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CANONICAL EQUIVARIANT COHOMOLOGY CLASSES GENERATING ZETA VALUES OF TOTALLY REAL FIELDS

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Abstract. It is known that the special values at nonpositive integers of a Dirichlet L-function may be expressed using the generalized Bernoulli numbers, which are defined by a generating function. The purpose of this article is to consider the generalization of this classical result to the case of Hecke L-functions of totally real fields. Hecke L-functions may be expressed canonically as a finite sum of zeta functions of Lerch type. By combining the non-canonical multivariable generating functions constructed by Shintani [J. Fac. Sci. Univ. Tokyo Sect. IA Math. 23 (1976), pp. 393–417], we newly construct a canonical class, which we call the *Shintani generating class*, in the equivariant cohomology of an algebraic torus associated to the totally real field. Our main result states that the specializations at torsion points of the derivatives of the Shintani generating class give values at nonpositive integers of the zeta functions of Lerch type. This result gives the insight that the correct framework in the higher dimensional case is to consider higher equivariant cohomology classes instead of functions.

Contents

1. Introduction

It is classically known that the special values at nonpositive integers of a Dirichlet L-function may be expressed using the generalized Bernoulli numbers, which are defined by a rational generating function. This simple but significant result is the basis of the deep connection between the special values of Dirichlet L-functions and important arithmetic invariants pertaining to the abelian extensions of $\mathbb{Q}$. In his ground-breaking article [27], Shintani generalized this result to the case of Hecke L-functions of totally real fields. His approach consists of two steps: The decomposition of a Hecke L-function into a finite sum of zeta functions – the *Shintani*

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zeta functions – associated to certain cones, and the construction of a multivariable generating function for special values of each Shintani zeta function. Although this method attained certain success, including the construction by Barsky [4] and Cassou-Noguès [8] of the p-adic L-functions for totally real fields, the decomposition step above requires a choice of cones, and the resulting generating function is non-canonical. A canonical object behind these generating functions, which is independent of this choice of cones, remained to be found.

The purpose of this article is to construct geometrically such a canonical object, which we call the *Shintani generating class*, through the combination of the following three ideas. We let g be the degree of the totally real field. First, the Hecke L-functions are expressed canonically in terms of the *zeta functions of Lerch type* (cf. Definition 1.2), or simply *Lerch zeta functions*, which are defined for finite additive characters parameterized by torsion points of a certain algebraic torus of dimension g, originally considered by Katz [21], associated to the totally real field. Second, via a Čech resolution, the multivariable generating functions constructed by Shintani for various cones may beautifully be combined to form the Shintani generating class, a canonical cohomology class in the $(g-1)$-st cohomology group of the algebraic torus minus the identity. Third, the class descends into the equivariant cohomology with respect to the action of totally positive units, which successfully allows for nontrivial specializations of the class and its derivatives at torsion points. Our main result, Theorem 5.1, states that the specializations at nontrivial torsion points of the derivatives of the Shintani generating class give values at nonpositive integers of the Lerch zeta functions associated to the totally real field.

The classical result for $\mathbb{Q}$ that we generalize, viewed through our emphasis on Lerch zeta functions, is as follows. The Dirichlet L-function may canonically be expressed as a finite linear combination of the classical *Lerch zeta functions*, defined by the series

$$\mathcal{L}(\xi,s) := \sum_{n=1}^{\infty} \xi(n) n^{-s} \tag{1}$$

for finite characters $\xi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{C}^\times)$. The series (1) converges for any $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$ and has an analytic continuation to the whole complex plane, holomorphic if $\xi \neq 1$. When $\xi = 1$, the function $\mathcal{L}(1,s)$ coincides with the Riemann zeta function $\zeta(s)$, hence has a simple pole at $s = 1$. A crucial property of the Lerch zeta functions is that it has a canonical generating function $\mathcal{G}(t)$, which single-handedly captures for *all* nontrivial finite characters ξ the values of Lerch zeta functions at nonpositive integers.

Let $\mathbb{G}_m := \mathrm{Spec}\,\mathbb{Z}[t,t^{-1}]$ be the multiplicative group, and let $\mathcal{G}(t)$ be the rational function

$$\mathcal{G}(t) := \frac{t}{1-t} \in \Gamma\big(U, \mathscr{O}_{\mathbb{G}_m}\big),$$

where $U := \mathbb{G}_m \setminus \{1\}$. We denote by ∂ the algebraic differential operator $\partial := t\frac{d}{dt}$, referred to as the "magic stick" in [20, 1.1.7]. Note that any $\xi \in \mathbb{G}_m(\mathbb{C})$ corresponds to an additive character $\xi\colon \mathbb{Z} \to \mathbb{C}^\times$ given by $\xi(n) := \xi^n$ for any $n \in \mathbb{Z}$. Then we have the following.

Theorem 1.1. *For any nontrivial torsion point ξ of $\mathbb{G}_m$ and $k \in \mathbb{N}$, we have*

$$\mathcal{L}(\xi,-k) = \partial^k \mathcal{G}(t)\big|_{t=\xi} \in \mathbb{Q}(\xi).$$

In particular, the values $\mathcal{L}(\xi,-k)$ for any $k\in\mathbb{N}$ are all algebraic.

The purpose of this article is to generalize the above result to the case of totally real fields. Let F be a totally real field of degree g, and let $\mathcal{O}_F$ be its ring of integers. We denote by $\mathcal{O}_{F+}$ the set of totally positive integers and by $\Delta:=\mathcal{O}_{F+}^\times$ the set of totally positive units of F. Let $\mathbb{T}:=\mathrm{Hom}_\mathbb{Z}(\mathcal{O}_F,\mathbb{G}_m)$ be an algebraic torus defined over $\mathbb{Z}$ which represents the functor associating to any $\mathbb{Z}$-algebra R the group $\mathbb{T}(R)=\mathrm{Hom}_\mathbb{Z}(\mathcal{O}_F,R^\times)$. Such a torus was used by Katz [21] to reinterpret the construction by Barsky [4] and Cassou-Noguès [8] of the p-adic L-function of totally real fields. For the case $F=\mathbb{Q}$, we have $\mathbb{T}=\mathrm{Hom}_\mathbb{Z}(\mathbb{Z},\mathbb{G}_m)=\mathbb{G}_m$, hence $\mathbb{T}$ is a natural generalization of the multiplicative group. For an additive character $\xi\colon\mathcal{O}_F\to R^\times$ and $\varepsilon\in\Delta$, we let ξ^ε be the character defined by $\xi^\varepsilon(\alpha):=\xi(\varepsilon\alpha)$ for any $\alpha\in\mathcal{O}_F$. This gives an action of Δ on the set of additive characters $\mathbb{T}(R)$.

We consider the following zeta function, which we regard as the generalization of the classical Lerch zeta function to the case of totally real fields.

Definition 1.2. For any torsion point $\xi\in\mathbb{T}(\mathbb{C})=\mathrm{Hom}_\mathbb{Z}(\mathcal{O}_F,\mathbb{C}^\times)$, we define the *zeta function of Lerch type*, or simply the *Lerch zeta function*, by

$$\mathcal{L}(\xi\Delta,s):=\sum_{\alpha\in\Delta_\xi\backslash\mathcal{O}_{F+}}\xi(\alpha)N(\alpha)^{-s},\tag{2}$$

where $N(\alpha)$ is the norm of α, and $\Delta_\xi\subset\Delta$ is the isotropic subgroup of ξ, i.e. the subgroup consisting of $\varepsilon\in\Delta$ such that $\xi^\varepsilon=\xi$.

The notation $\mathcal{L}(\xi\Delta,s)$ is used since (2) depends only on the Δ-orbit of ξ. This series is known to converge for $\mathrm{Re}(s)>1$, and may be continued analytically to the whole complex plane. When the narrow class number of F is *one*, the Hecke L-function of a finite Hecke character of F may canonically be expressed as a finite linear sum of $\mathcal{L}(\xi\Delta,s)$ for suitable finite characters ξ (see Proposition 2.3).

The action of Δ on additive characters gives a right action of Δ on $\mathbb{T}$. The structure sheaf $\mathscr{O}_\mathbb{T}$ on $\mathbb{T}$ has a natural Δ-equivariant structure in the sense of Definition 3.2. Let $U:=\mathbb{T}\setminus\{1\}$. Our main results are as follows.

