominator remains unchanged and a term in the numerator changes from (5) to (6). We will not need a precise accounting of the constants but only the number of first and second derivatives of $A(r,\nu,z)$ that appear. We denote by $\nabla_{z,\bar z}A$ the vector of first derivatives with respect to both the z and $\bar z$ derivatives and by $\nabla^2_{z,\bar z}A$ all of the second order derivatives of A. By an abuse of notation, we write

$$\begin{aligned}D^I\Big\{\frac{1}{(A(r,\nu,z)-i\nu\cdot t)^{2n+m-1}}\Big\}&=c_{n,m,|I_2|}D^{I_1}\Big\{\frac{\nu^{I_2}}{(A(r,\nu,z)-i\nu\cdot t)^{2n+m-1+|I_2|}}\Big\}\\&=\sum_{\substack{(I_1',I_1'')\\|I_1'|+2|I_1''|=|I_1|}}c_{n,m,I_1',I_1'',|I_2|}\frac{\nu^{I_2}(\nabla_{z,\bar z}A(r,\nu,z))^{I_1'}(\nabla^2_{z,\bar z}A(r,\nu,z))^{I_1''}}{(A(r,\nu,z)-i\nu\cdot t)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}}.\end{aligned}$$

where $|I_1'|$ is the number of first order derivatives in z or $\bar z$ and where $|I_1''|$ is the number of second order derivatives in z and $\bar z$. Note that $|I_1'|+2|I_1''|=|I_1|$ and not $|I_1'|+|I_1''|$. For example, suppose that $I_1=(2,1,0,\dots,0,0)$, which is two z_1 factors and one $\bar z_1$ factor. Then

$$(\nabla_{z,\bar z}A(r,\nu,z))^{I_1}=\Big(\frac{\partial}{\partial z_1}A(r,\nu,z)\Big)^2\Big(\frac{\partial}{\partial\bar z_1}A(r,\nu,z)\Big),$$

and $|I_1'| = 1$, $|I_1''| = 1$ and $|I_1| = 3$. We analyze each piece of $D^I N$ separately and consequently, the integral to estimate is

$$
(8)\quad N_{I_1',I_1'',I_2}(z,t) = \sum_{L\in\mathcal{I}_q} \int_{\nu\in S^{m-1}} \det(\bar U(\nu)_{K,L})\, d\bar Z(\nu,z)^L \times \int_{r=0}^1 \Bigg(\prod_{\substack{j\in L^c\cap P\\ j\in L\cap P^c}} \frac{r^{|\mu_j^\nu|}|\mu_j^\nu|}{(1-r^{|\mu_j^\nu|})} \prod_{\substack{k\in L\cap P\\ k\in L^c\cap P^c}} \frac{|\mu_k^\nu|}{(1-r^{|\mu_k^\nu|})}\Bigg) \times \frac{\nu^{I_2}(\nabla_{z,\bar z}A(r,\nu,z))^{I_1'}(\nabla^2_{z,\bar z}A(r,\nu,z))^{I_1''}}{(A(r,\nu,z)-i\nu\cdot t)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}}\frac{d\nu\,dr}{r}.
$$

4. The case when $|t| \geq |z|^2$, $q \neq n$

The tricky case is $|t| > |z|^2$ and so we will factor out a $|t|^{2n+m-1+|I_1'|+|I_1''|+|I_2|}$ from the denominator and we will rotate ν coordinates via an orthogonal matrix M_t chosen so that $M_t(t/|t|)$ is the unit vector in the ν_1 direction (so in the new coordinates, $\nu\cdot t = \nu_1|t|$). We also set $\nu^t = M_t^{-1}\nu$ and

$$
\hat q = \frac{z}{|t|^{1/2}} \in \mathbb{C}^{2n}, \quad\text{and}\quad Q(\nu^t,\hat q) = \frac{Z(\nu^t,z)}{|t|^{1/2}} = \frac{U(\nu^t)^*\cdot z}{|t|^{1/2}}.
$$

Note that $|Q(\nu^t,\hat q)|^2 = |\hat q|^2$ since U_{ν^t} is unitary.

Since $(\nabla_{z,\bar z}A(r,\nu^t,z))^{I_1'}$ contains a monomial in $z,\bar z$ of degree I_1', we obtain

$$
\begin{aligned}
N_{I_1',I_1'',I_2}(z,t) &= |t|^{-(2n+m-1+\frac12|I_1'|+|I_1''|+|I_2|)} N_{I_1',I_1'',I_2}(q)\\
&= |t|^{-(2n+m-1+\frac12\langle I\rangle)} N_{I_1',I_1'',I_2}(\hat q)
\end{aligned}
$$

where

$$
(9)\quad N_{I_1',I_1'',I_2}(\hat q) = \sum_{L\in\mathcal{I}_q} \int_{\nu^t\in S^{m-1}}\int_{r=0}^1 \det(\bar U(\nu^t)_{K,L})\, d\bar Z(\nu^t,z)^L B_L(r,\nu^t) \times \frac{(\nu^t)^{I_2}(\nabla_{z,\bar z}A(r,\nu^t,\hat q))^{I_1'}(\nabla^2_{z,\bar z}A(r,\nu^t,\hat q))^{I_1''}}{(A(r,\nu^t,\hat q)-i\nu_1)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}}\frac{d\nu\,dr}{r}
$$

and

$$
(10)\quad B_L(r,\nu) = \prod_{\substack{j\in L^c\cap P\\ j\in L\cap P^c}} \frac{r^{|\mu_j^\nu|}|\mu_j^\nu|}{1-r^{|\mu_j^\nu|}} \prod_{\substack{k\in L\cap P\\ k\in L^c\cap P^c}} \frac{|\mu_k^\nu|}{1-r^{|\mu_k^\nu|}}
$$

$$
(11)\quad A(r,\nu,\hat q) = \sum_{j=1}^{2n} |\mu_j^\nu| \left(\frac{1+r^{|\mu_j^\nu|}}{1-r^{|\mu_j^\nu|}}\right) |Q_j(\nu,\hat q)|^2.
$$

To prove Theorem 2.1 in the case that $|t| \geq |z|^2$ and $q \neq n$, it suffices to prove Theorem 4.1.

Theorem 4.1. *There is a uniform constant $C > 0$ so that $|N_{I_1',I_1'',I_2}(\hat q)| \leq C$ for all $\hat q \in \mathbb{C}^{2n}$.*

There are two primary terms which need to be analyzed: $B_L(r,\nu)$, and $A(r,\nu,\hat{q})$. We first concentrate on the singularity at $r=1$. The singularity at $r=0$ is easier and is handled in Section 10.

5. Analysis of $B_L(r,\nu)$ in the case $r>1/2$, $q\neq n$

It turns out that the key to analyzing $B_L(r,\nu)$ is $B_\emptyset(r,\nu)$. To this end, for $0<r<1$ and $u\in\mathbb{R}$, let

$$f(r,u)=\frac{ur^u}{(1-r^u)} \quad g(r,u)=f(r,u)+u=\frac{u}{(1-r^u)}. \tag{12}$$

Note that $g(r,u)=f(r,-u)$. Since $\mu_j^\nu>0$ for $j\in P$ and $\mu_k^\nu<0$ for $k\in P^c$, we can write

$$B(r,\nu)=B_\emptyset(r,\nu)=\prod_{j\in P}\frac{r^{|\mu_j^\nu|}|\mu_j^\nu|}{(1-r^{|\mu_j^\nu|})}\prod_{k\in P^c}\frac{|\mu_k^\nu|}{(1-r^{|\mu_k^\nu|})}$$

then

$$B(r,\nu)\frac{dr}{r}=\prod_{j\in P}f(r,\mu_j^\nu)\prod_{k\in P^c}g(r,-\mu_k^\nu)\frac{dr}{r} \tag{13}$$

$$=\prod_{j=1}^{2n}f(r,\mu_j^\nu)\frac{dr}{r}. \tag{14}$$

Both descriptions of this term are useful. Note that the eigenvalues μ_j^ν are not necessarily smooth in $\nu\in S^{m-1}$ (though they are continuous). However as the next lemma shows, $B(r,\nu)$ is real analytic in both $0<r<1$ and in $\nu\in S^{m-1}$ and this uses the fact that the eigenvalues are bounded away from zero.

Lemma 5.1. *The function $B(r,\nu)=\prod_{j\in P}f(r,\mu_j^\nu)\prod_{k\in P^c}g(r,-\mu_k^\nu)$ is real analytic in both $0<r<1$ and in $\nu\in S^{m-1}$.*

Proof. Using (13), write

$$B(r,\nu)=B^+(r,\nu)\cdot B^-(r,\nu) \quad \text{where}$$
$$B^+(r,\nu)=\prod_{j\in P}f(r,\mu_j^\nu);\quad B^-(r,\nu)=\prod_{k\in P^c}g(r,-\mu_k^\nu).$$

It suffices to show that $\ln B^+(r,\nu)$ and $\ln B^-(r,\nu)$ are real analytic in $0<r<1$ and in $\nu\in S^{m-1}$. We have

$$\ln B^+(r,\nu)=\sum_{j\in P}\ln\tilde{f}(r,\mu_j^\nu)$$

where $\tilde{f}(r,z)=\frac{zr^z}{(1-r^z)}$ for $z=u+iv$. Since $\tilde{f}(r,z)>0$ for $z=u>0$, $\ln(\tilde{f}(r,z))$ is real analytic in $0<r<1$ and complex analytic as a function of $z=u+iv$ in a neighborhood, $U\subset\mathbb{C}$ containing the set $\{u+i0;\ u>0\}$. Note that by hypothesis, there is a compact set $K\subset\{u+i0;\ u>0\}$ which contains all the positive eigenvalues μ_j^ν for $j\in P$ and $\nu\in S^{m-1}$. Let $\gamma\in U$ be a smooth simple closed curve which contains K. Let $D(\nu,z)=\det(A_\nu-zI)$ where recall that A_ν is the Hermitian matrix for $\phi_\nu(z,z)$. The eigenvalues μ_j^ν, $j\in P$ are the roots of the analytic function $z \to D(\nu, z)$ that lie inside γ. By standard Residue theory, we have

$$\ln B^+(r,\nu) = \sum_{j\in P} \ln \tilde{f}(r,\mu_j^\nu) = \frac{1}{2\pi i}\oint_{z\in\gamma} \frac{\ln \tilde{f}(r,z) D'(\nu,z)\,dz}{D(\nu,z)}$$

where $D'(\nu, z)$ refers to the z-derivative of $D(\nu, z)$. Now observe that the right side is real analytic in $\nu \in S^{m-1}$ since $\nu \to A_\nu$ is real analytic in ν (and $D(\nu, z) \neq 0$ for $z \in \gamma$). The proof of the analyticity of $\ln B^-(r, \nu)$ is similar. This completes the proof of the lemma. □

We observe that

$$B_L(r,\nu) = B(r,\nu)\prod_{j\in L\cap P^c} \frac{f(r,-\mu_j^\nu)}{f(r,\mu_j^\nu)} \prod_{k\in L\cap P} \frac{g(r,\mu_k^\nu)}{g(r,-\mu_k^\nu)} = B(r,\nu)\prod_{j\in L} r^{-\mu_j^\nu}.$$

We need the following piece of notation for the next lemma. For $J \in \mathcal{I}_q$ and $(\ell_1, \ldots, \ell_q) \in \mathbb{N}^q$, set $\epsilon_J^{(\ell_1,\ldots,\ell_q)} = (-1)^{|\sigma|}$ if $\{\ell_1, \ldots, \ell_q\} = J$ as sets and $|\sigma|$ is the length of the permutation that takes $(j_1, \ldots, j_q)$ to J. Set $\epsilon_J^{(\ell_1,\ldots,\ell_q)} = 0$ otherwise.

It may be the case the $B_L(r, \nu)$ is *not* analytic, however, we have Lemma 5.2. We also use the notation that if M is a matrix and $J, L \in \mathcal{I}_q$, the $M_{J,L}$ is the $q \times q$ minor of M with entries $M_{j\ell}$, $j \in J$, $\ell \in L$.

Lemma 5.2. *The function*

$$\nu \mapsto \sum_{L\in\mathcal{I}_q} \det(\bar{U}(\nu)_{K,L})\, d\bar{Z}(\nu,z)^L \prod_{j\in L} r^{-\mu_j^\nu}$$

is real analytic in both $0 < r < 1$ *and in* $\nu \in S^{m-1}$. *Moreover,*

$$\sum_{L\in\mathcal{I}_q} \det(\bar{U}(\nu)_{K,L})\, d\bar{Z}^L(\nu,z) \prod_{j\in L} r^{-\mu_j^\nu} = \sum_{J\in\mathcal{I}_q} \det([r^{-\bar{A}_\nu}]_{K,J})\, d\bar{z}^J. \tag{15}$$

Remark 5.3. In view of the above expression for $B_L(r, \nu)$, we record the following equation for future reference

$$\sum_{L\in\mathcal{I}_q} \det(\bar{U}(\nu)_{K,L})\, d\bar{Z}^L(\nu,z) B_L(r,\nu) = \sum_{J\in\mathcal{I}_q} \det([r^{-\bar{A}_\nu}]_{K,J}) B(r,\nu)\, d\bar{z}^J, \tag{16}$$

which is real analytic in $0 < r < 1$, $\nu \in S^{m-1}$ in view of Lemma 5.1.

Proof. Once we show (15), the analyticity statement follows immediately from the fact that $\bar{A}_\nu$ depends analytically on ν and therefore the matrix $r^{-\bar{A}_\nu}$ will also depend analytically on ν.

First, we record two basic equations. Suppose M is a $N \times N$ matrix with complex entries and consider $w = Mz$, where $w, z \in \mathbb{C}^N$. If $1 \le q \le N$ and $K \in \mathcal{I}_q$, then

$$d\bar{w}^K = \sum_{J\in I_q} \det(\bar{M}_{K,J})\, d\bar{z}^J. \tag{17}$$

This is easily established using standard multilinear algebra.