Theorem 1.3.

(1) *(Proposition 4.2) There exists a canonical class*

$$\mathcal{G}\in H^{g-1}(U/\Delta,\mathscr{O}_\mathbb{T}),$$

where $H^{g-1}(U/\Delta,\mathscr{O}_\mathbb{T})$ is the equivariant cohomology of U with coefficients in $\mathscr{O}_\mathbb{T}$ (see §3 for the precise definition.)

(2) *(Theorem 5.1) For any nontrivial torsion point ξ of $\mathbb{T}$, we have a canonical isomorphism*

$$H^{g-1}(\xi/\Delta_\xi,\mathscr{O}_\xi)\cong\mathbb{Q}(\xi).$$

Through this isomorphism, for any integer $k\geq 0$, we have

$$\mathcal{L}(\xi\Delta,-k)=\partial^k\mathcal{G}(\xi)\in\mathbb{Q}(\xi),$$

where $\partial\colon H^{g-1}(U/\Delta,\mathscr{O}_\mathbb{T})\to H^{g-1}(U/\Delta,\mathscr{O}_\mathbb{T})$ is a certain differential operator given in (19), and $\partial^k\mathcal{G}(\xi)$ is the image of $\partial^k\mathcal{G}$ with respect to the specialization map $H^{g-1}(U/\Delta,\mathscr{O}_\mathbb{T})\to H^{g-1}(\xi/\Delta_\xi,\mathscr{O}_\xi)$ induced by the equivariant morphism $\xi\to U$.

We refer to the class $\mathcal{G}$ as the *Shintani generating class*. If $F = \mathbb{Q}$, then we have $\Delta = \{1\}$, and the class $\mathcal{G}$ is simply the rational function $\mathcal{G}(t) = t/(1-t) \in H^0(U, \mathscr{O}_{\mathbb{G}_m}) = \Gamma(U, \mathscr{O}_{\mathbb{G}_m})$. Thus Theorem 1.3 (2) coincides with Theorem 1.1 in this case. For the case $F = \mathbb{Q}$ and also for the case of imaginary quadratic fields (see for example [11,12]), canonical algebraic generating functions of special values of Hecke L-functions play a crucial role in relating the special values of Hecke L-functions to arithmetic invariants. However, up until now, the discovery of such a *canonical* generating function has been elusive in the higher dimensional cases. Our result suggests that the correct framework in the higher dimensional case is to consider equivariant cohomology classes instead of functions. It is noteworthy that such a framework is also adopted by Kings–Sprang [22] for the study of the critical Hecke L-values of totally imaginary fields, which is a higher dimensional extension of the work of Bannai–Kobayashi [1].

We remark that there are other approaches to L-values of totally real fields via cohomology classes, by Sczech [26], Solomon [28,29], Hu–Solomon [17], Hill [16], Spiess [30], Charollois–Dasgupta [9], Charollois–Dasgupta–Greenberg [10], and Bergeron–Charollois–Garcia [6]. They mainly study certain group cocycles on $\mathrm{GL}_n(\mathbb{Q})$ called the Eisenstein cocycles or the Shintani cocycles, while we stress a more algebro-geometric viewpoint by treating the geometric object $\mathbb{T}$ and equivariant sheaf cohomology classes. We also note that Bekki [7] studies L-values of general number fields by incorporating the ideas of both the above-mentioned works and this article.

As another related result, the relation of special values of Hecke L-functions of totally real fields to the topological polylogarithm on a torus was studied by Beilinson–Kings–Levin [5]. The polylogarithms for general commutative group schemes were constructed by Huber–Kings [18]. Our discovery of the Shintani generating class arose from our attempt to explicitly describe various realizations of the Δ-equivariant version of the polylogarithm for the algebraic torus $\mathbb{T}$. In subsequent research, we will explore the arithmetic implications of our insight (see for example [2]).

The content of this article is as follows. In §2, we will introduce the Lerch zeta function $\mathcal{L}(\xi\Delta, s)$ and show that this function may be expressed non-canonically as a linear sum of Shintani zeta functions. We will then review the multivariable generating function constructed by Shintani of the special values of Shintani zeta functions. In §3, we will define the equivariant cohomology of a scheme with an action of a group, and will construct the equivariant Čech complex $C^\bullet(\mathfrak{U}/\Delta, \mathscr{F})$ which calculates the equivariant cohomology of $U := \mathbb{T} \setminus \{1\}$ with coefficients in an equivariant coherent sheaf $\mathscr{F}$ on U. In §4, we will define in Proposition 4.2 the Shintani generating class $\mathcal{G}$, and in Lemma 4.6 give the definition of the derivatives. Finally in §5, we will give the proof of our main theorem, Theorem 5.1, which coincides with Theorem 1.3 (2).

2. Lerch Zeta Function

In this section, we first introduce the Lerch zeta function for totally real fields. Then we will define the Shintani zeta function associated to a cone σ and a function $\phi\colon \mathcal{O}_F \to \mathbb{C}$ which factors through $\mathcal{O}_F/\mathfrak{f}$ for some nonzero ideal $\mathfrak{f} \subset \mathcal{O}_F$. We will then describe the generating function of its values at nonpositive integers when ϕ is a finite additive character.

Let $\xi \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{C}^\times)$ be a $\mathbb{C}$-valued character on $\mathcal{O}_F$ of finite order. As in Definition 1.2 of §1, we define the Lerch zeta function for totally real fields by the series

$$\mathcal{L}(\xi\Delta, s) := \sum_{\alpha \in \Delta_\xi \backslash \mathcal{O}_{F+}} \xi(\alpha) N(\alpha)^{-s},$$

where $\Delta_\xi := \{\varepsilon \in \Delta \mid \xi^\varepsilon = \xi\}$, which may be continued analytically to the whole complex plane.

Remark 2.1. Note that we have

$$\mathcal{L}(\xi\Delta, s) = \sum_{\alpha \in \Delta \backslash \mathcal{O}_{F+}} \sum_{\varepsilon \in \Delta_\xi \backslash \Delta} \xi(\varepsilon\alpha) N(\alpha)^{-s}.$$

Even though $\xi(\alpha)$ is not well-defined for $\alpha \in \Delta\backslash\mathcal{O}_{F+}$, the sum $\sum_{\varepsilon \in \Delta_\xi \backslash \Delta} \xi(\varepsilon\alpha)$ is well-defined for $\alpha \in \Delta\backslash\mathcal{O}_{F+}$.

The importance of $\mathcal{L}(\xi\Delta, s)$ is in its relation to the Hecke L-functions of F. Let $\mathfrak{f}$ be a nonzero integral ideal of F. We denote by $\mathrm{Cl}^+_F(\mathfrak{f}) := I_\mathfrak{f}/P^+_\mathfrak{f}$ the strict ray class group modulo $\mathfrak{f}$ of F, where $I_\mathfrak{f}$ is the group of fractional ideals of F prime to $\mathfrak{f}$ and $P^+_\mathfrak{f} := \{(\alpha) \mid \alpha \in F_+, \alpha \equiv 1 \bmod^\times \mathfrak{f}\}$. A finite Hecke character of F of conductor $\mathfrak{f}$ is a character

$$\chi\colon \mathrm{Cl}^+_F(\mathfrak{f}) \to \mathbb{C}^\times.$$

By [24, Chapter VII (6.9) Proposition], there exists a unique character $\chi_{\mathrm{fin}}\colon (\mathcal{O}_F/\mathfrak{f})^\times \to \mathbb{C}^\times$ associated to χ such that $\chi((\alpha)) = \chi_{\mathrm{fin}}(\alpha)$ for any $\alpha \in \mathcal{O}_{F+}$ prime to $\mathfrak{f}$. In particular, we have $\chi_{\mathrm{fin}}(\varepsilon) = 1$ for any $\varepsilon \in \Delta$. Extending by *zero*, we regard χ_{fin} as functions on $\mathcal{O}_F/\mathfrak{f}$ and $\mathcal{O}_F$ with values in $\mathbb{C}$.