Second, conjugation by $U(\nu)$ diagonalizes the matrix A_ν, and diagonalizes r^{-A_ν}. In particular,

$$R^{-\mu^\nu} = U(\nu)^T r^{-\bar{A}_\nu} \bar{U}(\nu) \tag{18}$$

where $R^{-\mu^\nu}$ is the $(2n) \times (2n)$ matrix with real entries, $r^{-\mu_j^\nu}$, on the diagonal and zeros off of the diagonal.

Now we start with the left side of (15):

$$\begin{aligned}\sum_{L\in\mathcal{I}_q}\det(\bar U(\nu)_{K,L})\,d\bar Z^L(\nu,z)\prod_{j\in L}r^{-\mu_j^\nu} &= \sum_{L\in\mathcal{I}_q}\det(\bar U(\nu)_{K,L})\det(R_{L,L}^{-\mu^\nu})\,d\bar Z^L(\nu,z)\\ &= \sum_{L\in\mathcal{I}_q}\det([\bar U(\nu)R^{-\mu^\nu}]_{K,L})\,d\bar Z^L(\nu,z)\end{aligned}$$

where the second equation uses the fact that $R^{-\mu^\nu}$ is a diagonal matrix. Now use (18) and the fact that $\bar U(\nu)U(\nu)^T = I$ to conclude that

$$\text{Left side of (15)} = \sum_{L\in\mathcal{I}_q}\det[r^{-\bar A_\nu}\bar U(\nu)]_{K,L}\,d\bar Z^L(\nu,z) = d(r^{-\bar A_\nu}\bar z)^K$$

where the last equality uses the equation $z = U(\nu)Z(\nu,z)$ as well as (17) with $w = r^{-A_\nu}z$. Now (15) follows by using (17) to expand out the right side of the above equation in terms of $d\bar z^J$. □

We make the following change of variables for $s > 1$:

$$r = r(s) = \frac{s-1}{s+1} \quad\text{or equivalently}\quad s = \frac{r+1}{1-r} \quad\text{with}\quad \frac{dr}{r} = \frac{2\,ds}{(s^2-1)}. \tag{19}$$

Note that $1/2 \le r < 1$ transforms to $s \ge 3$.

Our goal for the remainder of the section is to prove Proposition 5.4.

Proposition 5.4.

(1) *The expansion of* $\frac{B(r(s),\nu)r'(s)}{r(s)}$ *around* $s=\infty$ *is*

$$\frac{B(r(s),\nu)r'(s)}{r(s)} = \frac{2}{2^{2n}(1-1/s^2)}\left[\sum_{\ell=0}^{2n-1}P_\ell(\nu)s^{2n-\ell-2} + \frac{O(s,\nu)}{s^2}\right]. \tag{20}$$

(21)

$$\textit{Typical monomial in } P_\ell(\nu) = \nu^{\ell-e};\quad \textit{where } e \textit{ is even with } 0\le e\le \ell.$$

Here, $P_\ell(\nu)$ *is a polynomial in* $\nu = (\nu_1,\dots \nu_m)\in S^{m-1}$ *of total degree* ℓ. *By an abuse of notation, the term,* $\nu^{\ell-e}$, *in* (21) *stands for a monomial in the coordinates of* ν *of total degree* $\ell - e$.

Additionally, the (Taylor) remainder $O(s,\nu)$ *is real analytic in* $s>1$ *and* $\nu\in S^{m-1}$. *Furthermore* $O(s,\nu)$ *is bounded in* $s>1$.

(2) *Modulo coefficients (that are computable but not relevant to the estimate), the expansion of* $\det([r(s)^{-\bar A_\nu}]_{K,J})$ *around* $s=\infty$ *is comprised of a sums of terms*

(22)

$$\frac{\nu^{\ell'-e'}}{s^{\ell'}}\ \textit{where}\ \ell'\ge 1,\ e'\ \textit{is an even integer with}\ 0\le e'\le\ell',\ \textit{and}$$

$$\nu^{\ell'-e'}\ \textit{is a monomial of degree}\ \ell'-e'\ \textit{in the coordinates of}\ \nu\in S^{m-1}.$$

To start the proof of Proposition 5.4, let

$$F(s,u) = f(r(s),u),\quad G(s,u) = g(r(s),u).$$

Using (14), we obtain

$$\frac{B(r(s),\nu)r'(s)}{r(s)} = 2\prod_{j=1}^{2n}F(s,\mu_j^\nu)\frac{1}{(s^2-1)}. \tag{23}$$

We will need to Taylor expand $B(r(s),\nu)$ in s about $s=\infty$, which is equivalent to letting $s=1/w$ and expanding about $w=0$. To this end, let

$$\tilde{F}(w,u)=w[F(1/w,u)+u/2]=w\left[g\left(\frac{1-w}{1+w},u\right)-\frac{u}{2}\right]. \tag{24}$$

Lemma 5.5. *$\tilde{F}(w,u)$ is a real analytic function of w and u for $-1<w<1$ and $u\in\mathbb{R}$. In addition,*

1. *For each fixed u, the function $w\to\tilde{F}(w,u)$ is an even function of w;*
2. *For each fixed w, the function $u\to\tilde{F}(w,u)$ is an even function of u;*
3. *The coefficients in the Taylor series expansions of $\tilde{F}(w,u)$ in w about $w=0$ are of the form:*

$$j\text{th coefficient}=\begin{cases}0 & \text{if } j \text{ is odd}\\ P_j(u) & \text{if } j \text{ is even}\end{cases}$$

where $P_j(u)$ is a polynomial of degree j in u that involves only even powers of u.

Proof. We have

$$\begin{aligned}\tilde{F}(w,u)&=\frac{wu}{1-\left(\frac{1-w}{1+w}\right)^{u}}-\frac{wu}{2}\\ &=\frac{wu}{1-e^{u\ln\left(\frac{1-w}{1+w}\right)}}-\frac{wu}{2}.\end{aligned} \tag{25}$$

Since $1-e^z$ vanishes to first order in z at the origin, the (u,w) power series expansion of the denominator has a factor of uw, which cancels with the uw in the numerator. The resulting term is analytic and nonvanishing in a neighborhood of the origin. Hence $\tilde{F}$ is real analytic. Part (2) follows easily from (24). Parts (1) follows by a calculation (Maple helps). For Part (3), we expand the exponential term appearing in (25) and cancel the common factor of uw to obtain

$$\tilde{F}(w,u)=\left[\frac{1}{L(w)+\frac{uw}{2!}L(w)^2+\frac{(uw)^2}{3!}L(w)^3+\dots}\right]-\frac{wu}{2}$$

where $L(w)=w^{-1}\ln\left(\frac{1-w}{1+w}\right)$ is analytic on $-1<w<1$. From repeated w-differentiations of $\tilde{F}$, one can see that the jth w-derivative of $\tilde{F}$ at $w=0$ is a polynomial expression in u of degree j. In view of Part (2), this expression is zero if j is odd and only involves even powers of u when j is even as stated in Part (3). This concludes the proof of the lemma. □

We let $w=1/s$ and unravel this lemma to imply the following expansions for $F(s,u)$.

$$F(s,u)=\frac{s}{2}-\frac{u}{2}+\frac{u^2-1}{6s}-\frac{u^4-5u^2+4}{90s^3}+\sum_{j=3}^{\infty}\frac{p_{2j}(u)}{s^{2j-1}} \tag{26}$$

where $p_{2j}(u)$ is a polynomial in u of degree $2j$ with only even powers of u. The above series converges uniformly on any closed subset of $\{s>1\}$. Note that F has the linear term $u/2$ and that all other terms involve only even powers of u.

Our next task is to use (26) to expand the expression $B(r(s),\nu)$ given in (23) in powers of $1/s$ (about $s=\infty$). To get started, here are the first few terms (in order of decreasing powers of s):

$$\frac{B(r(s),\nu)r'(s)}{r(s)} = \frac{2}{(s^2-1)}\prod_{j=1}^{2n} F(s,\mu_j^\nu) \tag{27}$$

$$= \frac{2s^{2n}}{2^{2n}(s^2-1)}\prod_{j=1}^{2n}\left[1-\frac{\mu_j^\nu}{s}+\frac{(\mu_j^\nu)^2-1}{3s^2}+\sum_{k=2}^{\infty}\frac{p_{2k}(\mu_j^\nu)}{s^{2k}}\right] \tag{28}$$

where $p_{2k}(u)$ is a polynomial of degree $2k$ with only even powers of u.

Now, we expand the product on the right (denoted by Product) in terms of symmetric polynomials in the variables $\mu_1^\nu,\dots,\mu_{2n}^\nu$. First, a definition.

Definition 5.6. A *symmetric polynomial* of degree m on $\mathbb{R}^N$ is a polynomial P of degree m in the variables $(u_1,\dots u_N)\in\mathbb{R}^N$ such that

$$P(u_1,\dots u_N) = P(u_{\sigma(1)},\dots u_{\sigma(N)})$$

for all permutations σ on $\{1,2,\dots,N\}$.

An *allowable* multiindex $\alpha=(\alpha_1,\dots,\alpha_N)$ is a nonincreasing N-tuple of nonnegative integers, that is, integers α_j, $1\le j\le N$, satisfying $\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_N\ge 0$. Let $|\alpha|=\alpha_1+\cdots+\alpha_N$ and define

$$S^\alpha(u_1,\dots,u_N) = \sum_{i_1,\dots,i_N}{}' u_{i_1}^{\alpha_1}\dots u_{i_N}^{\alpha_N}$$

where the sum is taken over all *distinct* indices $i_1,\dots,i_N$ each ranging from 1 to N.

Note the prime over the sum emphasizes that the indices i_j are distinct. Also for clarity, if the $2n$-tuple α ends with multiple zeros, we stop writing after the first zero. For example, we write $S^{1,0}(\mu_1^\nu,\dots,\mu_{2n}^\nu)$ for $S^{1,0,\dots,0}(\mu_1^\nu,\dots,\mu_{2n}^\nu)$. Clearly each $S^\alpha(u)$ is a symmetric polynomial of degree $|\alpha|$. For a fixed $m>0$, the collection of $S^\alpha(u)$ over all allowable multiindices α with $|\alpha|=m$ forms a basis of the space of symmetric polynomials of degree m on $\mathbb{R}^N$.

From an examination of the product in (28) and using the fact that $p_{2k}(u)$ is a polynomial of degree $2k$ with only even powers of u, we obtain Lemma 5.7.

Lemma 5.7. *For $\ell\ge 0$, the coefficient of $\frac{1}{s^\ell}$ in the product on the right side of* (28) *is a linear combination of*

$$S^\alpha(\mu_1^\nu,\dots,\mu_{2n}^\nu),\quad \textit{with } |\alpha|=\ell,\ \ell-2,\ \ell-4,\dots,\ell-e$$

where e is the largest even integer which is less than or equal to ℓ.

As an illustration of this lemma, we write out the first few terms of the product on the right side of (28)

$$\begin{aligned}\text{Product} &= 1-s^{-1}\sum_{k=1}^{2n}\mu_k^\nu + s^{-2}\left((1/3)\sum_{k=1}^{2n}[(\mu_k^\nu)^2-1]+\sum_{j\neq k}\mu_j^\nu\mu_k^\nu\right)+\dots\\ &= 1-\frac{1}{s}S^{1,0}(\mu^\nu)+\frac{1}{s^2}\Big((1/3)(S^{2,0}(\mu^\nu)-2n)+S^{1,1,0}(\mu^\nu)\Big)+\dots.\end{aligned}$$

Now we need transform the $S^\alpha(\mu^\nu)$ into a more useful basis involving elementary symmetric functions.

Definition 5.8. For $0 \le \ell \le N$, the elementary symmetric function of degree ℓ in $\mathbb{R}^N$ is

$$E_\ell(u) = \sum_{(j_1,\dots,j_\ell)\in\mathcal{I}_\ell} u_{j_1}\cdots u_{j_\ell}. \tag{29}$$

With $N = 2n$, the key fact about the $E_\ell(\mu^\nu)$ is that they appear as coefficients in the characteristic polynomial for A_ν:

$$\det(A_\nu - \lambda I) = \lambda^{2n} + \sum_{\ell=1}^{2n}(-1)^\ell E_\ell(\mu^\nu)\lambda^{2n-\ell}. \tag{30}$$

Note that each row of A_ν depends linearly and homogeneously on ν which yields the following consequence for the coefficient, $E_\ell(\mu^\nu)$, of $\lambda^{2n-\ell}$.

Lemma 5.9. *$E_\ell(\mu^\nu)$ is a homogenous polynomial of degree ℓ in the coordinates of $\nu = (\nu_1,\dots,\nu_m) \in S^{m-1}$.*

In particular, $E_\ell(\mu^\nu)$ is analytic in ν even though the eigenvalues μ_j^ν are not necessarily differentiable in ν.

Definition 5.10. Suppose $L = (\ell_1,\dots,\ell_j,\dots)$ is a multiindex (of indeterminate length) with $\ell_j \ge \ell_{j+1}$ and only a finite number of the ℓ_j are nonzero. For $u = (u_1,\dots,u_N)$, define

$$E^L(u) = E_{\ell_1}(u)\cdot E_{\ell_2}(u)\cdots E_{\ell_N}(u),$$

$E^L(u)$ is a symmetric polynomial of degree $|L| = \ell_1 + \cdots + \ell_j + \dots$

The next theorem is [Sta99, Theorem 7.4.4].

Theorem 5.11. *For a given integer, $m \ge 1$, the collection of*

$$\{E^L(u);\ |L| = m;\ u \in \mathbb{R}^N\}$$

is a basis for the space of symmetric polynomials of degree m on $\mathbb{R}^N$.

Corollary 5.12 follows from this theorem and Lemma 5.7.

Corollary 5.12. *In the expansion of $B(r(s),\nu)\frac{r'(s)}{r(s)}$ given in (28), the coefficient of $s^{2n-2-\ell}$ is expressible as a linear combination of*

$$E^L(\mu^\nu) = E_k(\mu^\nu)^{n_k}\cdots E_2(\mu^\nu)^{n_2}E_1(\mu^\nu)^{n_1}\dots,\quad k \ge 1$$

where $L = (n_k,\dots,n_1)$ with $|L| = n_1 + 2n_2 + \dots kn_k = \ell - e$, where e is an even integer with $0 \le e \le \ell$. Moreover, this coefficient is a linear combination of monomials in the components of $\nu = (\nu_1,\dots,\nu_m) \in S^{m-1}$ each having degree $\ell - e$.

We will not need to know the exact values of the coefficients in this expansion. Rather, the key phrase is the last sentence in the above corollary: *the coefficient of $s^{2n-2-\ell}$ is a linear combination of monomials in the components of $\nu = (\nu_1,\dots,\nu_m) \in S^{m-1}$ each having degree $\ell - e$.*

Proof of Proposition 5.4. In view of Corollary 5.12 and (28), equations (20) and (21) both hold. Additionally, the real analyticity of the Taylor remainder term $O(s,\nu)$ for $s>1$ and $\nu \in S^{m-1}$ is assured from Lemma 5.1, the (Taylor) remainder $O(s,\nu)$. Furthermore $O(s,\nu)$ is bounded in $s>1$.

The proof of Part 2 is simpler. An expansion for $r(s)^{-u}$ about $s=\infty$ yields

$$r(s)^{-u} = \Big(\frac{s-1}{s+1}\Big)^{-u} = 1 - \frac{2u}{s} + \frac{2u^2}{s^2} - \frac{2u(1+2u^2)}{s^3} + \sum_{k=4}^{\infty} \frac{\tilde{p}_k(u)}{s^k}$$

where $\tilde{p}_k(u)$ is a a polynomial that has only odd powers of u if k is odd and even powers of u if k is even (this fact can be proven by setting $w=\frac{1}{s}$, and Taylor expansion around $w=0$, and an induction argument on the form of the derivatives). This means

$$r(s)^{-A_\nu} = 1 - \frac{2A_\nu}{s} + \frac{2A_\nu^2}{s^2} - \frac{2A_\nu(1+2A_\nu^2)}{s^3} + \sum_{k=4}^{\infty} \frac{\tilde{p}_k(A_\nu)}{s^k}.$$

Equation (22) now follows from expanding the appropriate $q\times q$ minor determinant. □

6. Expansion of A in denominator in the case $1/2 \le r < 1$, $q \neq n$

The formula for $A(r,\nu,\hat{q})$ is given in (11). Using (12), the coefficient function in front of $|Q_j(\nu,\hat{q})|^2$ is

$$f(r,\mu_j^\nu) + g(r,\mu_j^\nu) = 2f(r,\mu_j^\nu) + \mu_j^\nu$$

which in the s variables (where $r = r(s) = \frac{s-1}{s+1}$), using (26), this becomes

$$\text{Coefficient of } |Q_j|^2 \ = 2F(s,\mu_j^\nu) + \mu_j^\nu = s + \sum_{k=1}^{\infty} \frac{p_{2k}(\mu_j^\nu)}{s^{2k-1}} \tag{31}$$

where $p_{2k}(u)$ is a polynomial of degree $2k$ with only even powers of u. From (26), the first two terms are

$$p_2(u) = \frac{u^2-1}{3}; \quad p_4(u) = -\frac{u^4-5u^2+4}{45}.$$

Now using (11), (31), and (7), we obtain

$$\begin{aligned} A(r(s),\nu,\hat{q}) - i\nu_1 &= s|\hat{q}|^2 + \sum_{j=1}^{2n}\sum_{k=1}^{\infty} \frac{p_{2k}(\mu_j^\nu)}{s^{2k-1}} |Q_j(\nu,\hat{q})|^2 - i\nu_1 \\ &= (s|\hat{q}|^2 - i\nu_1) + \sum_{k=1}^{\infty} \frac{\hat{q}^* \cdot p_{2k}(A_\nu)\cdot \hat{q}}{s^{2k-1}}. \end{aligned} \tag{32}$$

We denote by $e_j \in \mathbb{C}^{2n}$ the jth unit vector $e_j = (0,\ldots,1,\ldots,0)$ (1 in the j^{th} position). We observe that

$$\frac{\partial A}{\partial z_j}\Big|_{(r(s),\nu,\hat{q})} = s\hat{q}^*\cdot e_j + \sum_{k=1}^{\infty} \frac{\hat{q}^*\cdot p_{2k}(A_\nu)\cdot e_j}{s^{2k-1}} \tag{33}$$

and

$$\frac{\partial^2 A}{\partial z_{j_1}\partial \bar{z}_{j_2}}\Big|_{(r(s),\nu,\hat{q})} = s e_{j_2}^*\cdot e_{j_1} + \sum_{k=1}^{\infty} \frac{e_{j_2}^*\cdot p_{2k}(A_\nu)\cdot e_{j_1}}{s^{2k-1}} \tag{34}$$

since the substitution of $\hat{q}$ for z comes *after* the differentiation.

Note the coefficients $\hat{q}^* \cdot p_{2k}(A_\nu) \cdot \hat{q}$ consist of quadratic terms in $\hat{q}$ and $\bar{\hat{q}}$ together with a linear combination of monomial terms in the coordinates of ν of degree $2k-e$ where e is even with $0 \le e \le 2k$.

7. Expanding the kernel for N in the case $1/2 \le r < 1$, $q \ne n$

From (9) and (16), to estimate $N_{I_1', I_1'', I_2}(\hat{q})$, we must investigate the integrands

$$N_{K,J}(\hat{q}, s, \nu) = \det([r(s)^{-\bar{A}_\nu}]_{K,J}) \frac{B(r(s), \nu^t) r'(s)}{r(s)} \times \frac{(\nu^t)^{I_2} (\nabla_{z,\bar{z}} A(r(s), \nu^t, \hat{q}))^{I_1'} (\nabla^2_{z,\bar{z}} A(r(s), \nu^t, \hat{q}))^{I_1''}}{(A(r(s), \nu^t, \hat{q}) - i\nu_1)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}}. \tag{35}$$

For nonzero $V \in \mathbb{C}$, consider the Taylor expansion

$$\frac{1}{(V+\zeta)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}} = \frac{1}{V^{2n+m-1+|I_1'|+|I_1''|+|I_2|}} + \sum_{j=1}^{\infty} \alpha_j \frac{\zeta^j}{V^{2n+m-1+|I_1'|+|I_1''|+|I_2|+j}}$$

which converges uniformly for $|\zeta| \le |V|/2$ (the values of α_j are unimportant). We make use of the following expansions: From (32) with $V = s|q|^2 - i\nu_1$ and $\zeta = \sum_{k=1}^{\infty} \frac{\hat{q}^* \cdot p_{2k}(A_{\nu^t}) \cdot \hat{q}}{s^{2k-1}}$ we have

$$\begin{aligned} &\frac{(\nu^t)^{I_2} (\nabla_{z,\bar{z}} A(r(s), \nu^t, \hat{q}))^{I_1'} (\nabla^2_{z,\bar{z}} A(r(s), \nu^t, \hat{q}))^{I_1''}}{(A(r(s), \nu^t, \hat{q}) - i\nu_1)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}} \\ &= (\nu^t)^{I_2} \Bigg[\frac{1}{(s|\hat{q}|^2 - i\nu_1)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}} \\ &\quad + \sum_{j=1}^{\infty} \frac{\alpha_j \left[\sum_{k=1}^{\infty} \frac{\hat{q}^* \cdot p_{2k}(A_{\nu^t}) \cdot \hat{q}}{s^{2k-1}} \right]^j}{(s|\hat{q}|^2 - i\nu_1)^{2n+m-1+j+|I_1'|+|I_1''|+|I_2|}} \Bigg] \\ &\quad \times (\nabla_{z,\bar{z}} A(r(s), \nu^t, \hat{q}))^{I_1'} (\nabla^2_{z,\bar{z}} A(r(s), \nu^t, \hat{q}))^{I_1''}. \end{aligned} \tag{36}$$

Carefully writing out $(\nabla_{z,\bar{z}} A(r(s), \nu^t, \hat{q}))^{I_1'}$ and $(\nabla^2_{z,\bar{z}} A(r(s), \nu^t, \hat{q}))^{I_1''}$ would be more confusing than useful, as we only need the lead term and the generic expression for the higher order terms. Using (33), we write

(37)

$$\begin{aligned} &(\nabla_{z,\bar{z}} A(r(s), \nu^t, \hat{q}))^{I_1'} \\ &= s^{|I_1'|} C_{0,I_1'} \left((\hat{q}, \bar{\hat{q}})^{|I_1'|} \right) \\ &\quad + \sum_{K=1}^{\infty} \sum_{\substack{k_1+\cdots+k_{|I_1'|}=K \\ k_j \ge 0, \text{ all } j}} s^{|I_1'|-2K} C_{k_1,\ldots,k_{|I_1'|},I_1'} \left((\hat{q}, \bar{\hat{q}})^{|I_1'|} \right) p_{2k_1,I_1'}(\nu^t) \cdots p_{2k_{|I_1'|},I_1'}(\nu^t) \end{aligned}$$

and using (34), we have

(38)

$$(\nabla^2_{z,\bar z}A(r(s),\nu^t,\hat q))^{I_1''} = s^{|I_1''|}C_{0,I_1''} + \sum_{K=1}^{\infty}\sum_{\substack{k_1+\cdots+k_{|I_1''|}=K\\ k_j\ge 0,\ \text{all } j}} s^{|I_1''|-2K}C_{k_1,\dots,k_{I_1''},I_1''}p_{2k_1,I_1''}(\nu^t)\cdots p_{2k_{|I_1''|},I_1''}(\nu^t).$$

Here, $C_{0,I_1'}\big((\hat q,\bar{\hat q})^{|I_1'|}\big)$ and $C_{k_1,\dots,k_{I_1'},I_1'}\big((\hat q,\bar{\hat q})^{|I_1'|}\big)$ denote polynomial expressions involving $\hat q$ and $\bar{\hat q}$ of degree $|I_1'|$ and $C_{k_1,\dots,k_{I_1''},I_1''}$ are constants (independent of $\hat q$).

From the derivative products (37) and (38), a typical term in $(\nabla_{z,\bar z}A)^{I_1'}(\nabla^2_{z,\bar z}A)^{I_1''}$ is of the form

$$s^{|I_1'|+|I_1''|-2K}C_K\big((\hat q,\bar{\hat q})^{|I_1|}\big)p_{2k_1}(\nu^t)\cdots p_{2k_{|I_1|}}(\nu^t) \tag{39}$$

where $C_K\big((\hat q,\bar{\hat q})^{|I_1|}\big)$ is a polynomial in $\hat q$ and $\bar{\hat q}$ of degree at most $|I_1|$ and $k_1+\cdots+k_{|I_1'|+|I_1''|}=K\ge 1$ and each k_j is a nonnegative integer.

The main term is the lowest degree term in $\frac{1}{s}$ and is given by

$$(\nu^t)^{I_2}\frac{s^{|I_1'|+|I_1''|}C((\hat q,\bar{\hat q})^{|I_1'|})}{(s|\hat q|^2-i\nu_1)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}}$$

where $C((\hat q,\bar{\hat q})^{|I_1'|})$ is a monomial in terms in the coordinates for $(\hat q,\bar{\hat q})$ of degree $|I_1'|$. Its exact expression is possible to compute but not relevant for this calculation.

Letting

$$K_{j,I_1',I_1''}=k_1+\cdots+k_j+k_{I_1}+\cdots+k_{|I_1'|+|I_1''|}\ge 1+|I_1'|+|I_1''|,$$

a typical term from the expansion of (36) is

(40) Typical Term in (36)

$$= (\nu^t)^{I_2}s^{|I_1'|+|I_1''|-2K_{j,I_1',I_1''}}C_{\tilde K}\big((\hat q,\bar{\hat q})^{|I_1'|}\big)p_{2\tilde k_1}(\nu^t)\cdots p_{2\tilde k_{|I_1'|+|I_1''|}}(\nu^t) \times \frac{C(\hat q^j,\bar{\hat q}^j)\tilde p_{2k_1-e_1}(\nu^t)\dots\tilde p_{2k_j-e_j}(\nu^t)}{(s|\hat q|^2-i\nu_1)^{2n+m-1+j+|I_1'|+|I_1''|+|I_2|}s^{(2k_1-1)+\cdots+(2k_j-1)}}$$

where $k_\ell\ge 1$ if $j\ge 1$ (and does not appear if $j=0$) $C(\hat q^j,\bar{\hat q}^j)$ stands for monomial terms in the coordinates for $\hat q$ of degree j and $\bar{\hat q}$, of degree j, and where each $\tilde p_{2k_a-e_a}(\nu)$, $1\le a\le j$ is a monomial in the coordinates of ν of degree $2k_a-e_a$. Here, e_a is an even integer with $0\le e_a\le 2k_a$. Set $E_j=e_1+\cdots+e_j$ and incorporate the matrix M_t^{-1} into the $C(\hat q^j,\bar{\hat q}^j)$ term to obtain

(41)

$$\text{Typical Term in (40)} = \frac{C_t((\hat q,\bar{\hat q})^{2j+|I_1'|})\nu^{2K_{j,I_1',I_1''}-E_j}\nu^{I_2}}{(s|\hat q|^2-i\nu_1)^{2n+m-1+j+|I_1'|+|I_1''|+|I_2|}s^{2K_{j,I_1',I_1''}-(j+|I_1'|+|I_1''|)}}$$

and E_j is a an even integer with $0\le E_j\le 2(k_1+\cdots+k_j)$. Note that we have used the same abuse of notation with $\nu^{2K_j-E_j}$ as we did in (21) and the dependence on t is a (possibly nonsmooth but certainly bounded) dependence on $t/|t|$. We will not need all the terms in the expansion–just up through $j+|I_1'|+|I_1''|=2n-1$ with a remainder term involving $j+|I_1'|+|I_1''|=2n$ (and therefore $K_{j,I_1',I_1''}:=K_{2n}\ge 2n$).