In what follows, we let $\mathbb{T}[\mathfrak{f}] := \mathrm{Hom}(\mathcal{O}_F/\mathfrak{f}, \overline{\mathbb{Q}}^\times) \subset \mathbb{T}(\overline{\mathbb{Q}})$ be the set of $\mathfrak{f}$-torsion points of $\mathbb{T}$. We say that a character χ, χ_{fin} or $\xi \in \mathbb{T}[\mathfrak{f}]$ is *primitive*, if it does not factor respectively through $\mathrm{Cl}^+_F(\mathfrak{f}')$, $(\mathcal{O}_F/\mathfrak{f}')^\times$ or $\mathcal{O}_F/\mathfrak{f}'$ for any integral ideal $\mathfrak{f}' \neq \mathfrak{f}$ such that $\mathfrak{f}'|\mathfrak{f}$. Then we have the following.

Lemma 2.2. *For any* $\xi \in \mathbb{T}[\mathfrak{f}]$, *let*

$$c_\chi(\xi) := \frac{1}{\boldsymbol{N}(\mathfrak{f})} \sum_{\beta \in \mathcal{O}_F/\mathfrak{f}} \chi_{\mathrm{fin}}(\beta)\xi(-\beta).$$

Then we have

$$\chi_{\mathrm{fin}}(\alpha) = \sum_{\xi \in \mathbb{T}[\mathfrak{f}]} c_\chi(\xi)\xi(\alpha).$$

Moreover, if χ_{fin} *is primitive, then we have* $c_\chi(\xi) = 0$ *for any non-primitive* ξ.

Proof. The first statement follows from

$$\sum_{\xi \in \mathbb{T}[\mathfrak{f}]} c_\chi(\xi)\xi(\alpha) = \frac{1}{\boldsymbol{N}(\mathfrak{f})} \sum_{\beta \in \mathcal{O}_F/\mathfrak{f}} \chi_{\mathrm{fin}}(\beta) \Bigg(\sum_{\xi \in \mathbb{T}[\mathfrak{f}]} \xi(\alpha - \beta) \Bigg) = \chi_{\mathrm{fin}}(\alpha),$$

where the last equality follows from the fact that $\sum_{\xi \in \mathbb{T}[\mathfrak{f}]} \xi(\alpha) = N(\mathfrak{f})$ if $\alpha \equiv 0 \pmod{\mathfrak{f}}$ and $\sum_{\xi \in \mathbb{T}[\mathfrak{f}]} \xi(\alpha) = 0$ if $\alpha \not\equiv 0 \pmod{\mathfrak{f}}$. Next, suppose χ_{fin} is primitive, and let $\mathfrak{f}' \neq \mathfrak{f}$ be an integral ideal of F such that $\mathfrak{f}'|\mathfrak{f}$ and $\xi \in \mathbb{T}[\mathfrak{f}']$. Since χ_{fin} is primitive, it does not factor through $\mathcal{O}_F/\mathfrak{f}'$, hence there exists an element $\gamma \in \mathcal{O}_F$

prime to $\mathfrak{f}$ such that $\gamma \equiv 1 \pmod{\mathfrak{f}'}$ and $\chi_{\mathrm{fin}}(\gamma) \neq 1$. Then since $\xi \in \mathbb{T}[\mathfrak{f}']$, we have $\xi(\gamma\alpha) = \xi(\alpha)$ for any $\alpha \in \mathcal{O}_F$. This gives

$$\begin{aligned} c_\chi(\xi) = \frac{1}{\boldsymbol{N}(\mathfrak{f})} \sum_{\beta \in \mathcal{O}_F/\mathfrak{f}} \chi_{\mathrm{fin}}(\beta)\xi(-\beta) &= \frac{1}{\boldsymbol{N}(\mathfrak{f})} \sum_{\beta \in \mathcal{O}_F/\mathfrak{f}} \chi_{\mathrm{fin}}(\beta)\xi(-\gamma\beta) \\ &= \frac{\overline{\chi}_{\mathrm{fin}}(\gamma)}{\boldsymbol{N}(\mathfrak{f})} \sum_{\beta \in \mathcal{O}_F/\mathfrak{f}} \chi_{\mathrm{fin}}(\gamma\beta)\xi(-\gamma\beta) = \overline{\chi}_{\mathrm{fin}}(\gamma) c_\chi(\xi). \end{aligned}$$

Since $\chi_{\mathrm{fin}}(\gamma) \neq 1$, we have $c_\chi(\xi) = 0$ as desired. □

Note that since multiplication by $\varepsilon \in \Delta$ is bijective on $\mathcal{O}_F/\mathfrak{f}$ and since $\chi_{\mathrm{fin}}(\varepsilon) = 1$, we have $c_\chi(\xi^\varepsilon) = c_\chi(\xi)$. Then we have the following.

Proposition 2.3. *Assume that the narrow class number of F is one, and let $\chi\colon \mathrm{Cl}^+_F(\mathfrak{f}) \to \mathbb{C}^\times$ be a finite primitive Hecke character of F of conductor $\mathfrak{f} \neq (1)$. Then for $U[\mathfrak{f}] := \mathbb{T}[\mathfrak{f}] \setminus \{1\}$, we have*

$$L(\chi, s) = \sum_{\xi \in U[\mathfrak{f}]/\Delta} c_\chi(\xi) \mathcal{L}(\xi\Delta, s).$$

Proof. By definition and Lemma 2.2, we have

$$\begin{aligned} \sum_{\xi \in \mathbb{T}[\mathfrak{f}]/\Delta} c_\chi(\xi)\mathcal{L}(\xi\Delta, s) &= \sum_{\xi \in \mathbb{T}[\mathfrak{f}]/\Delta} \sum_{\alpha \in \Delta\backslash \mathcal{O}_{F+}} \sum_{\varepsilon \in \Delta_\xi \backslash \Delta} c_\chi(\xi)\xi(\varepsilon\alpha)\boldsymbol{N}(\alpha)^{-s} \\ &= \sum_{\alpha \in \Delta\backslash \mathcal{O}_{F+}} \sum_{\xi \in \mathbb{T}[\mathfrak{f}]/\Delta} \sum_{\varepsilon \in \Delta_\xi \backslash \Delta} c_\chi(\xi^\varepsilon)\xi^\varepsilon(\alpha)\boldsymbol{N}(\alpha)^{-s} \\ &= \sum_{\alpha \in \Delta\backslash \mathcal{O}_{F+}} \sum_{\xi \in \mathbb{T}[\mathfrak{f}]} c_\chi(\xi)\xi(\alpha)\boldsymbol{N}(\alpha)^{-s} \\ &= \sum_{\alpha \in \Delta\backslash \mathcal{O}_{F+}} \chi_{\mathrm{fin}}(\alpha)\boldsymbol{N}(\alpha)^{-s} = \sum_{\mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a})\boldsymbol{N}\mathfrak{a}^{-s}. \end{aligned}$$

Our assertion follows from the definition of the Hecke L-function and the fact that $c_\chi(\xi) = 0$ for $\xi = 1$. □

Remark 2.4. We assumed the condition on the narrow class number for simplicity. By considering the Lerch zeta functions corresponding to additive characters in $\mathrm{Hom}_\mathbb{Z}(\mathfrak{a}, \mathbb{C}^\times)$ for general fractional ideals $\mathfrak{a}$ of F, we may express the Hecke L-functions when the narrow class number of F is greater than *one*.

We will next define the Shintani zeta function associated to a cone. Note that we have a canonical isomorphism

$$F \otimes \mathbb{R} \cong \mathbb{R}^I := \prod_{\tau \in I} \mathbb{R}, \qquad \alpha \otimes 1 \mapsto (\alpha^\tau),$$

where I is the set of embeddings $\tau\colon F \hookrightarrow \mathbb{R}$ and we let $\alpha^\tau := \tau(\alpha)$ for any embedding $\tau \in I$. We denote by $\mathbb{R}^I_+ := \prod_{\tau \in I} \mathbb{R}_+$ the set of totally positive elements of $\mathbb{R}^I$, where $\mathbb{R}_+$ is the set of positive real numbers.