In particular, using (37) and (38),

$$\text{(42)} \qquad \text{Typical Remainder Term in (36)} = \frac{O_t(1)O(\nu,s)}{(s|\hat{q}|^2 - i\nu_1)^{4n+m-1} s^{2K_{2n}-2n}}$$

where $O_t(1)$ is a real analytic function of the coordinates of $\hat{q}$ and $\bar{\hat{q}}$ that may depend on t. Also, $O(\nu, s)$ stands for a real analytic function in $\nu \in S^{m-1}$ and $s > 1$ and bounded in s. Note that the power of s in the denominator is at least $2n$ since $K_{2n} \geq 2n$, as mentioned above.

In the expansions of $B(r(s),\nu)\frac{r'(s)}{r(s)}$ and $\det([r^{-\bar{A}_\nu}]_{K,J})$ given in (20) and (22), respectively, writing $(1-1/s^2)^{-1} = \sum_{j'=0}^{\infty} s^{-2j'}$. Therefore, by (21), we see that up to the coefficients of some polynomials, a typical term in the expansion of $\det([r^{-\bar{A}_\nu}]_{K,J})\frac{B(r(s),\nu)r'(s)}{r(s)}$ is

$$\text{(43)} \qquad \text{Typical Term of } \det([r^{-\bar{A}_\nu}]_{K,J})\frac{B(r(s),\nu)r'(s)}{r(s)} = s^{2n-2j'-2-\ell'}\nu^{\ell'-e'}$$

together with a remainder of the form $\frac{O(\nu,s)}{s^{2+2j'}}$. Note e' is even and $0 \leq e' \leq \ell'$.

Now the typical term of $N_{K,J}$ is the product of a term in (41) with a term in (43). Therefore

$$\text{(44)} \quad \text{Typical Term in } N_{K,J} = C(\hat{q},\bar{\hat{q}})^{2j+|I_1'|}\frac{s^{N-2-\ell}\nu^{\ell-e}(\nu^t)^{I_2}}{(s|\hat{q}|^2 - i\nu_1)^{N+m-1+|I_2|}} \quad \text{where}$$

$$\text{(45)} \qquad N = 2n + j + |I_1'| + |I_1''|, \quad \ell = \ell' + 2j' + 2K_{j+|I_1'|+|I_1''|}, \quad e = e' + E_j + 2j'.$$

Note that e is even with $0 \leq e \leq \ell$, due to the constraints listed in on the indices in (41) and (43).

The remainder term for $N_{K,J}$ is the product of the remainders given in (42) and the remainder given just after (43): a typical term comprising the remainder is

$$\text{(46)} \quad \text{Typical Remainder Term for } N_{K,J} = \frac{O(\hat{q}^{4n})O(\nu,s)}{(s|\hat{q}|^2 - i\nu_1)^{4n+m-1} s^{2K_{2n}-2n+2+2j'}}$$

where $O(\nu, s)$ is real analytic function in $\nu \in S^{m-1}$ and $s \geq 3$ and bounded in s. Note that the exponent in s in the denominator is at least 2 since $K_j \geq j,\ j \geq 1$. We will now show that the integral (over $\nu \in S^{m-1}$, and $s \geq 1$) of the typical term in (44) is bounded in $\hat{q}$. We will also show the same for a remainder term in (46).

As to the first task, let $\hat{r} = |\hat{q}|^2 > 0$ and define

$$H_{N,\ell,m,e,I_2}(\hat{r},s,\nu) = \frac{s^{N-2-\ell}\nu^{\ell+I_2-e}}{(s\hat{r} - i\nu_1)^{N+m-1+|I_2|}}.$$

To establish Theorem 4.1 over the region $1/2 \leq r < 1$, we need to show that for each $\ell \geq 0$, there is a uniform constant C such that

$$\text{(47)} \qquad \Big| \int_{\nu \in S^{m-1}} \int_{s=3}^{\infty} H_{N,\ell,m,e,I_2}(\hat{r},s,\nu)\, ds\, d\nu \Big| \leq C$$

for all $\hat{r} > 0$ near zero.

8. Unit sphere integrals

To compute the integral of $H_{N,\ell,m,e,I_2}(\hat{r},s,\nu)$ over the unit sphere, S^{m-1} in $\mathbb{R}^m$, we need to use some easy facts about spherical integrals:

(1) Let $\nu = (\nu_1, \nu') \in S^{m-1}$, then ν' belongs to a $m-2$ dimensional sphere in $\mathbb{R}^{m-1}$ of radius $|\nu'| = \sqrt{1-\nu_1^2}$.
(2) Let θ be the "angle" between ν and the ν' plane; note that $\nu_1 = \sin(\theta)$, $-\pi/2 \leq \theta \leq \pi/2$; and $|\nu'| = \cos\theta$.
(3) Surface measure on the unit sphere in $\mathbb{R}^m$ is $d\nu = (\cos\theta)^{m-2}\, d\theta\, d\nu'$ where $d\nu'$ is surface measure on, S^{m-2}, the unit sphere in R^{m-1}.
(4) The integral of any odd function of ν' over S^{m-2}, the unit sphere in R^{m-1}, will be zero.

Using the last fact, we claim that we can assume the monomial $\nu^{\ell+I_2-e}$ depends on ν_1 only. To see this, write $\nu^{\ell+I_2-e} = (\nu')^a \nu_1^b$ with $|a| + b = |\ell| + |I_2| - |e|$. By (4), if $|a|$ were odd, then the ν'-integral would be zero. Thus we can assume $a = e'$ where $|e'|$ is even, which implies $|b| = |\ell| + |I_2| - (|e| + |e'|) = |\ell| + |I_2| - |E|$ with $|E|$ even. We can then factor out the $(\nu')^a$ from the ν_1 integral and we are left with $\nu_1^{\ell+I_2-E}$ within the ν_1 integral.

We now change variables and let $x = \nu_1 = \sin\theta$, $-1 \leq x \leq 1$. Note that $\cos\theta = \sqrt{1-x^2}$ and $d\theta = \frac{dx}{\sqrt{1-x^2}}$. Therefore

$$d\nu = (1-x^2)^{(m-3)/2}\, dx\, d\nu' \tag{48}$$

where $d\nu'$ is surface measure on S^{m-2}. The desired estimate in (47) will follow from Lemma 8.1:

Lemma 8.1. *For any nonnegative integers N, m and ℓ with $m \geq 2$ and any even integer E with $0 \leq E \leq |\ell| + |I_2|$, let*

$$A_{N,m}^{\ell,E,I_2}(\hat{r}) = \int_{x=-1}^{1} \int_{s=3}^{\infty} \frac{(1-x^2)^{(m-3)/2} s^{N-2-\ell} x^{\ell-E+|I_2|}\, ds\, dx}{(s\hat{r} - ix)^{N+m-1+|I_2|}}$$

then $A_{N,m,I_2}^{\ell,E,I_2}(\hat{r})$ is a smooth function of $\hat{r} > 0$ up to $\hat{r} = 0$.

As shown in the proof, the lemma is not true if E is odd.

Proof of Lemma 8.1. First write

$$A_{N,m}^{\ell,E,I_2}(\hat{r}) = C_{N,\ell,I_2} D_{\hat{r}}^{N-(2+\ell)} \left\{ B_{m,I_2}^{\ell,E}(\hat{r}) \right\}$$

where $C_{N,\ell}$ is a constant and

$$B_{m,I_2}^{\ell,E}(\hat{r}) = \int_{x=-1}^{1} \int_{s=3}^{\infty} \frac{(1-x^2)^{(m-3)/2} x^{\ell-E+|I_2|}\, ds\, dx}{(s\hat{r} - ix)^{m+\ell+1+|I_2|}}.$$

Here, $D_{\hat{r}}^j$ indicates the j^{th} derivative with respect to $\hat{r}$. The index j is allowed to be negative in which case this means the $|j|^{th}$ anti-derivative with respect to $\hat{r}$ (with a particular initial condition specified at a fixed value of $\hat{r} = \hat{r}_0 > 0$).

The proof of the fact will be complete once we show $B_{m,I_2}^{\ell,E}(\hat{r})$ is smooth for $\hat{r} > 0$ up to $\hat{r} = 0$. The s-integral can be computed to give:

$$B_{m,I_2}^{\ell,E}(\hat{r}) = \frac{1}{\hat{r}(m+\ell+|I_2|)} b_{m,I_2}^{\ell,E}(\hat{r}) \tag{49}$$

where

$$b_{m,I_2}^{\ell,E}(\hat{r}) = \int_{x=-1}^{1} \frac{(1-x^2)^{(m-3)/2} x^{\ell-E+|I_2|}\, dx}{(3\hat{r} - ix)^{m+\ell+|I_2|}}.$$

We need to show $b^{\ell,E}_{m,I_2}(\hat{r})$ is smooth in $\hat{r}>0$ up to $\hat{r}=0$ and that

$$b^{\ell,E}_{m,I_2}(\hat{r}=0)=0 \tag{50}$$

for then (49) will imply that $B^{\ell,E}_{m,I_2}(\hat{r})$ is smooth in $\hat{r}>0$ up to $\hat{r}=0$. To this end, we note that for $\hat{r}>0$, the integrand of $b^{\ell,E}_{m,I_2}(\hat{r})$ has an analytic extension in x to the upper half of the complex plane. So we can deform the integral using Cauchy to the top half of the unit circle, denoted by C^+ from $z=-1$ to $z=+1$ to obtain

$$b^{\ell,E}_{m,I_2}(\hat{r})=\int_{z\in C^+}\frac{(1-z^2)^{(m-3)/2}z^{\ell-E+|I_2|}\,dz}{(3\hat{r}-iz)^{m+\ell+|I_2|}}.$$

This expression shows that $b^{\ell,E}_{m,I_2}(\hat{r})$ extends smoothly (in fact, analytically) in $\hat{r}$ to a neighborhood of $\hat{r}=0$. All that remains to show is that $b^{\ell,E}_{m,I_2}(\hat{r}=0)=0$. We have

$$(-i)^{m+\ell+|I_2|}b^{\ell,E}_{m,I_2}(\hat{r}=0)=\int_{z\in C^+}\frac{(1-z^2)^{(m-3)/2}\,dz}{z^{m+E}}. \tag{51}$$

If $m=3$, then this integral is $\int_{z\in C^+}\frac{dz}{z^{3+E}}=0$ since e is even. If m is odd and greater than 3, then this integral can be reduced using integration by parts with $dv=1/z^{m+E}\,dz$ and $u=(1-z^2)^{(m-3)/2}$ (note there are no boundary terms at $z=\pm1$) to obtain

$$b^{\ell,E}_{m}(\hat{r}=0)=c_{m,\ell}\int_{z\in C^+}\frac{(1-z^2)^{(m-5)/2}\,dz}{z^{m+E-2}}.$$

One can continue integrating by parts this until the power of $(1-z^2)$ is zero to obtain

$$b^{\ell,E}_{m,I_2}(\hat{r}=0)=\tilde{c}_{m,\ell,I_2}\int_{z\in C^+}\frac{dz}{z^{3+E}}=0. \tag{52}$$

This establishes (50) for m odd. (Note clearly, the above integral is *not* zero if E is odd, which is why this assumption is so necessary).

If $m\geq 4$ is even, then we can integrate by parts until we obtain

$$b^{\ell,E}_{m,I_2}(\hat{r}=0)=\tilde{c}_{m,\ell,I_2}\int_{z\in C^+}\frac{\sqrt{1-z^2}\,dz}{z^{4+E}}.$$

Since E is even, let $E=2k$ for a nonnegative integer k. Amazingly, there is a closed-form antiderivative:

$$\int\frac{\sqrt{1-z^2}}{z^{4+E}}\,dz=-\sum_{j=0}^{k}\frac{(1-z^2)^{j+3/2}}{z^{2j+3}}\binom{k}{j}\frac{1}{(2j+3)}. \tag{53}$$

Clearly this antiderivative vanishes at both $z=\pm1$. If $m=2$, then one can integrate by parts in (51) with $dv=\frac{z\,dz}{\sqrt{1-z^2}}$ and reduce to this integral to (53). Thus, Lemma 8.1 and hence (47) are proved. □

9. The remainder term, $q\neq n$

To restate the remainder in (46)

$$\text{Remainder}=\frac{O(\nu,s)}{(s|\hat{q}|^2-i\nu_1)^{4n+m-1}s^J},\quad \text{with } J\geq 2.$$

We use the facts (1)–(3) about spherical integrals in the previous section with $x = \nu_1$. Since s^{-J} is integrable over $\{s \geq 3\}$ and since $O(\nu', \nu_1, x)$ is real analytic (and hence uniformly bounded) in $\nu' \in (\sqrt{1-x^2})S^{m-2}$ ($m-2$ dimensional sphere of radius $\sqrt{1-x^2}$), it suffices to prove the following lemma, which will finish the proof of Theorem 4.1 for the integral over the region $1/2 \leq r < 1$.

Lemma 9.1. *For $m \geq 2$, let*

$$R(s, \hat{r}, \nu') = \int_{x=-1}^{1} \frac{(1-x^2)^{\frac{m-3}{2}} O(\nu', x, s)\, dx}{(s\hat{r} - ix)^{4n+m-1}}.$$

Then $R(s, \hat{r}, \nu')$ is uniformly bounded for $s \geq 3$, $\hat{r} \geq 0$, and $\nu' \in (\sqrt{1-x^2})S^{m-2}$.

Proof. Divide up the interval $-1 \leq x \leq 1$ into $\{|x| \geq 1/2\}$ and $\{|x| \leq 1/2\}$. The denominator is bounded below on $\{|x| \geq 1/2\}$. The numerator is also bounded except in the case $m = 2$ in which case $(1-x^2)^{\frac{m-3}{2}}$ has an integrable singularity at $x = \pm 1$.