Definition 2.5. A rational closed polyhedral cone in $\mathbb{R}^I_+ \cup \{0\}$, which we simply call a cone, is any set of the form

$$\sigma_{\boldsymbol{\alpha}} := \{x_1\alpha_1 + \cdots + x_m\alpha_m \mid x_1, \dots, x_m \in \mathbb{R}_{\geq 0}\}$$

for some $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathcal{O}_{F+}^m$. In this case, we say that $\boldsymbol{\alpha}$ is a generator of $\sigma_{\boldsymbol{\alpha}}$. By considering the case $m = 0$, we see that $\sigma = \{0\}$ is a cone.

We define the dimension $\dim \sigma$ of a cone σ to be the dimension of the $\mathbb{R}$-vector space generated by σ. In what follows, we fix an element $\tau_0 \in I$. For any subset $D \subset \mathbb{R}_+^I$, we let

$$\breve{D} := \{u \in \mathbb{R}_+^I \mid \exists \delta > 0,\, 0 < \forall \delta' < \delta,\, u - \delta' e_0 \in D\}, \tag{3}$$

where e_0 denotes the unit vector in $\mathbb{R}^I$ whose τ_0 component is 1 and others are 0.

Definition 2.6. Let σ be a cone, and let $\phi\colon \mathcal{O}_F \to \mathbb{C}$ be a $\mathbb{C}$-valued function on $\mathcal{O}_F$ which factors through $\mathcal{O}_F/\mathfrak{f}$ for some nonzero ideal $\mathfrak{f} \subset \mathcal{O}_F$. We define the *Shintani zeta function* $\zeta_\sigma(\phi, \boldsymbol{s})$ associated to a cone σ and function ϕ by the series

$$\zeta_\sigma(\phi, \boldsymbol{s}) := \sum_{\alpha \in \breve{\sigma} \cap \mathcal{O}_F} \phi(\alpha) \alpha^{-\boldsymbol{s}}, \tag{4}$$

where $\boldsymbol{s} = (s_\tau) \in \mathbb{C}^I$ and $\alpha^{-\boldsymbol{s}} := \prod_{\tau \in I} (\alpha^\tau)^{-s_\tau}$. The series (4) converges if $\mathrm{Re}(s_\tau) > 1$ for any $\tau \in I$.

By [27, Proposition 1], the function $\zeta_\sigma(\phi, \boldsymbol{s})$ has a meromorphic continuation to any $\boldsymbol{s} \in \mathbb{C}^I$. If we let $\boldsymbol{s} = (s, \dots, s)$ for $s \in \mathbb{C}$, then we have

$$\zeta_\sigma(\phi, (s, \dots, s)) = \sum_{\alpha \in \breve{\sigma} \cap \mathcal{O}_F} \phi(\alpha) N(\alpha)^{-s}. \tag{5}$$

Shintani constructed the generating function of values of $\zeta_\sigma(\xi, \boldsymbol{s})$ at nonpositive integers for additive characters $\xi\colon \mathcal{O}_F \to \mathbb{C}^\times$ of finite order, given as follows. In what follows, we view $z \in F \otimes \mathbb{C}$ as an element $z = (z_\tau) \in \mathbb{C}^I$ through the canonical isomorphism $F \otimes \mathbb{C} \cong \mathbb{C}^I$.

Definition 2.7. Let $\sigma = \sigma_{\boldsymbol{\alpha}}$ be a g-dimensional cone generated by $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g) \in \mathcal{O}_{F+}^g$, and we let $P_{\boldsymbol{\alpha}} := \{x_1\alpha_1 + \cdots + x_g\alpha_g \mid \forall i\ 0 \le x_i < 1\}$ be the parallelepiped spanned by $\alpha_1, \dots, \alpha_g$. We define $\mathscr{G}_\sigma(z)$ to be the meromorphic function on $F \otimes \mathbb{C} \cong \mathbb{C}^I$ given by

$$\mathscr{G}_\sigma(z) := \frac{\sum_{\alpha \in \breve{P}_{\boldsymbol{\alpha}} \cap \mathcal{O}_F} e^{2\pi i \mathrm{Tr}(\alpha z)}}{\left(1 - e^{2\pi i \mathrm{Tr}(\alpha_1 z)}\right) \cdots \left(1 - e^{2\pi i \mathrm{Tr}(\alpha_g z)}\right)},$$

where $\mathrm{Tr}(\alpha z) := \sum_{\tau \in I} \alpha^\tau z_\tau$ for any $\alpha \in \mathcal{O}_F$. The definition of $\mathscr{G}_\sigma(z)$ depends only on the cone and is independent of the choice of the generator $\boldsymbol{\alpha}$.

Remark 2.8. If $F = \mathbb{Q}$ and $\sigma = \mathbb{R}_{\ge 0}$, then we have $\mathscr{G}_\sigma(z) = \frac{e^{2\pi i z}}{1 - e^{2\pi i z}}$.

For $\boldsymbol{k} = (k_\tau) \in \mathbb{N}^I$, we denote $\partial^{\boldsymbol{k}} := \prod_{\tau \in I} \partial_\tau^{k_\tau}$, where $\partial_\tau := \frac{1}{2\pi i} \frac{\partial}{\partial z_\tau}$. For $u \in F$, we let ξ_u be the finite additive character on $\mathcal{O}_F$ defined by $\xi_u(\alpha) := e^{2\pi i \mathrm{Tr}(\alpha u)}$. We note that any additive character on $\mathcal{O}_F$ with values in $\mathbb{C}^\times$ of finite order is of this form for some $u \in F$. Theorem 2.9, based on the work of Shintani, is standard (see for example [8, Théorème 5], [13, Lemme 3.2]).

Theorem 2.9. *Let $\boldsymbol{\alpha}$ and σ be as in Definition 2.7. For any $u \in F$ satisfying $\xi_u(\alpha_j) \neq 1$ for $j = 1, \dots, g$, we have*

$$\partial^{\boldsymbol{k}} \mathscr{G}_\sigma(z)\big|_{z = u \otimes 1} = \zeta_\sigma(\xi_u, -\boldsymbol{k}).$$

Note that the condition $\xi_u(\alpha_j) \neq 1$ for $j = 1, \ldots, g$ ensures that $z = u \otimes 1$ does not lie on the poles of the function $\mathscr{G}_\sigma(z)$.

The Lerch zeta function $\mathcal{L}(\xi\Delta, s)$ may be expressed as a finite sum of functions $\zeta_\sigma(\xi, (s, \ldots, s))$ using the Shintani decomposition. We first review the definition of the Shintani decomposition. We say that a cone σ is *simplicial*, if there exists a generator of σ that is linearly independent over $\mathbb{R}$. Any cone generated by a subset of such a generator is called a *face* of σ. A simplicial fan Φ is a set of simplicial cones such that for any $\sigma \in \Phi$, any face of σ is also in Φ, and for any cones $\sigma, \sigma' \in \Phi$, the intersection $\sigma \cap \sigma'$ is a common face of σ and σ'.

A version of Shintani decomposition that we will use in this article is as follows.

Definition 2.10. A *Shintani decomposition* is a simplicial fan Φ satisfying the following properties.

(1) $\mathbb{R}^I_+ \cup \{0\} = \coprod_{\sigma \in \Phi} \sigma^\circ$, where σ° is the relative interior of σ, i.e., the interior of σ in the $\mathbb{R}$-linear span of σ.
(2) For any $\sigma \in \Phi$ and $\varepsilon \in \Delta$, we have $\varepsilon\sigma \in \Phi$.
(3) The quotient $\Delta\backslash\Phi$ is a finite set.

We may obtain such decomposition by slightly modifying the construction of Shintani [27, Theorem 1] (see also [15, §2.7 Theorem 1], [31, Theorem 4.1]). Another construction was given by Ishida [19, p.84]. For any integer $q \geq 0$, we denote by Φ_{q+1} the subset of Φ consisting of cones of dimension $q + 1$. Note that by [31, Proposition 5.6], Φ_g satisfies

$$\mathbb{R}^I_+ = \coprod_{\sigma \in \Phi_g} \breve{\sigma}. \tag{6}$$

This gives the following result.