For the interval $\{|x| \leq 1/2\}$, we replace x by $z \in \mathbb{C}$ and note that the integrand can be extended to analytic function z in a complex neighborhood of the interval $-1/2 \leq x \leq 1/2$. Let C be a path in this neighborhood and in the upper half plane which connects $z = -1/2$ to $z = 1/2$ and otherwise does not intersect the real axis. Using Cauchy's Theorem, we have

$$\int_{x=-1/2}^{1/2} \frac{(1-x^2)^{\frac{m-3}{2}} O(\nu', x, s)\, dx}{(sr - ix)^{4n+m-1}} = \int_{z \in C} \frac{(1-z^2)^{\frac{m-3}{2}} O(\nu', z, s)\, dz}{(sr - iz)^{4n+m-1}}.$$

Since the denominator is uniformly bounded away from zero, for $z \in C$, $s \geq 3$ and $r = |\hat{q}|^2 > 0$, the integral on the right is uniformly bounded in ν', r, and s. This completes the proof. $\square$

10. The case $0 \leq r \leq 1/2$, $q \neq n$

Our starting point is (9) which equates to (35) but we wish to remain in the r variable. We fix K, restrict the r integral to $0 \leq r \leq 1/2$ and examine

$$\begin{aligned} &N^A_{I_1', I_1'', I_2}(\hat{q}) \\ &= \sum_{L \in \mathcal{I}_q} \int_{\nu \in S^{m-1}} \int_{r=0}^{\frac{1}{2}} \det(\bar{U}(\nu)_{K,L})\, d\bar{Z}(z, \nu^t)^L B_L(r, \nu^t) \\ &\quad \times \frac{(\nu^t)^{I_2} (\nabla_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1'} (\nabla^2_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1''}}{(A(r, \nu^t, \hat{q}) - i\nu_1)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}} \frac{d\nu\, dr}{r} \end{aligned} \tag{54}$$

$$\begin{aligned} &= \sum_{J \in \mathcal{I}_q} d\bar{z}^J \Bigg[\int_{\nu \in S^{m-1}} \int_{r=0}^{\frac{1}{2}} \det([r^{-\bar{A}_\nu}]_{K,J}) B(r, \nu^t) \\ &\quad \times \frac{(\nu^t)^{I_2} (\nabla_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1'} (\nabla^2_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1''}}{(A(r, \nu^t, \hat{q}) - i\nu_1)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}} \Bigg] \frac{d\nu\, dr}{r}. \end{aligned} \tag{55}$$

We denote by $N^J_{I_1', I_1'', I_2}(\hat{q})$ the $d\bar{z}^J$ coefficient of $N^A_{I_1', I_1'', I_2}(\hat{q})$.

We devote the remainder of this section to the proof of Lemma 10.1, which will establish Theorem 4.1 for the integral over the region $0 < r < 1/2$.

Lemma 10.1.

$$|N^J_{I'_1,I''_1,I_2}(\hat{q})| \leq C \quad \textit{for all } \hat{q} = \frac{z}{|t|^{1/2}} \in \mathbb{C}^n \tag{56}$$

where C is a uniform constant.

Recall from (10) that

$$B_L(r,\nu) = \prod_{\substack{j\in L^c\cap P \\ j\in L\cap P^c}} \frac{r^{|\mu_j^\nu|}|\mu_j^\nu|}{1-r^{|\mu_j^\nu|}} \prod_{\substack{k\in L\cap P \\ k\in L^c\cap P^c}} \frac{|\mu_k^\nu|}{1-r^{|\mu_k^\nu|}}.$$

Since $L \in \mathcal{I}_q$ and $q \neq n$, at least one of $L^c \cap P$ or $L \cap P^c$ is nonempty. This means there exist constants $C > 0$ and

$$c_0 = \min\Big\{ \sum_{\substack{j\in L^c\cap P \\ j\in L\cap P^c}} |\mu_j^\nu| : \nu \in S^{m-1} \text{ and } L \in \mathcal{I}_q \Big\}$$

so that

$$\Big|\frac{B(r,\nu)}{r}\Big| \leq C r^{c_0-1} \quad \text{for } 0 < r < 1/2. \tag{57}$$

From this estimate, it follows that the integrals in (54) and therefore in (55) over $\{0 \leq r \leq 1/2\} \times \{|\nu_1| \geq 1/2\}$ are uniformly bounded for $\hat{q} \in \mathbb{C}^n$. Moreover, we know from (16) and the accompanying remark that the integrand of $N^J_{I'_1,I''_1,I_2}(\hat{q})$ is real analytic in $\nu \in S^{m-1}$ and $0 < r \leq 1/2$.

We now concentrate on the ν_1-integral over $|\nu_1| \leq 1/2$. We have

$$A(r,\nu,\hat{q}) = \sum_{j=1}^{2n} |\mu_j^\nu| \left(\frac{1+r^{|\mu_j^\nu|}}{1-r^{|\mu_j^\nu|}}\right) |Q_j(\nu,\hat{q})|^2$$

with

$$Q(\nu,\hat{q}) = |t|^{-1/2} Z(\nu,z) = U(\nu)^* \cdot \hat{q}$$

where $U(\nu)$ is the unitary matrix which diagonalizes the scalar valued Levi form in the normal direction ν. For $u \in \mathbb{R}$, let

$$\Lambda(u) = |u| \left(\frac{1+r^{|u|}}{1-r^{|u|}}\right). \tag{58}$$

As a generalization of (7), we have

$$A(r,\nu,\hat{q}) = \sum_{j=1}^{2n} \Lambda(\mu_j)|Q_j(\nu,\hat{q})|^2 = \hat{q}^* \cdot \Lambda(A_\nu) \cdot \hat{q} \tag{59}$$

where A_ν is the Hessian matrix of $\Phi(z,z)\cdot\nu$. Here $\Lambda(A_\nu)$ is computed by replacing $|u|$ by $\sqrt{A_\nu^2}$ in (58) and where $\left(I - r^{\sqrt{A_\nu^2}}\right)^{-1}$ is the matrix inverse of $I - r^{\sqrt{A_\nu^2}}$. Furthermore, $r^{\sqrt{A_\nu^2}}$ is defined as $\exp\left(\ln r \sqrt{A_\nu^2}\right)$. Note that since all the eigenvalues of A_ν are real and bounded away from zero, the (operator) norm of the matrix $r^{\sqrt{A_\nu^2}}$, for $0 \leq r \leq 1/2$, is less than one since $\ln r < 0$, guaranteeing the existence of the inverse of $I - r^{\sqrt{A_\nu^2}}$. For this analysis to work, we need to know the map $\nu \to \sqrt{A_\nu^2}$ is analytic in ν, established in Lemma 10.2.

Lemma 10.2. *The map $\nu \to \sqrt{A_\nu^2}$ is analytic for $\nu \in S^{m-1}$.*

Proof. Observe that the matrix $X = A_\nu^2$ is a Hermitian symmetric matrix with positive eigenvalues which are contained in a compact interval, say $[c_0, R] \subset \mathbb{R}$ with $R > c_0 > 0$, for all $\nu \in S^{m-1}$. So consider the power series for $\sqrt{X}$ about $X = RI$:

$$\sqrt{X} = \sum_{n=0}^{\infty} a_n [X - RI]^n$$

which has radius of convergence R. Since the open disc centered at $x = R$ of radius R contains all the eigenvalues of A_ν^2 in its interior, the following series converges uniformly in ν:

$$\sqrt{A_\nu^2} = \sum_{n=0}^{\infty} a_n [A_\nu^2 - RI]^n.$$

This series is clearly analytic in $\nu \in S^{m-1}$. □

Proof of Lemma 10.1. The expression for A given in (59) and the discussion following shows that

$$X(r, \nu, \hat{q}) := \left[\frac{\partial}{\partial \nu_1} \{A(r, \nu^t, \hat{q})\} - iI \right]^{-1}$$

is a smooth matrix on $\{0 < r \leq 1/2\} \times \{\nu \in S^{m-1}\}$ with $X(r, \nu, \hat{q}) \to 0$ as $r \to 0$ (due to a factor of $\ln r$ in the denominator). Moreover, since $\frac{dr^u}{du} = r^u \ln r$, differentiation of $A(r, \nu^t, \hat{q})$, $X(r, \nu, \hat{q})$, $B(r, \nu^t)$, or $\det([r^{-\bar{A}_\nu}]_{K,J})$ produces a term of the same size with a possible additional $\ln r$ term. However, from (57), there is always a r^{c_0-1} term, and $r^{c_0-1}|\ln r|^N$ is integrable at 0 for any power N since $c_0 > 0$.

Now view the set $\{\nu \in S^{m-1};\ |\nu_1| \leq 1/2\}$ as the graph over the set

$$\{\nu = (\nu_1, \nu');\ |\nu_1| \leq 1/2;\ \nu' \in S^{m-2}\}.$$

We use integration by parts for the integral over $|\nu_1| \leq 1/2$ of (55) as follows:

$$\begin{aligned}
&\int_{\nu_1=-1/2}^{\nu_1=1/2} \frac{(\nu^t)^{I_2} (\nabla_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1'} (\nabla^2_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1''}}{(A(r, \nu^t, \hat{q}) - i\nu_1)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}} \frac{\det([r^{-\bar{A}_\nu}]_{K,J}) B(r, \nu^t)}{r} \, d\nu_1 \\
&= \int_{\nu_1=-1/2}^{\nu_1=1/2} \frac{\partial}{\partial \nu_1} \left\{ \frac{-(2n+m+|I_1'|+|I_1''|+|I_2|-2)^{-1}}{(A(r, \nu^t, \hat{q}) - i\nu_1)^{2n+m+|I_1'|+|I_1''|+|I_2|-2}} \right\} \\
&\times \frac{(\nu^t)^{I_2} (\nabla_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1'} (\nabla^2_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1''} X(r, \nu^t, \hat{q}) \det([r^{-\bar{A}_\nu}]_{K,J}) B(r, \nu^t)}{r} \, d\nu_1 \\
&= -\int_{\nu_1=-1/2}^{\nu_1=1/2} \frac{-(2n+m+|I_1'|+|I_1''|+|I_2|-2)^{-1}}{(A(r, \nu^t, \hat{q}) - i\nu_1)^{2n+m+|I_1'|+|I_1''|+|I_2|-2}} \\
&\times \frac{\partial}{\partial \nu_1} \left\{ \frac{(\nu^t)^{I_2} (\nabla_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1'} (\nabla^2_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1''} X(r, \nu^t, \hat{q}) \det([r^{-\bar{A}_\nu}]_{K,J}) B(r, \nu^t)}{r} \right\} d\nu_1 \\
&+ \text{Boundary Terms at } |\nu_1| = 1/2.
\end{aligned}$$

The power of $(A(r, \nu^t, \hat{q}) - i\nu_1)$ in the denominator has been reduced by one. As discussed above and analogous to (57), we have

$$\left| \frac{\partial}{\partial \nu_1} \left\{ \frac{X(r, \nu^t, \hat{q}) \det([r^{-\bar{A}_\nu}]_{K,J}) B(r, \nu^t) (\nu^t)^{I_2} (\nabla_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1'} (\nabla^2_{z,\bar{z}} A(r, \nu^t, \hat{q}))^{I_1''}}{r} \right\} \right| \leq C |\ln r| r^{c_0 - 1}$$

which is integrable in $0 \le r \le 1/2$. The boundary terms are also controlled by a similar estimate.

We continue integrating by parts in ν_1 until we reduce the fractional expression involving $(A(r,\nu^t,\hat{q}) - i\nu_1)$ to a log-term, to obtain:

$$\begin{aligned}
&\int_{\nu_1=-1/2}^{\nu_1=1/2} \frac{(\nu^t)^{I_2}(\nabla_{z,\bar{z}}A(r,\nu^t,\hat{q}))^{I_1'}(\nabla^2_{z,\bar{z}}A(r,\nu^t,\hat{q}))^{I_1''}}{(A(r,\nu^t,\hat{q}) - i\nu_1)^{2n+m-1+|I_1'|+|I_1''|+|I_2|}} \frac{\det([r^{-\bar{A}_\nu}]_{K,J})B(r,\nu^t)}{r}\, d\nu_1 \\
&= c_{n,m,I_1',I_1'',I_2} \int_{\nu_1=-1/2}^{\nu_1=1/2} \ln\left[A(r,\nu^t,\hat{q}) - i\nu_1\right] \\
&\qquad \times \frac{E[X(r,\nu^t,\hat{q}), \det([r^{-\bar{A}_\nu}]_{K,J}), B(r,\nu^t), A(r,\nu^t,\hat{q}), I_1', I_1'', I_2]}{r}\, d\nu_1 \\
&+ \text{Boundary Terms at } |\nu_1| = 1/2
\end{aligned}$$

where c_{n,m,I_1',I_1'',I_2} is a constant depending only on n, m, I_1', I_1'', I_2; ln is the principal branch of the logarithm defined on the right half-plane (note $A(r,\nu^t,\hat{q}) > 0$); and the function $E[X(r,\nu^t,\hat{q}), \det([r^{-\bar{A}_\nu}]_{K,J}), B(r,\nu^t), A(r,\nu^t,\hat{q}), I_1', I_1'', I_2]$ is an expression involve a sum of products of ν_1-derivatives of $X(r,\nu^t,\hat{q})$, $\det([r^{-\bar{A}_\nu}]_{K,J})$, and $B(r,\nu^t)$ where the total number of derivatives is $2n+m+|I_1'|+|I_1''|+|I_2|-1$. Note that $|\ln[A(r,\nu^t,\hat{q}) - i\nu_1]|$ is integrable in ν_1 uniformly in the other variables $\nu' \in S^{m-2}$ and $0 \le r \le 1/2$. In addition

$$\left|\frac{E\left[X(r,\nu^t,\hat{q}),\ B(r,\nu^t)\right]}{r}\right| \le C|\ln r|^{2n+m+|I_1'|+|I_1''|+|I_2|-1} r^{c_0-1}$$

which is also integrable $\nu' \in S^{m-2}$ and $0 \le r \le 1/2$. Similar estimates hold for the boundary terms. This establishes (56) and completes the proof of Lemma 10.1. This also concludes the proof of Theorem 4.1 and hence establishes Theorem 2.1 when $|t| \ge |z|^2$. □

11. The $|z|^2 \ge |t|$ case, $q \ne n$

To complete the proof of Theorem 2.1, we have left to check the case when $|z|^2 \ge |t|$. As before, we break the integral up into two cases: $0 < r \le \frac{1}{2}$ and $\frac{1}{2} < r \le 1$.