Proposition 2.11. *Let $\xi \colon \mathcal{O}_F \to \mathbb{C}^\times$ be a character of finite order, and $\Delta_\xi \subset \Delta$ its isotropic subgroup. If Φ is a Shintani decomposition, then we have*

$$\mathcal{L}(\xi\Delta, s) = \sum_{\sigma \in \Delta_\xi\backslash\Phi_g} \zeta_\sigma(\xi, (s, \ldots, s)). \tag{7}$$

Proof. By (6), if C is a representative of $\Delta_\xi\backslash\Phi_g$, then $\coprod_{\sigma\in C} \breve{\sigma}$ is a representative of the set $\Delta_\xi\backslash\mathbb{R}^I_+$. Our result follows from the definition of the Lerch zeta function and (5). □

The expression (7) is non-canonical, since it depends on the choice of the Shintani decomposition.

3. Equivariant Coherent Cohomology

In this section, we will first give the definition of equivariant sheaves and equivariant cohomology of a scheme with an action of a group. As in §1, we let

$$\mathbb{T} := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{G}_m) \tag{8}$$

be the algebraic torus over $\mathbb{Z}$ defined by Katz [21, §1], satisfying $\mathbb{T}(R) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, R^\times)$ for any $\mathbb{Z}$-algebra R. We will then construct the *equivariant Čech complex*, which is an explicit complex which may be used to describe equivariant cohomology of $U := \mathbb{T} \setminus \{1\}$ with action of Δ.

Remark 3.1. In order to consider the values of Hecke L-functions when the narrow class number of F is greater than *one* (cf. Remark 2.4), then it would be necessary to consider the algebraic tori

$$\mathbb{T}_{\mathfrak{a}} := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}, \mathbb{G}_m)$$

for general fractional ideals $\mathfrak{a}$ of F.

We first review the basic facts concerning sheaves on schemes that are equivariant with respect to an action of a group. Let G be a group with identity e. A G-scheme is a scheme X equipped with a right action of G. We denote by $[u]\colon X \to X$ the action of $u \in G$, so that $[uv] = [v] \circ [u]$ for any $u, v \in G$ holds. In what follows, we let X be a G-scheme.

Definition 3.2. A *G-equivariant structure* on an $\mathscr{O}_X$-module $\mathscr{F}$ is a family of isomorphisms

$$\iota_u \colon [u]^*\mathscr{F} \xrightarrow{\cong} \mathscr{F}$$

for $u \in G$, such that $\iota_e = \mathrm{id}_{\mathscr{F}}$ and the diagram

$$\begin{tikzcd} {[uv]^*\mathscr{F}} \arrow[r, "\iota_{uv}"] \arrow[d, equal] & \mathscr{F} \\ {[u]^*[v]^*\mathscr{F}} \arrow[r, "{[u]^*\iota_v}"] & {[u]^*\mathscr{F}} \arrow[u, "\iota_u"'] \end{tikzcd}$$

is commutative. We call $\mathscr{F}$ equipped with a G-equivariant structure a *G-equivariant sheaf.*

Note that the structure sheaf $\mathscr{O}_X$ itself is naturally a G-equivariant sheaf. For any G-equivariant sheaf $\mathscr{F}$ on X, we define the equivariant global section by $\Gamma(X/G, \mathscr{F}) := \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Gamma(X, \mathscr{F})) = \Gamma(X, \mathscr{F})^G$. Then the equivariant cohomology $H^m(X/G, -)$ is defined to be the m-th right derived functor of $\Gamma(X/G, -)$ (note that the category of G-equivariant sheaves has enough injectives by [14, Proposition 5.1.1 and Théorème 1.10.1]).

Suppose we have a group homomorphism $\pi\colon G \to H$. For a G-scheme X and an H-scheme Y, we say that a morphism $f\colon X \to Y$ of schemes is *equivariant* with respect to π, if we have $f \circ [u] = [\pi(u)] \circ f$ for any $u \in G$. If $\mathscr{F}$ is a H-equivariant sheaf on Y and f is equivariant, then $f^*\mathscr{F}$ is naturally an G-equivariant sheaf on X with the equivariant structure given by $f^*\iota_{\pi(u)}\colon [u]^*(f^*\mathscr{F}) = f^*([\pi(u)]^*\mathscr{F}) \to f^*\mathscr{F}$ for any $u \in G$, and f induces the pull-back homomorphism

$$f^*\colon H^m(Y/H, \mathscr{F}) \to H^m(X/G, f^*\mathscr{F}) \tag{9}$$

on equivariant cohomology.

We now consider our case of the algebraic torus $\mathbb{T}$. For any $\alpha \in \mathcal{O}_F$, the morphism $\mathbb{T}(R) \to R^\times$ defined by mapping $\xi \in \mathbb{T}(R)$ to $\xi(\alpha) \in R^\times$ induces a morphism of group schemes $t^\alpha\colon \mathbb{T} \to \mathbb{G}_m$, which gives a rational function of $\mathbb{T}$. Then we have

$$\mathbb{T} = \mathrm{Spec}\,\mathbb{Z}[t^\alpha \mid \alpha \in \mathcal{O}_F],$$

where $t^\alpha, t^{\alpha'}$ satisfies the relation $t^\alpha t^{\alpha'} = t^{\alpha+\alpha'}$ for any $\alpha, \alpha' \in \mathcal{O}_F$. If we take a basis $\alpha_1, \ldots, \alpha_g$ of $\mathcal{O}_F$ as a $\mathbb{Z}$-module, then we have

$$\mathrm{Spec}\,\mathbb{Z}[t^\alpha \mid \alpha \in \mathcal{O}_F] = \mathrm{Spec}\,\mathbb{Z}[t^{\pm\alpha_1}, \ldots, t^{\pm\alpha_g}] \cong \mathbb{G}_m^g.$$

The action of Δ on $\mathcal{O}_F$ by multiplication induces an action of Δ on $\mathbb{T}$. Explicitly, the isomorphism $[\varepsilon]\colon \mathbb{T} \to \mathbb{T}$ for $\varepsilon \in \Delta$ is given by $t^\alpha \mapsto t^{\varepsilon\alpha}$ for any $\alpha \in \mathcal{O}_F$.

Definition 3.3. Let $\widetilde{F}$ be the composite of F^τ for all $\tau \in I$. For any $\boldsymbol{k} = (k_\tau) \in \mathbb{Z}^I$, we define a Δ-equivariant sheaf $\mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\boldsymbol{k})$ on $\mathbb{T}$ as follows. As an $\mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}$-module we let $\mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\boldsymbol{k}) := \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}$. The Δ-equivariant structure

$$\iota_\varepsilon \colon [\varepsilon]^*\mathscr{O}_{\mathbb{T}\otimes\widetilde{F}} \xrightarrow{\cong} \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}$$

is given by multiplication by $\varepsilon^{-\boldsymbol{k}} := \prod_{\tau\in I}(\varepsilon^\tau)^{-k_\tau}$ for any $\varepsilon \in \Delta$. Note that for $\boldsymbol{k}, \boldsymbol{k}' \in \mathbb{Z}^I$, we have $\mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\boldsymbol{k}) \otimes \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\boldsymbol{k}') = \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\boldsymbol{k}+\boldsymbol{k}')$. For the case $\boldsymbol{k} = (k, \dots, k)$, we have $\varepsilon^{-\boldsymbol{k}} = N(\varepsilon)^{-k} = 1$ for any $\varepsilon \in \Delta$, hence $\mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\boldsymbol{k}) = \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}$.

The open subscheme $U := \mathbb{T} \setminus \{1\}$ also carries a natural Δ-scheme structure. We will now construct the *equivariant Čech complex*, which may be used to express the cohomology of U with coefficients in a Δ-equivariant $\mathscr{O}_U$-module $\mathscr{F}$. For any $\alpha \in \mathcal{O}_F$, we let $U_\alpha := \mathbb{T} \setminus \{t^\alpha = 1\}$. Then any $\varepsilon \in \Delta$ induces an isomorphism $[\varepsilon]\colon U_{\varepsilon\alpha} \to U_\alpha$. We say that $\alpha \in \mathcal{O}_{F+}$ is *primitive* if $\alpha/N \notin \mathcal{O}_{F+}$ for any integer $N > 1$. In what follows, we let $A \subset \mathcal{O}_{F+}$ be the set of primitive elements of $\mathcal{O}_{F+}$. Then

(1) $\varepsilon A = A$ for any $\varepsilon \in \Delta$.
(2) The set $\mathfrak{U} := \{U_\alpha\}_{\alpha\in A}$ gives an affine open covering of U.

We note that for any simplicial cone σ of dimension m, there exists a generator $\boldsymbol{\alpha} \in A^m$, unique up to permutation of the components.

Let q be an integer ≥ 0. For any $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_q) \in A^{q+1}$, we let $U_{\boldsymbol{\alpha}} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$, and we denote by $j_{\boldsymbol{\alpha}} \colon U_{\boldsymbol{\alpha}} \hookrightarrow U$ the inclusion. We let

$$\mathscr{C}^q(\mathfrak{U}, \mathscr{F}) := \prod_{\boldsymbol{\alpha}\in A^{q+1}}^{\mathrm{alt}} j_{\boldsymbol{\alpha}*} j_{\boldsymbol{\alpha}}^* \mathscr{F}$$

be the subsheaf of $\prod_{\boldsymbol{\alpha}\in A^{q+1}} j_{\boldsymbol{\alpha}*} j_{\boldsymbol{\alpha}}^* \mathscr{F}$ consisting of sections $\boldsymbol{s} = (s_{\boldsymbol{\alpha}})$ such that $s_{\rho(\boldsymbol{\alpha})} = \mathrm{sgn}(\rho) s_{\boldsymbol{\alpha}}$ for any $\rho \in \mathfrak{S}_{q+1}$ and $s_{\boldsymbol{\alpha}} = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$. We define the differential $d^q \colon \mathscr{C}^q(\mathfrak{U}, \mathscr{F}) \to \mathscr{C}^{q+1}(\mathfrak{U}, \mathscr{F})$ to be the usual alternating sum

$$(d^q f)_{\alpha_0\cdots\alpha_{q+1}} := \sum_{j=0}^{q+1} (-1)^j f_{\alpha_0\cdots\check{\alpha}_j\cdots\alpha_{q+1}}\big|_{U_{(\alpha_0,\dots,\alpha_{q+1})}\cap V} \tag{10}$$

for any section $(f_{\boldsymbol{\alpha}})$ of $\mathscr{C}^q(\mathfrak{U}, \mathscr{F})$ of each open set $V \subset U$. If we let $\mathscr{F} \hookrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F})$ be the natural inclusion, then we have the exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{d^0} \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{d^1} \cdots \xrightarrow{d^{q-1}} \mathscr{C}^q(\mathfrak{U}, \mathscr{F}) \xrightarrow{d^q} \cdots .$$

We next consider the action of Δ. For any $\boldsymbol{\alpha} \in A^{q+1}$ and $\varepsilon \in \Delta$, we have a commutative diagram

$$\begin{tikzcd} U_{\varepsilon\boldsymbol{\alpha}} \arrow[r, hook, "j_{\varepsilon\boldsymbol{\alpha}}"] \arrow[d, "{[\varepsilon]}"', "\cong"] & U \arrow[d, "{[\varepsilon]}"', "\cong"] \\ U_{\boldsymbol{\alpha}} \arrow[r, hook, "j_{\boldsymbol{\alpha}}"] & U, \end{tikzcd}$$

where $\varepsilon\boldsymbol{\alpha} := (\varepsilon\alpha_0, \dots, \varepsilon\alpha_q)$. Then we have an isomorphism

$$[\varepsilon]^* j_{\boldsymbol{\alpha}*} j_{\boldsymbol{\alpha}}^* \mathscr{F} \cong j_{\varepsilon\boldsymbol{\alpha}*} j_{\varepsilon\boldsymbol{\alpha}}^* [\varepsilon]^* \mathscr{F} \xrightarrow{\cong} j_{\varepsilon\boldsymbol{\alpha}*} j_{\varepsilon\boldsymbol{\alpha}}^* \mathscr{F},$$

where the last isomorphism is induced by the Δ-equivariant structure $\iota_\varepsilon\colon [\varepsilon]^*\mathscr{F} \cong \mathscr{F}$. This induces an isomorphism $\iota_\varepsilon\colon [\varepsilon]^*\mathscr{C}^q(\mathfrak{U},\mathscr{F}) \xrightarrow{\cong} \mathscr{C}^q(\mathfrak{U},\mathscr{F})$, which is compatible with the differential (10). Hence $\mathscr{C}^\bullet(\mathfrak{U},\mathscr{F})$ is a complex of Δ-equivariant sheaves on U.

Proposition 3.4. *If $\mathscr{F}$ is quasi-coherent as an $\mathscr{O}_U$-module, the sheaf $\mathscr{C}^q(\mathfrak{U},\mathscr{F})$ is acyclic with respect to the functor $\Gamma(U/\Delta,-)$.*

Proof. By definition, the functor $\Gamma(U/\Delta,-)$ is the composite of the functors $\Gamma(U,-)$ and $\mathrm{Hom}_{\mathbb{Z}[\Delta]}(\mathbb{Z},-)$. Since $\Gamma(U,-)$ sends injective Δ-equivariant $\mathscr{O}_U$-modules to Δ-modules acyclic for $\mathrm{Hom}_{\mathbb{Z}[\Delta]}(\mathbb{Z},-)$ (cf. [14, Lemme 5.6.1]), we have a spectral sequence

$$E_2^{a,b} = H^a\big(\Delta, H^b(U,\mathscr{C}^q(\mathfrak{U},\mathscr{F}))\big) \Rightarrow H^{a+b}(U/\Delta,\mathscr{C}^q(\mathfrak{U},\mathscr{F})).$$

We first prove that $H^b(U,\mathscr{C}^q(\mathfrak{U},\mathscr{F}))=0$ if $b\neq 0$. If we fix some total order on the set A, then we have

$$\mathscr{C}^q(\mathfrak{U},\mathscr{F}) \cong \prod_{\alpha_0<\cdots<\alpha_q} j_{\boldsymbol{\alpha}*}j^*_{\boldsymbol{\alpha}}\mathscr{F},$$

and each component $j_{\boldsymbol{\alpha}*}j^*_{\boldsymbol{\alpha}}\mathscr{F}$ is acyclic for the functor $\Gamma(U,-)$ since $U_{\boldsymbol{\alpha}}$ is affine (notice that the above product is taken in the category of $\mathscr{O}_U$-modules, not of quasi-coherent ones). Therefore $\mathscr{C}^q(\mathfrak{U},\mathscr{F})$ is acyclic by Lemma 3.5. It is now sufficient to prove that $H^a\big(\Delta, H^0(U,\mathscr{C}^q(\mathfrak{U},\mathscr{F}))\big)=0$ for any integer $a\neq 0$, where

$$H^0(U,\mathscr{C}^q(\mathfrak{U},\mathscr{F})) = \prod^{\mathrm{alt}}_{\boldsymbol{\alpha}\in A^{q+1}} \Gamma(U, j_{\boldsymbol{\alpha}*}j^*_{\boldsymbol{\alpha}}\mathscr{F}) \cong \prod_{\alpha_0<\cdots<\alpha_q} \Gamma(U_{\boldsymbol{\alpha}},\mathscr{F}).$$