11.1. The case $\frac{1}{2} < r \le 1$. As before, we start with the harder case. Fortunately, though, the bulk of the preliminary computations still hold. We take (41) as our starting point. The differences between the $|t|$ and $|z|^2$ dominant cases, though, is that in the manipulations leading to (41) we do not want to factor $|t|$ out of the integral and replace z by $\hat{q}$. We also worry about the $C_t((\hat{q},\bar{\hat{q}})^{2j+|I_1'|}$ term which is now a $C(z,\bar{z})^{2j+|I_1'|}$ term. We will use size estimates and ignore completely the (uniformly bounded) ν terms. Thus, (41) simplifies to

$$|\text{Typical Term}| \le \frac{C(z,\bar{z})^{2j+|I_1'|}}{(s|z|^2)^{2n+m-1+j+|I_1'|+|I_1''|+|I_2|} s^{2K_{j,I_1',I_1''}-(j+|I_1'|+|I_1''|)}} \tag{60}$$

and we estimate

$$\int_{s=3}^{\infty}\int_{\nu\in S^{m-1}} \frac{C(z,\bar z)^{2j+|I_1'|}}{(s|z|^2)^{2n+m-1+j+|I_1'|+|I_1''|+|I_2|}s^{2K_{j,I_1',I_1''}-(j+|I_1'|+|I_1''|)}}\,d\nu\,ds$$
$$\le C_{j,I_1',I_1'',I_2}\frac{|z|^{2j+|I_1'|}}{|z|^{2(2n+m-1+j+|I_1'|+|I_1''|+|I_2|)}} = C_{j,I_1',I_1'',I_2}\frac{1}{|z|^{2(2n+m-1+\frac{1}{2}\langle I\rangle)}}.$$

Since $\langle I\rangle = |I_1'| + 2|I_1''| + 2|I_2|$, this establishes the estimate in Theorem 2.1 for this term. The remainder term (46) has a similarly straightforward adaptation and estimate.

11.2. **The case** $0 < r \le \frac{1}{2}$**.** The estimates in this case will also follow from size estimates. We established the key estimate on $B(r,\nu)$ in (57). Moreover, since the eigenvalues for $\nu \in S^{m-1}$ are bounded away from zero (say by c), we have

$$\frac{1}{1-r^u} \le \frac{1}{1-(1/2)^c} \le C.$$

It therefore follows that for $j = 0, 1, 2$

$$|\nabla^j_{z,\bar z} A(r,\nu,z)| \sim C|z|^{2-j}.$$

Consequently, we ignore the t-term and estimate (8) directly by

$$|N_{I_1',I_1'',I_2}(z,t)| \le C_{I_1',I_1'',I_2}\int_{r=0}^{\frac{1}{2}}\int_{\nu\in S^{m-1}} r^{c_0-1}\frac{|z|^{|I_1'|}d\nu\,dr}{|z|^{2(2n+m-1+|I_1'|+|I_1''|+|I_2|)}}$$
$$= \frac{C_{I_1',I_1'',I_2}}{|z|^{2(2n+m-1+\frac{1}{2}\langle I\rangle)}}.$$

This establishes the desired estimate in the case when $|z|^2 \ge |t|$ and hence concludes the proof of Theorem 2.1.

12. The case $q = n$

The techniques that prove the estimates in the $q \ne n$ case are robust enough to work in the $q = n$ case, as well. However, the non-triviality of $\ker \Box_b$ changes for the formula for $N_K(z,t)$, and in this section, we sketch the argument for the $I = \emptyset$ case, which is when there are no derivatives. We also assume, without loss of generality, that the set of positive indices $P = \{1, 2, \ldots, n\}$.

We computed the relative solution to $\Box_b$ in the case $q = n$ given by $\int_0^\infty e^{-s\Box_b}(I - S_n)\,ds$ in [BR22]. Following the notation of [BR22], for each q-tuple $L \in \mathcal{I}_q$, we set

$$\Gamma_L = \{\alpha \in S^{m-1} : \mu^\alpha_\ell > 0 \text{ for all } \ell \in L \text{ and } \mu^\alpha_\ell < 0 \text{ for all } \ell \notin L\}.$$

If $L \in \mathcal{I}_n$, then $\Gamma_L = \emptyset$, unless $L = P$, in which case $\Gamma_P = S^{m-1}$. Therefore, from [BR22, Theorem 2.2, Part 3], if $K \in \mathcal{I}_n$, then

(61)
$$N_K(z,t) = K_{n,m}\Big[\sum_{L\in\mathcal{I}_n, L\neq P}\int_{\nu\in S^{m-1}} \det(\bar{U}(\nu)_{K,L})\, d\bar{Z}(z,\nu)^L \times \int_{r=0}^1 \frac{B_L(r,\nu)}{(A(r,\nu,z)-i\nu\cdot t)^{2n+m-1}}\frac{dr\, d\nu}{r}$$

(62)
$$+\int_{\nu\in S^{m-1}} \det(\bar{U}(\nu)_{K,P})\, d\bar{Z}(z,\nu)^P \times \int_{r=0}^1 \left[\frac{B_P(r,\nu)}{(A(r,\nu,z)-i\nu\cdot t)^{2n+m-1}} - \frac{|\det A_\nu|}{(A(0,\nu,z)-i\nu\cdot t)^{2n+m-1}}\right]\frac{dr\, d\nu}{r}\Big].$$

12.1. **The case $1/2 < r < 1$.** As above, we split up the r-integral into $0 < r \leq 1/2$ and $1/2 < r < 1$. The challenge is in the region when $1/2 < r < 1$ which is where we concentrate our efforts. In this case, the first fraction of the integrand in (62) can be combined with the terms in (61) so that the sum can range over all $L \in I_n$ in (61). After factoring out $|t|$ and rotating coordinates so that $\nu\cdot t/|t| = \nu_1$, we set $\hat{q} = \frac{z}{|t|^{1/2}}$ and rewrite the integral over $1/2 \leq r \leq 1$ as follows:

$$|t|^{2n+m-1}N_K^1$$

(63)
$$= \sum_{L\in\mathcal{I}_n}\int_{\nu\in S^{m-1}} \det(\bar{U}(\nu_t)_{K,L})\, d\bar{Z}(\hat{q},\nu_t)^L \int_{r=1/2}^1 \frac{B_L(r,\nu_t)}{(A(r,\nu_t,\hat{q})-i\nu_1)^{2n+m-1}}\frac{dr\, d\nu}{r}$$

(64)
$$-\int_{\nu\in S^{m-1}} \det(\bar{U}(\nu_t)_{K,P})\, d\bar{Z}(\hat{q},\nu_t)^P \int_{r=1/2}^1 \frac{|\det A_{\nu_t}|}{(A(0,\nu_t,\hat{q})-i\nu_1)^{2n+m-1}}\frac{dr\, d\nu}{r}$$

and where (as above) $\nu_t = M_t^{-1}(\nu)$ and M_t is an orthogonal transformation on $\mathbb{R}^m$ with $M_t(t/|t|) = e_1$. The analysis of (63) is precisely the same as we carried out in Sections 3–11. Thus we focus on (64). We show the following

Proposition 12.1. *Let*

(65)
$$N_K^2(\hat{q},t) = \int_{\nu\in S^{m-1}} \det(\bar{U}(\nu_t)_{K,P}) d\bar{Z}(\hat{q},\nu_t)^P \int_{r=1/2}^1 \frac{|\det A_{\nu_t}|}{(A(0,\nu_t,\hat{q})-i\nu_1)^{2n+m-1}}\frac{dr d\nu}{r}.$$

Then there are positive constants c_0 and C_0 such that $|N_K^2(\hat{q},t)| \leq C_0$ for all $|\hat{q}| \leq c_0$.

Remark 12.2. Note that the case $|\hat{q}| > c_0$ falls into the $|z|$ dominant case which is much easier to handle.

We devote the remainder of this section to proving this proposition. To prove the proposition, we need the following analyticity lemma.

Lemma 12.3. *The following functions are analytic as a function of $\nu \in S^{m-1}$:*

- $\nu \to |\det(A_\nu)|$
- $\nu \to A(0,\nu,\hat{q}) = \sum_{j=1}^{2n} |\mu_j^\nu||\hat{q}_j^\nu|^2$

- $\nu \to \det(\bar{U}(\nu)_{K,P})\, d\bar{Z}(\hat{q},\nu)^P = \sum_{J\in\mathcal{I}_n} \det(\bar{U}(\nu)_{K,P}) \det[U(\nu)_{P,J}]^T\, d\bar{z}^J$

Proof. For the first bullet, note that A_ν has n positive and n negative eigenvalues and so $|\det A_\nu| = (-1)^n \det A_\nu$. So the expression in the first bullet is analytic since A_ν is linear in ν.

For the second bullet, note that A_ν and $|A_\nu| = \sqrt{A_\nu^2}$ have the same eigenvectors. Therefore

$$\sum_{j=1}^{2n} |\mu_j^\nu||\hat{q}_j^\nu|^2 = \sum_{j=1}^{2n} |\mu_j^\nu||Q_j(\nu,\hat{q})|^2 = \hat{q}^* U(\nu)\cdot |D_\nu| \cdot U(\nu)^*\hat{q} = \hat{q}^*\sqrt{A_\nu^2}\,\hat{q} \tag{66}$$

which is analytic in ν. Here, D_ν is the diagonal matrix with the eigenvalues of A_ν as its diagonal entriesl, and $\sqrt{\ }$ is the principal branch of the square root of a positive definite Hermitian symmetric matrix.

Showing the expression in the third bullet is analytic in ν is equivalent to showing that the expression

$$\det(\bar{U}(\nu)_{K,P}) \det[U(\nu)_{P,J}]^T \tag{67}$$

is real analytic in $\nu \in S^{m-1}$ for each J, K in $\mathcal{I}_n$.

We shall need the standard branch of the function $\arctan z$, which is holomorphic on $\mathbb{C} \setminus \{z = iy : x = 0 \text{ and } |y| \geq 1\}$. Let I_{2n} be the $2n \times 2n$ identity matrix and consider the sequence

$$\frac{1}{\pi}\Big(\arctan(nA^\nu) + \frac{\pi}{2} I_{2n}\Big)$$

for $j = 1, 2, \ldots$. Since the eigenvalues of A^ν are bounded away from zero, each of these matrices in this sequence is analytic in $\nu \in S^{m-1}$ and is diagonalized by $U(\nu)$ and $U(\nu)^*$. Furthermore, this sequence converges uniformly in ν as $j \to \infty$ to a matrix A_0^ν, which is analytic in ν with n eigenvalues equal to 1 and n eigenvalues equal to -1. Also A_0^ν is diagonalized by $U(\nu)$ and $[U(\nu)]^*$.

Now consider

$$\tilde{A}_0^\nu = \frac{1}{2}(A_0^\nu + I).$$

$\tilde{A}_0^\nu$ is analytic in ν and has n eigenvalues equal to 1 and n eigenvalues equal to zero. It is also diagonalized by $U(\nu)$ and $[U(\nu)]^*$. Therefore,

$$[U(\nu)]^* \tilde{A}_0^\nu U(\nu) = D^0, \quad \text{where}$$

$$D^0 = \begin{pmatrix} I_n & 0_n \\ 0_n & 0_n \end{pmatrix}$$

and where I_n is the $n\times n$ identity matrix and 0_n is the $n\times n$ zero matrix. Therefore

$$\overline{\tilde{A}_0^\nu} = \bar{U}(\nu) D^0 [U(\nu)]^T.$$

Taking determinants, we have

$$\det[\overline{\tilde{A}_0^\nu}]_{KJ} = \sum_{L,L'} \det(\bar{U}(\nu)_{K,L}) \det(D^0_{L,L'}) \det([U(\nu)_{L',J}]^T).$$

Given that D_0 is diagonal, the only nonzero contributions to this sum occur when $L = L' = P$, which is the set of positive indices $= 1, \ldots, n$. We obtain

$$\det[\overline{\tilde{A}_0^\nu}]_{KJ} = \det(\bar{U}(\nu)_{K,P}) \det([U(\nu)_{P,J}]^T).$$

Since the left side is analytic in ν, so is the right side and this establishes the analyticity of (67) and thus concludes the proof of the lemma. □

Proof of Proposition 12.1. Note that the r-integral in (65) can be computed exactly (as $\ln(2)$), so we need only examine the ν-integral. Clearly, the integral over $|\nu_1| \geq 1/2$ clearly bounded uniformly in q and t. So we restrict attention to the region $\{\nu \in S^{m-1};\ |\nu_1| \leq 1/2\}$. The key is to examine the integral over ν_1-slices of this region. Without loss of generality, let us assume we are on a region, V, of the sphere where $\nu_m = h(\nu_1, \nu')$ can be written as an analytic function of the other variables (ν_1, ν') with $\nu' = (\nu_2, \ldots, \nu_{m-1})$. We may further assume that the "cap" V is large enough so that projection of V onto $\nu_m = 0$ contains the disk $\{(\nu_1, \nu') \in \mathbb{R}^{m-1} : |\nu_1| \leq \frac{1}{2} \text{ and } |\nu'| \leq \frac{1}{2}\}$. We also write $d\nu = g(\nu_1, \nu') d\nu_1 d\nu'$ where g is an analytic function on V. Let

$$G(\hat{q}, t, \nu_1, \nu') = \det(\bar{U}^{\tilde{\nu}_t}_{K,P})\, d\bar{Z}(\hat{q}, \tilde{\nu}_t)^P |\det A_{\tilde{\nu}_t}|\, g(\nu_1, \nu') \qquad \text{where } \tilde{\nu}_t = M_t^{-1}(\nu_1, \nu', h(\nu_1, \nu')).$$