Assume that the total order on A is preserved by the action of Δ (for example, we may take the order on $\mathbb{R}$ through an embedding $\tau\colon A\hookrightarrow\mathbb{R}$ for some $\tau\in I$). Let B be the subset of A^{q+1} consisting of elements $\boldsymbol{\alpha}=(\alpha_0,\dots,\alpha_q)$ such that $\alpha_0<\cdots<\alpha_q$. Then action of Δ on B is free. We denote by B_0 a subset of B representing the set $\Delta\backslash B$, so that any $\boldsymbol{\alpha}\in B$ may be written uniquely as $\boldsymbol{\alpha}=\varepsilon\boldsymbol{\alpha}_0$ for some $\varepsilon\in\Delta$ and $\boldsymbol{\alpha}_0\in B_0$. We let

$$M := \prod_{\boldsymbol{\alpha}\in B_0} \Gamma(U_{\boldsymbol{\alpha}},\mathscr{F}),$$

and we let $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\Delta],M)$ be the coinduced module of M, with the action of Δ given for any $\varphi\in\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\Delta],M)$ by $\varepsilon\varphi(u)=\varphi(u\varepsilon)$ for any $u\in\mathbb{Z}[\Delta]$ and $\varepsilon\in\Delta$. Then we have a $\mathbb{Z}[\Delta]$-linear isomorphism

$$H^0(U,\mathscr{C}^q(\mathfrak{U},\mathscr{F})) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\Delta],M) \tag{11}$$

given by mapping any $(s_{\boldsymbol{\alpha}})\in H^0(U,\mathscr{C}^q(A,\mathscr{F}))$ to the $\mathbb{Z}$-linear homomorphism

$$\varphi_{(s_{\boldsymbol{\alpha}})}(\delta) := \Big(\iota_\delta\big([\delta]^* s_{\delta^{-1}\boldsymbol{\alpha}_0}\big)\Big) \in M$$

for any $\delta\in\Delta$. The compatibility of (11) with the action of Δ is seen as follows. By definition, the action of $\varepsilon\in\Delta$ on $(s_{\boldsymbol{\alpha}})\in H^0(U,\mathscr{C}^q(\mathfrak{U},\mathscr{F}))$ is given by $\varepsilon\big((s_{\boldsymbol{\alpha}})\big) = \big(\iota_\varepsilon([\varepsilon]^* s_{\varepsilon^{-1}\boldsymbol{\alpha}})\big)$. Hence noting that

$$\iota_\delta\circ[\delta]^*\iota_\varepsilon = \iota_{\delta\varepsilon}\colon \Gamma(U_{\boldsymbol{\alpha}},[\delta\varepsilon]^*\mathscr{F}) \to \Gamma(U_{\boldsymbol{\alpha}},\mathscr{F})$$

and $[\delta]^* \circ \iota_\varepsilon = [\delta]^*\iota_\varepsilon \circ [\delta]^* \colon \Gamma(U_{\boldsymbol{\alpha}}, [\varepsilon]^*\mathscr{F}) \to \Gamma(U_{\boldsymbol{\alpha}}, [\delta]^*\mathscr{F})$ for any $\delta \in \Delta$, we have

$$\begin{aligned}\varphi_{\varepsilon(s_{\boldsymbol{\alpha}})}(\delta) &= \Big(\iota_\delta\big([\delta]^*(\iota_\varepsilon([\varepsilon]^* s_{\varepsilon^{-1}\delta^{-1}\boldsymbol{\alpha}_0}))\big)\Big) = \Big((\iota_\delta \circ [\delta]^*\iota_\varepsilon)\big([\delta\varepsilon]^* s_{\varepsilon^{-1}\delta^{-1}\boldsymbol{\alpha}_0}\big)\Big) \\ &= \big(\iota_{\delta\varepsilon}([\delta\varepsilon]^* s_{\varepsilon^{-1}\delta^{-1}\boldsymbol{\alpha}_0})\big) = \varphi_{(s_{\boldsymbol{\alpha}})}(\delta\varepsilon)\end{aligned}$$

as desired. The fact that (11) is an isomorphism follows from the fact that B_0 is a representative of $\Delta\backslash B$. By (11) and Shapiro's lemma, we have

$$H^a(\Delta, H^0(U, \mathscr{C}^q(\mathfrak{U}, \mathscr{F}))) \cong H^a(\{1\}, M) = 0$$

for $a \neq 0$ as desired. □

Lemma 3.5 was used in the proof of Proposition 3.4.

Lemma 3.5. *Let X be a scheme and let $(\mathscr{F}_\lambda)_{\lambda\in\Lambda}$ be a family of quasi-coherent sheaves on X. Then for any integer $m \geq 0$, we have*

$$H^m\Big(X, \prod_{\lambda\in\Lambda} \mathscr{F}_\lambda\Big) \cong \prod_{\lambda\in\Lambda} H^m(X, \mathscr{F}_\lambda).$$

Here the product $\prod_{\lambda\in\Lambda} \mathscr{F}_\lambda$ is taken in the category of $\mathscr{O}_X$-modules.

Proof. Take an injective resolution $0 \to \mathscr{F}_\lambda \to I^\bullet_\lambda$ for each $\lambda \in \Lambda$. We will prove that $0 \to \prod_{\lambda\in\Lambda} \mathscr{F}_\lambda \to \prod_{\lambda\in\Lambda} I^\bullet_\lambda$ is an injective resolution. Since the product of injective objects is injective, it is sufficient to prove that $0 \to \prod_{\lambda\in\Lambda} \mathscr{F}_\lambda \to \prod_{\lambda\in\Lambda} I^\bullet_\lambda$ is exact.

For any affine open set V of X, by affine vanishing, the global section $0 \to \mathscr{F}_\lambda(V) \to I^\bullet_\lambda(V)$ is exact, hence the product

$$0 \to \prod_{\lambda\in\Lambda} \mathscr{F}_\lambda(V) \to \prod_{\lambda\in\Lambda} I^\bullet_\lambda(V) \tag{12}$$

is also exact. For any $x \in X$, if we take the direct limit of (12) with respect to open affine neighborhoods of x, then we obtain the exact sequence

$$0 \to \Big(\prod_{\lambda\in\Lambda} \mathscr{F}_\lambda\Big)_x \to \Big(\prod_{\lambda\in\Lambda} I^\bullet_\lambda\Big)_x.$$

This shows that $0 \to \prod_{\lambda\in\Lambda} \mathscr{F}_\lambda \to \prod_{\lambda\in\Lambda} I^\bullet_\lambda$ is exact as desired. □

Proposition 3.4 gives Corollary 3.6.

Corollary 3.6. *We let $C^\bullet(\mathfrak{U}/\Delta, \mathscr{F}) := \Gamma(U/\Delta, \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}))$. Then for any integer $m \geq 0$, the equivariant cohomology $H^m(U/\Delta, \mathscr{F})$ is given as*

$$H^m(U/\Delta, \mathscr{F}) = H^m(C^\bullet(\mathfrak{U}/\Delta, \mathscr{F})).$$

By definition, for any integer $q \in \mathbb{Z}$, we have

$$C^q(\mathfrak{U}/\Delta, \mathscr{F}) = \Big(\prod^{\mathrm{alt}}_{\boldsymbol{\alpha}\in A^{q+1}} \Gamma(U_{\boldsymbol{\alpha}}, \mathscr{F})\Big)^\Delta.$$

4. Shintani Generating Class

We let $\mathbb{T}$ be the algebraic torus of (8), and let $U = \mathbb{T} \setminus \{1\}$. In this section, we will use the description of equivariant cohomology of Corollary 3.6 to define the Shintani generating class as a class in $H^{g-1}(U/\Delta, \mathscr{O}_{\mathbb{T}})$. We will then consider the action of the differential operators ∂_τ on this class.

We first interpret the generating functions $\mathscr{G}_\sigma(z)$ of Definition 2.7 as rational functions on $\mathbb{T}$. Let $\mathfrak{D}^{-1} := \{u \in F \mid \mathrm{Tr}_{F/\mathbb{Q}}(u\mathcal{O}_F) \subset \mathbb{Z}\}$ be the inverse different of F. Then there exists an isomorphism

$$(F \otimes \mathbb{C})/\mathfrak{D}^{-1} \xrightarrow{\cong} \mathbb{T}(\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{C}^\times), \qquad z \mapsto \xi_z \tag{13}$$

given by mapping any $z \in F \otimes \mathbb{C}$ to the character $\xi_z(\alpha) := e^{2\pi i \mathrm{Tr}(\alpha z)}$ in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{C}^\times)$. The function t^α on $\mathbb{T}(\mathbb{C})$ corresponds through the isomorphism (13) to the function $e^{2\pi i \mathrm{Tr}(\alpha z)}$ for any $\alpha \in \mathcal{O}_F$. Thus the following holds.