From Lemma 12.3, $G(\hat{q}, t, \nu_1, \nu')$ is analytic in ν_1, ν' and uniformly bounded in $\nu \in V \subset S^{m-1}$, $\hat{q}$, and t. We need to show that there are positive constants c_0 and C_0 so that

$$\Big| \int_{|\nu_1| \leq 1/2} \frac{G(\hat{q}, t, \nu_1, \nu')\, d\nu_1}{(A(0, \tilde{\nu}_t, \hat{q}) - i\nu_1)^{2n+m-1}} \Big| \leq C_0 \quad \text{for all } |\nu'| \leq \frac{1}{2} \text{ and } |\hat{q}| \leq c_0. \tag{68}$$

We shall proceed by using Cauchy's Theorem to bump the contour of integration around the potential singularity at $\nu_1 = 0$. First, let $\delta_0 > 0$ be chosen small enough to that $G(\hat{q}, t, \nu_1, \nu')$ and $A(0, \tilde{\nu}_t, \hat{q})$ analytically continue from $\{\nu_1 \in \mathbb{R};\ |\nu_1| \leq 1/2\}$ to a neighborhood of the rectangle $\tilde{V}_1 = \{\zeta_1 = \nu_1 + i\eta_1 \in \mathbb{C};\ |\nu_1| \leq 1/2 \text{ and } 0 \leq \eta_1 \leq \delta_0\}$ in the upper half plane and for all $(\nu_1, \nu', h(\nu_1, \nu')) \in V$. Also note from (66) that $A(0, \nu, \hat{q}) = \hat{q}^* \sqrt{A_\nu^2}\, \hat{q} \geq 0$ for $\nu \in V$. In addition, the analytic extension of $A(0, \nu, \hat{q})$ to $\tilde{V}_1$ is the function

$$A(0, \tilde{\nu}_t(\zeta_1), \hat{q}) := A(0, M_t^{-1}(\zeta_1, \nu', h(\zeta_1, \nu_t), \hat{q}).$$

Furthermore, its ζ_1 derivative is uniformly bounded by $\tilde{C}|\hat{q}|^2$ for $\zeta_1 \in \tilde{V}_1$ and $\nu \in V$ where $\tilde{C} > 0$ is a uniform constant. The following estimate now follows:

$$\operatorname{Re} A(0, \tilde{\nu}_t(\zeta_1), \hat{q}) \geq -\tilde{C}|\hat{q}|^2 \eta_1 \quad \text{for } \zeta_1 = \nu_1 + i\eta_1 \in \tilde{V}_1.$$

This inequality implies

$$|A(0, \tilde{\nu}_t(\zeta_1), \hat{q}) - i\zeta_1| \geq (1 - \tilde{C}|\hat{q}|^2)\eta_1 \quad \text{for } \zeta_1 = \nu_1 + i\eta_1 \in \tilde{V}_1.$$

Let γ_1 be the upper three sides of the boundary of the rectangle of $\tilde{V}_1$, i.e. the union of the three line segments, respectively, from $-1/2$ to $-1/2 + i\delta$; from $-1/2 + i\delta$ to $1/2 + i\delta$, and from $1/2 + i\delta$ to $1/2$. The above inequality shows that there is a constant $c_0 > 0$ such that if $|\hat{q}| < c_0$, then

$$\begin{aligned} &|A(0, \tilde{\nu}_t(\zeta_1), \hat{q}) - i\zeta_1| > 0 \quad \text{for } \zeta_1 \text{ inside } \tilde{V}_1 \text{ and} \\ &|A(0, \tilde{\nu}_t(\zeta_1), \hat{q}) - i\zeta_1| \geq c_0 \quad \text{for } \zeta_1 \in \gamma_1\ . \end{aligned}$$

Now we can use Cauchy's Theorem to deform the path of integration in (68) to γ_1 and the proof of the estimate in (68) easily follows. This concludes the proof of the proposition. ∎

12.2. **The cases** $0 < r < 1/2$ **and** $|z|^2 > |t|$**.** The estimates of N_K for the interval $0 < r < 1/2$ follow the same arguments as given in Section 10. The extra term arising from S_n in (62) eliminates the convergence issues at $r = 0$. Lemma 12.3 and the earlier analyticity lemmas show that all the components of the integrands are analytic in ν. Since $L \neq P$ in the sum in (61), the numerator of its integrand contains a positive power of r. In addition, the term in brackets [ ] in (62) vanishes at $r = 0$, so both integrands are integrable in r near $r = 0$. Therefore, the same integration by parts argument from Section 10 applies to reduce the power of the denominator terms (down to a log-term) to prove the desired estimates.

The case when $|z|^2 > |t|$ is handled using the techniques of Section 11. This completes the proof of Theorem 2.1.

13. Examples

Here, we record four examples with complex tangent dimension $2n \geq 2$ and higher codimension $m \geq 2$ in cases where the eigenvalues are always nonzero. These examples piggy back on the following standard example in the case of $2n = 2$ and $m = 2$ originally computed in [BR13, BR20]:

Example 13.1. $2n = 2$, $m = 2$, and $q = 0$. Consider $\Phi(z, z) = (\phi_1(z, z), \phi_2(z, z))$ where

$$\phi_1(z, z) = 2\,\mathrm{Re}(z_1\bar{z}_2)$$
$$\phi_2(z, z) = |z_1|^2 - |z_2|^2.$$

The eigenvalues of the A_ν (the Hessian of $\Phi(z, z) \cdot \nu$) are $+1$ and -1. We use formula (3) for N_L with $L = \emptyset$ and so $\varepsilon_1 = -1$ and $\varepsilon_2 = +1$. Since $m = 2$, S^{m-1} is just the unit circle parameterized by $\nu = (\cos\theta, \sin\theta)$, $0 \leq \theta \leq 2\pi$ and $d\nu = d\theta$. We rotate θ coordinates so that $\nu \cdot t$ becomes $|t|\sin\theta$. From (3), we obtain

$$N(z, t) = \frac{4^2}{2(2\pi)^4} \int_0^1 \int_0^{2\pi} \frac{r}{(1-r)^2} \frac{2!\, d\theta}{\left[\left(\frac{1+r}{1-r}\right)|z|^2 - i|t|\sin\theta\right]^3} \frac{dr}{r}.$$

We let $s = \frac{r+1}{1-r}$, $ds = \frac{2\,dr}{(1-r)^2}$ to obtain

$$\begin{aligned} N(z, t) &= \frac{4^2}{2(2\pi)^4} \int_0^{2\pi} \int_1^{\infty} \frac{ds\, d\theta}{[s|z|^2 - i|t|\sin\theta]^3} \\ &= \frac{4}{(2\pi)^4} \frac{1}{|z|^2|t|^2} \int_0^{2\pi} \frac{d\theta}{[|\hat{q}|^2 - i\sin\theta]^2} \quad \text{with } \hat{q} = z/|t|^{1/2} \\ &= \frac{1}{2\pi^3} \frac{1}{[|z|^4 + |t|^2]^{3/2}} \approx \frac{1}{(|z| + |t|^{1/2})^6} \approx \frac{1}{\rho(z, t)^6} \end{aligned}$$

as indicated by Theorem 2.1.

Example 13.2. $2n = 2$ and $m = 3$. Consider $\Phi(z, z) = (\phi_1(z, z), \phi_2(z, z), \phi_3(z, z))$ where

$$\phi_1(z, z) = 2\,\mathrm{Re}(z_1\bar{z}_2)$$
$$\phi_2(z, z) = |z_1|^2 - |z_2|^2$$
$$\phi_3(z, z) = 2\,\mathrm{Im}(z_1\bar{z}_2).$$

Let $\nu = (\nu_1, \nu_2, \nu_3)$ be a unit vector in $\mathbb{R}^3$. Then

$$A_\nu = \begin{pmatrix} \nu_2 & \nu_1 - i\nu_3 \\ \nu_1 + i\nu_3 & -\nu_2 \end{pmatrix}.$$

The characteristic equation for the eigenvalues is $\det(A_\nu - \lambda I) = \lambda^2 - |\nu|^2 = \lambda^2 - 1$ with eigenvalues $\lambda = +1,\ -1$.

From (3) with $2n = 2$ and $m = 3$, we obtain

$$N(z,t) = \frac{4^2 3!}{2(2\pi)^4} \int_0^1 \int_{\nu \in S^2} \frac{r}{(1-r)^2} \frac{d\nu\, dr}{\left[\left(\frac{1+r}{1-r}\right)|z|^2 - i|t|\nu_1\right]^4} \frac{dr}{r}.$$

We now let $s = \frac{r+1}{r-1}$ as before and let $x = \nu_1$. Using (48), we write $d\nu = dx\, d\phi$ where ϕ is the angular measure of the S^1 copy of the equator of S^2. Since ϕ does not appear in the integrand, its integral provides a factor of 2π. We obtain

$$\begin{aligned} N(z,t) &= \frac{4^2 3!}{4(2\pi)^4} \int_1^\infty \int_{x=-1}^1 \frac{2\pi\, dx\, ds}{[s|z|^2 - i|t|x]^4} \\ &= \frac{2}{\pi^3} \frac{1}{(|z|^4 + |t|^2)^2} \approx \frac{1}{(|z| + |t|^{1/2})^8} \approx \frac{1}{\rho(z,t)^8} \end{aligned}$$

as indicated by Theorem 2.1.

Example 13.3. $2n = 4$, $m = 4$, $q = 0$. Consider

$$\Phi(z,z) = (\phi_1(z,z), \phi_2(z,z), \phi_3(z,z), \phi_4(z,z))$$

where

$$\begin{aligned} \phi_1(z,z) &= 2\,\mathrm{Re}(z_1\bar{z}_2) + 2\,\mathrm{Re}(z_3\bar{z}_4) \\ \phi_2(z,z) &= 2\,\mathrm{Re}(z_2\bar{z}_3) - 2\,\mathrm{Re}(z_1\bar{z}_4) \\ \phi_3(z,z) &= 2\,\mathrm{Im}(z_1\bar{z}_2) - 2\,\mathrm{Im}(z_3\bar{z}_4) \\ \phi_4(z,z) &= -2\,\mathrm{Im}(z_2\bar{z}_3) + 2\,\mathrm{Im}(z_1\bar{z}_4). \end{aligned}$$

Let $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$ be a unit vector in $\mathbb{R}^4$. Then

$$A_\nu = \begin{pmatrix} 0 & \nu_1 - i\nu_3 & 0 & -\nu_2 - i\nu_4 \\ \nu_1 + i\nu_3 & 0 & \nu_2 + i\nu_4 & 0 \\ 0 & \nu_2 - i\nu_4 & 0 & \nu_1 + i\nu_3 \\ -\nu_2 + i\nu_4 & 0 & \nu_1 - i\nu_3 & 0 \end{pmatrix}. \tag{69}$$

The characteristic polynomial (in λ) is the quadratic polynomial $(\lambda^2 - 1)^2$ with eigenvalues $+1, +1, -1, -1$.

From (3) with $2n = 4$ and $m = 4$, we obtain

$$N(z,t) = \frac{4^4 6!}{2(2\pi)^8} \int_0^1 \int_{\nu \in S^3} \frac{r^2}{(1-r)^4} \frac{d\nu}{\left[\left(\frac{1+r}{1-r}\right)|z|^2 - i|t|\nu_1\right]^7} \frac{dr}{r}.$$

We let $\hat{q} = z/|t|^{1/2}$ and $s = \frac{1+r}{1-r}$ (as before) to obtain

$$N(z,t) = \frac{4^4 6!}{2^4 (2\pi)^8 |t|^7} \int_1^\infty \int_{\nu \in S^3} \frac{(s^2 - 1)\, d\nu\, ds}{(|\hat{q}|^2 s - i\nu_1)^7}.$$

Now we let $x = \nu_1$ and use (48) with $m = 4$ to write $d\nu = \sqrt{1-x^2}\,dx\,d\nu'$ where $d\nu'$ is surface measure on S^2 (the equator of S^3). When ν' is integrated out, this provides a factor of 4π. We obtain

$$\begin{aligned} N(z,t) &= \frac{4^4 6!(4\pi)}{2^4(2\pi)^8|t|^7}\int_1^\infty \int_{x=-1}^1 \frac{(s^2-1)\sqrt{1-x^2}\,dx\,ds}{(|\hat{q}|^2 s - ix)^7} \\ &= \frac{15}{2\pi^6}\frac{1}{(|z|^4+|t|^2)^{7/2}} \approx \frac{1}{(|z|+|t|^{1/2})^{14}} \approx \frac{1}{\rho(z,t)^{14}} \end{aligned}$$

as indicated by Theorem 2.1.

Example 13.4. This is the same example as Example 13.3, except with $q = 1$ and $K = \{1\}$. The matrix A_ν from (69) has associated eigensystem (with entries written as pairs $\{v, \lambda)\}$ where v is a unit eigenvector with eigenvalue λ is

$$\left\{ \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -\nu_1 - i\nu_3 \\ 0 \\ \nu_2 - i\nu_4 \end{pmatrix}, 1 \right\}, \right.$$
$$\left. \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu_2 + i\nu_4 \\ 1 \\ \nu_1 - i\nu_3 \end{pmatrix}, 1 \right\} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\nu_1 - i\nu_3 \\ 0 \\ \nu_2 - i\nu_4 \end{pmatrix}, -1 \right\} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu_2 + i\nu_4 \\ -1 \\ \nu_1 - i\nu_3 \end{pmatrix}, -1 \right\} \right\}.$$

We have

$$U(\nu) = \begin{pmatrix} v_1^\nu & v_2^\nu & v_3^\nu & v_4^\nu \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 & 0 \\ -\nu_1 - i\nu_3 & \nu_2 + i\nu_4 & -\nu_1 - i\nu_3 & \nu_2 + i\nu_4 \\ 0 & 1 & 0 & -1 \\ \nu_2 - i\nu_4 & \nu_1 - i\nu_3 & \nu_2 - i\nu_4 & \nu_1 - i\nu_3 \end{pmatrix}$$

where

$$v_1^\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -\nu_1 - i\nu_3 \\ 0 \\ \nu_2 - i\nu_4 \end{pmatrix},$$
$$v_2^\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu_2 + i\nu_4 \\ 1 \\ \nu_1 - i\nu_3 \end{pmatrix},$$
$$v_3^\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\nu_1 - i\nu_3 \\ 0 \\ \nu_2 - i\nu_4 \end{pmatrix},$$
$$v_4^\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu_2 + i\nu_4 \\ -1 \\ \nu_1 - i\nu_3 \end{pmatrix}$$

so that

$$U(\nu)^* A_\nu U(\nu) = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

where I_2 is the 2×2 identity matrix.

Since $|\mu_j^\nu| = 1$ for all j and $\nu \in S^{m-1}$,

$$A_\nu(r,z) = \sum_{j=1}^{2n} |\mu_j^\nu| \left(\frac{1 + r^{|\mu_j^\nu|}}{1 - r^{|\mu_j^\nu|}} \right) |z_j^\nu|^2 = \sum_{j=1}^{4} \left(\frac{1+r}{1-r} \right) |z_j^\nu|^2 = \frac{1+r}{1-r}|z|^2.$$

Next,

$$\prod_{j=1}^{2n} \frac{r^{(1/2)(1-\varepsilon_{j,L}^\nu)|\mu_j^\nu|}|\mu_j^\nu|}{(1 - r^{|\mu_j^\nu|})} = \frac{1}{(1-r)^4} \begin{cases} r & L \in \{1,2\} \\ r^3 & L \in \{3,4\} \end{cases}.$$

Next, we compute

$$d\bar{Z}(\nu, z) = U(\nu)^T \cdot d\bar{z} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -\nu_1 - i\nu_3 & 0 & \nu_2 - i\nu_4 \\ 0 & \nu_2 + i\nu_4 & 1 & \nu_1 - i\nu_3 \\ 1 & -\nu_1 - i\nu_3 & 0 & \nu_2 - i\nu_4 \\ 0 & \nu_2 + i\nu_4 & -1 & \nu_1 - i\nu_3 \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ d\bar{z}_2 \\ d\bar{z}_3 \\ d\bar{z}_4 \end{pmatrix}.$$

Next, $d\bar{Z}(\nu, z) = U(\nu)^T \cdot d\bar{z}$, then multiplying both sides by $\bar{U}(\nu)$ where $U(\nu)^T$ is the transpose of U produces

$$\bar{U}(\nu) \cdot d\bar{Z}(z, \nu) = \bar{U}(\nu) \cdot U(\nu)^T \cdot d\bar{z} = \overline{U(\nu) \cdot U(\nu)^*} \cdot d\bar{z} = d\bar{z}.$$

Also, $\det(\bar{U}(\nu)_{K',L}) = \bar{U}(\nu)_{k',\ell}$. From (3) and (4) and the computations in this example,

$$\begin{aligned} &N_1(z,t) \\ &= -\frac{K_{4,4}}{2} \int_{\nu \in S^3} \Big[-d\bar{z}_1 - (\nu_1 + i\nu_3)\, d\bar{z}_2 + (\nu_2 - i\nu_4) d\bar{z}_4 \Big] \\ &\quad \times \int_{r=0}^1 \frac{r}{(1-r)^4} \frac{d\nu}{(\frac{1+r}{1-r}|z|^2 - i\nu \cdot t)^7} \frac{dr}{r} \\ &\quad + \frac{K_{4,4}}{2} \int_{\nu \in S^3} \Big[d\bar{z}_1 - (\nu_1 + i\nu_3)\, d\bar{z}_2 + (\nu_2 - i\nu_4)\, d\bar{z}_4 \Big] \\ &\quad \times \int_{r=0}^1 \frac{r^3}{(1-r)^4} \frac{d\nu}{(\frac{1+r}{1-r}|z|^2 - i\nu \cdot t)^7} \frac{dr}{r} \end{aligned}$$

where $K_{4,4} = \frac{4^4(6!)}{2(2\pi)^8}$. Reorganizing, we have

$$\begin{aligned} N_1(z,t) = &\left[\frac{K_{4,4}}{2} \int_{\nu \in S^3} \int_{r=0}^1 \frac{1+r^2}{(1-r)^4} \frac{1}{(\frac{1+r}{1-r}|z|^2 - i\nu \cdot t)^7} \, dr\, d\nu \right] d\bar{z}_1 \\ &+ \left[\frac{K_{4,4}}{2} \int_{\nu \in S^3} \int_{r=0}^1 \frac{(\nu_1 + i\nu_3)(1-r^2)}{(1-r)^4} \frac{1}{(\frac{1+r}{1-r}|z|^2 - i\nu \cdot t)^7} \, dr\, d\nu \right] d\bar{z}_2 \\ &- \left[\frac{K_{4,4}}{2} \int_{\nu \in S^3} \int_{r=0}^1 \frac{(\nu_2 - i\nu_4)(1-r^2)}{(1-r)^4} \frac{1}{(\frac{1+r}{1-r}|z|^2 - i\nu \cdot t)^7} \, dr\, d\nu \right] d\bar{z}_4. \end{aligned} \tag{70}$$

We observe that with $\hat{q} = z/|t|^{1/2}$,

$$\int_0^1 \frac{1+r^2}{(1-r)^4(\frac{1+r}{1-r}|\hat{q}|^2 + ia)^7} \, dr = \frac{-a^2 + 6ia|\hat{q}|^2 + 25|\hat{q}|^4}{240|\hat{q}|^6(ia + |\hat{q}|^2)^6}$$

and

$$\int_0^1 \frac{1-r^2}{(1-r)^4(\frac{1+r}{1-r}|\hat{q}|^2+ia)^7}\,dr = \frac{ia+6|\hat{q}|^2}{60|\hat{q}|^4(ia+|\hat{q}|^2)^6}.$$

Let's also observe the estimate in the special case that $t=(|t|,0,\dots,0)$ and only the $d\bar{z}_1$ component (since $K=\{1\}$) and compute

$$\begin{aligned} I &= \int_{\nu\in S^3}\int_{r=0}^1 \frac{1+r^2}{(1-r)^4}\frac{1}{(\frac{1+r}{1-r}|z|^2-i\nu\cdot t)^7}\,dr\,d\nu \\ &= \frac{1}{|t|^7}\int_{\nu\in S^3}\int_{r=0}^1 \frac{1+r^2}{(1-r)^4}\frac{1}{(\frac{1+r}{1-r}|\hat{q}|^2-i\nu_1)^7}\,dr\,d\nu \\ &= \frac{1}{240|\hat{q}|^6|t|^7}\int_{\nu\in S^3}\frac{-\nu_1^2-6i\nu_1|\hat{q}|^2+25|\hat{q}|^4}{(|\hat{q}|^2-i\nu_1)^6}\,d\nu. \end{aligned}$$

Integrating in spherical coordinates, we compute

$$\begin{aligned} I &= \frac{1}{240|t|^4|z|^6}\int_0^\pi\int_0^\pi\int_0^{2\pi}\frac{-\cos^2\alpha_1-6i\cos\alpha_1|\hat{q}|^2+25|\hat{q}|^4}{(|\hat{q}|^2-i\cos\alpha_1)^6}\sin^2\alpha_1\sin\alpha_2\,d\alpha_3 d\alpha_2 d\alpha_1 \\ &= \frac{\pi}{120|t|^4|z|^6}\int_{-\pi}^{\pi}\frac{-\cos^2\alpha_1-6i\cos\alpha_1|\hat{q}|^2+25|\hat{q}|^4}{(|\hat{q}|^2-i\cos\alpha_1)^6}\sin^2\alpha_1\,d\alpha_1. \end{aligned}$$

The last integral follows from the fact that $\cos\alpha_1$ and $\sin^2\alpha_1$ are even functions. If $f(\cos\alpha_1,\sin\alpha_1)$ is the integrand, then

$$\int_{-\pi}^{\pi} f(\cos\alpha_1,\sin\alpha_1)\,d\alpha_1 = \oint_{|z|=1} f\Big(\frac{z+\frac{1}{z}}{2},\frac{z-\frac{1}{z}}{2i}\Big)\frac{1}{iz}\,dz.$$

If

$$g(z) = f\Big(\frac{z+\frac{1}{z}}{2},\frac{z-\frac{1}{z}}{2i}\Big)\frac{1}{iz}$$

then

$$g(z) = -\frac{4iz(-1+z^2)^2(-100|\hat{q}|^4z^2+(1+z^2)^2+12i|\hat{q}|^2(z+z^3))}{(2|\hat{q}|^2z-i(1+z^2))^6}$$

has poles at $z=i(-|\hat{q}|^2\pm\sqrt{|\hat{q}|^4+1})$. The pole at $z=i(-|\hat{q}|^2+\sqrt{|\hat{q}|^4+1})$ occurs inside the unit disk and is easily computed using Mathematica. In fact,

$$\operatorname{Res}(g,i(-|\hat{q}|^2+\sqrt{|\hat{q}|^4+1})) = -\frac{5|\hat{q}|^6 i(-2+5|\hat{q}|^4)}{2(1+|\hat{q}|^4)^{\frac{9}{2}}}.$$

In summary, the $d\bar{z}_1$ component of N_1 in (70) is

$$I = -2\pi i\frac{\pi}{120|t|^7}\frac{5i(-2+5|\hat{q}|^4)}{2(1+|\hat{q}|^4)^{\frac{9}{2}}}.$$

Similar calculations with $t=(|t|,0,0,0)$ show that the $d\bar{z}_2$ component of N_1 in (70) is

$$II = \int_{\nu\in S^3}\int_{r=0}^1 \frac{(\nu_1+i\nu_3)(1-r^2)}{(1-r)^4}\frac{1}{(\frac{1+r}{1-r}|z|^2-i\nu\cdot t)^7}\,dr\,d\nu = \frac{2\pi^2 i}{15|t|^7}\frac{35|\hat{q}|^2}{8(1+|\hat{q}|^4)^{\frac{9}{2}}},$$

and the $d\bar{z}_4$ component of N_1 in (70) is

$$III = \int_{\nu\in S^3}\int_{r=0}^1 \frac{(\nu_2-i\nu_4)(1-r^2)}{(1-r)^4}\frac{1}{(\frac{1+r}{1-r}|z|^2-i\nu\cdot t)^7}\,dr\,d\nu = 0.$$

In summary, we have computed that

$$N_1\big(z,(|t|,0,0,0)\big) = \frac{15}{\pi^6}\frac{1}{(|t|^2+|z|^4)^{\frac{7}{2}}}\left[\frac{-2|t|^2+5|z|^4}{2(|t|^2+|z|^4)}\,d\bar{z}_1 + i\frac{7|z|^2|t|}{(|t|^2+|z|^4)}\,d\bar{z}_2\right].$$

This expression has norm $\approx \rho(z,t)^{-14}$ as indicated by Theorem 2.1.

Example 13.5. This is a modification of Example 13.3 where the eigenvalues of A_ν do not depend analytically on ν. Let $\Phi(z,z) = (\phi_1(z,z), \phi_2(z,z), \phi_3(z,z), \phi_4(z,z))$ where

$$\begin{aligned}
\phi_1(z,z) &= 2\,\mathrm{Re}(z_1\bar{z}_2) + 2\,\mathrm{Re}(z_3\bar{z}_4)\\
\phi_2(z,z) &= 2\,\mathrm{Re}(z_2\bar{z}_3) - 2\,\mathrm{Re}(z_1\bar{z}_4)\\
\phi_3(z,z) &= 2\,\mathrm{Im}(z_1\bar{z}_2) - 2\,\mathrm{Im}(z_3\bar{z}_4)\\
\phi_4(z,z) &= -2\,\mathrm{Im}(z_2\bar{z}_3) + 2(1+b)\,\mathrm{Im}(z_1\bar{z}_4)
\end{aligned}$$

where b is a small real number. Let $\nu = (\nu_1,\nu_2,\nu_3,\nu_4)$ be a unit vector in $\mathbb{R}^4$. Then, it is easy to compute the complex Hessian of $\Phi_\nu = \Phi(z,z)\cdot\nu$:

$$A_\nu = \mathrm{Hessian}\Phi_\nu = \begin{pmatrix} 0 & \nu_1 - i\nu_3 & 0 & -\nu_2 - i(1+b)\nu_4\\ \nu_1 + i\nu_3 & 0 & \nu_2 + i\nu_4 & 0\\ 0 & \nu_2 - i\nu_4 & 0 & \nu_1 + i\nu_3\\ -\nu_2 + i(1+b)\nu_4 & 0 & \nu_1 - i\nu_3 & 0 \end{pmatrix}.$$

Note that when $b = 0$, then this is Example 13.3.

The characteristic polynomial (in λ) turns out to be a quadratic polynomial in λ^2 so the eigenvalues (though messy) can be computed as $\mu_1^\nu > 0$, $\mu_2^\nu > 0$, $\mu_3^\nu < 0$, $\mu_4^\nu < 0$ where:

$$\mu_1^\nu = \sqrt{\Lambda_+},\quad \mu_2^\nu = \sqrt{\Lambda_-},\quad \mu_3^\nu = -\sqrt{\Lambda_+},\quad \mu_4^\nu = -\sqrt{\Lambda_-}$$

and where

$$\begin{aligned}
\Lambda_\pm &= \nu_1^2 + \nu_2^2 + \nu_3^2 + (1/2)(b^2+2b+2)\nu_4^2\\
&\quad \pm |\nu_4||b|\left(\frac{(b+2)^2\nu_4^2}{4} + \nu_1^2 + \nu_3^2\right)^{1/2}.
\end{aligned}$$

Note that when $b = 0$, $\mu_1^\nu = \mu_2^\nu = 1$ and $\mu_3^\nu = \mu_4^\nu = -1$ as in Example 13.3. For b nonzero, but small, these eigenvalues are not smooth at $\nu_4 = 0$ and $\nu_1, \nu_3 \neq 0$ due to the presence of $|\nu_4|$.

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School of Mathematical and Statistical Sciences, Arizona State University, Physical Sciences Building A-Wing Rm. 216, 901 S. Palm Walk, Tempe, Arizona 85287-1804
Email address: `boggess@asu.edu`

Department of Mathematical Sciences, 1 University of Arkansas, SCEN 327, Fayetteville, Arkansas 72701
Email address: `araich@uark.edu`