Lemma 4.1. *For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g) \in A^g$ and $\sigma := \sigma_{\boldsymbol{\alpha}}$, consider the rational function*

$$\mathcal{G}_\sigma(t) := \frac{\sum_{\alpha \in \breve{P}_{\boldsymbol{\alpha}} \cap \mathcal{O}_F} t^\alpha}{\big(1 - t^{\alpha_1}\big) \cdots \big(1 - t^{\alpha_g}\big)}$$

on $\mathbb{T}$, where $P_{\boldsymbol{\alpha}}$ is again the parallelepiped spanned by $\alpha_1, \dots, \alpha_g$ (recall that $\breve{P}_{\boldsymbol{\alpha}}$ is defined by (3)*, depending on a fixed $\tau_0 \in I$). Then $\mathcal{G}_\sigma(t)$ corresponds to the function $\mathscr{G}_\sigma(z)$ of Definition* 2.7 *through the uniformization* (13)*. Note that by definition, if we let $R := \mathbb{Z}[t^\alpha \mid \alpha \in \mathcal{O}_{F+}]$, then we have*

$$\mathcal{G}_\sigma(t) \in R_{\boldsymbol{\alpha}} := R\Big[\frac{1}{1 - t^{\alpha_1}}, \dots, \frac{1}{1 - t^{\alpha_g}}\Big]. \tag{14}$$

From now on, we fix an ordering $I = \{\tau_1, \dots, \tau_g\}$ (thus the previously fixed element $\tau_0 \in I$ coincides with some τ_i, $i = 1, \dots, g$). For any $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g) \in \mathcal{O}_{F+}^g$, let $\big(\alpha_j^{\tau_i}\big)$ be the matrix in $M_g(\mathbb{R})$ whose (i,j)-component is $\alpha_j^{\tau_i}$. We let $\mathrm{sgn}(\boldsymbol{\alpha}) \in \{0, \pm 1\}$ be the signature of $\det\big(\alpha_j^{\tau_i}\big)$. We define the *Shintani generating class* $\mathcal{G}$ as follows.

Proposition 4.2. *For any $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g) \in A^g$, we let*

$$\mathcal{G}_{\boldsymbol{\alpha}} := \mathrm{sgn}(\boldsymbol{\alpha}) \mathcal{G}_{\sigma_{\boldsymbol{\alpha}}}(t) \in \Gamma(U_{\boldsymbol{\alpha}}, \mathscr{O}_{\mathbb{T}}).$$

Then we have $(\mathcal{G}_{\boldsymbol{\alpha}}) \in C^{g-1}(\mathfrak{U}/\Delta, \mathscr{O}_{\mathbb{T}})$. Moreover, $(\mathcal{G}_{\boldsymbol{\alpha}})$ satisfies the cocycle condition $d^{g-1}(\mathcal{G}_{\boldsymbol{\alpha}}) = 0$, hence defines a class

$$\mathcal{G} := [\mathcal{G}_{\boldsymbol{\alpha}}] \in H^{g-1}(U/\Delta, \mathscr{O}_{\mathbb{T}}).$$

We call this class the Shintani generating class.

Proof. By construction, $(\mathcal{G}_{\boldsymbol{\alpha}})$ defines an element in

$$\Gamma(U, \mathscr{C}^{g-1}(\mathfrak{U}, \mathscr{O}_{\mathbb{T}})) = \prod_{\boldsymbol{\alpha} \in A^g}^{\mathrm{alt}} \Gamma(U_{\boldsymbol{\alpha}}, \mathscr{O}_{\mathbb{T}}).$$

Since $[\varepsilon]^* \mathcal{G}_{\boldsymbol{\alpha}} = \mathcal{G}_{\varepsilon\boldsymbol{\alpha}}$ for any $\varepsilon \in \Delta$, we have

$$(\mathcal{G}_{\boldsymbol{\alpha}}) \in \Gamma\big(U, \mathscr{C}^{g-1}(\mathfrak{U}, \mathscr{O}_{\mathbb{T}})\big)^\Delta = C^{g-1}(\mathfrak{U}/\Delta, \mathscr{O}_{\mathbb{T}}).$$

To prove the cocycle condition $d^{g-1}(\mathcal{G}_{\boldsymbol{\alpha}})=0$, it is sufficient to check that

$$\sum_{j=0}^{g}(-1)^j\mathcal{G}_{(\alpha_0,\dots,\check{\alpha}_j,\dots\alpha_g)}=0 \tag{15}$$

for any $\alpha_0,\dots,\alpha_g\in A$. By definition, the rational function $\mathcal{G}_{\sigma_{\boldsymbol{\alpha}}}(t)$ maps to the formal power series

$$\mathcal{G}_{\sigma_{\boldsymbol{\alpha}}}(t)=\sum_{\alpha\in\check{\sigma}_{\boldsymbol{\alpha}}\cap\mathcal{O}_F}t^{\alpha}$$

by taking the formal completion $R_{\boldsymbol{\alpha}}\hookrightarrow\mathbb{Z}[\![t^{\alpha_1},\dots,t^{\alpha_g}]\!]$, where $R_{\boldsymbol{\alpha}}$ is the ring defined in (14). Since the map taking the formal completion is injective, it is sufficient to check (15) for the associated formal power series. By [31, Proposition 6.2], we have

$$\sum_{j=0}^{g}(-1)^j\mathrm{sgn}(\alpha_0,\dots,\check{\alpha}_j,\dots,\alpha_g)\mathbf{1}_{\check{\sigma}_{(\alpha_0,\dots,\check{\alpha}_j,\dots\alpha_g)}}\equiv 0 \tag{16}$$

as a function on $\mathbb{R}^I_+$, where $\mathbf{1}_D$ is the characteristic function of $D\subset\mathbb{R}^I_+$, i.e., $\mathbf{1}_D(x)=1$ if $x\in D$ and $\mathbf{1}_D(x)=0$ if $x\notin D$. Our assertion now follows by examining the formal power series expansion of $\mathcal{G}_{(\alpha_0,\dots,\check{\alpha}_j,\dots,\alpha_g)}$. □

Note that the above construction of the coclycle $(\mathcal{G}_{\boldsymbol{\alpha}})$ depends on the choice of $\tau_0\in I$ and the ordering $I=\{\tau_1,\dots,\tau_g\}$, used in the definition of $\check{P}_{\boldsymbol{\alpha}}$ and $\mathrm{sgn}(\boldsymbol{\alpha})$, respectively.

Proposition 4.3.

(1) *The cohomology class $\mathcal{G}$ does not depend on the choice of $\tau_0\in I$.*

(2) *If one uses another ordering $I=\{\tau'_1,\dots,\tau'_g\}$ in the construction of $\mathcal{G}$, the resulting class is $\mathrm{sgn}(\rho)\mathcal{G}$, where $\rho\in\mathfrak{S}_g$ is the permutation such that $\tau'_i=\tau_{\rho(i)}$.*

Proof. (2) is obvious from the identity

$$\mathrm{sgn}'(\boldsymbol{\alpha})=\mathrm{sgn}(\rho)\mathrm{sgn}(\boldsymbol{\alpha})$$

for $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_g)\in A^g$, where $\mathrm{sgn}'(\boldsymbol{\alpha})$ denotes the sign of $\det\bigl(\alpha_j^{\tau'_i}\bigr)$.

Let us prove (1). Since there is nothing to prove for $g=1$, we assume $g\geq 2$ and the two choices of τ_0 to be compared are τ_1 and τ_2. Let $(\mathcal{G}^1_{\boldsymbol{\alpha}})$ and $(\mathcal{G}^2_{\boldsymbol{\alpha}})$ denote the corresponding cocycles. First consider the case $g=2$. If we put

$$f_\alpha(t):=\frac{1}{1-t^\alpha}\qquad(\alpha\in A),$$

it is clear that $f:=(f_\alpha)\in C^0(\mathfrak{U}/\Delta,\mathscr{O}_{\mathbb{T}})$. Moreover, for $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)\in A^2$, we can verify the equality

$$\mathcal{G}^1_{\boldsymbol{\alpha}}-\mathcal{G}^2_{\boldsymbol{\alpha}}=f_{\alpha_1}-f_{\alpha_2}$$

by considering the relation among the characteristic functions of certain subsets of $\mathbb{R}^I_+$. Thus we have $(\mathcal{G}^1_{\boldsymbol{\alpha}})-(\mathcal{G}^2_{\boldsymbol{\alpha}})=-df$, which shows that the cocycles $(\mathcal{G}^1_{\boldsymbol{\alpha}})$ and $(\mathcal{G}^2_{\boldsymbol{\alpha}})$ are cohomologous. Next, consider the case $g\geq 3$. Let e_i $(i=1,2,3)$ denote the unit vector in $\mathbb{R}^I$ whose τ_i component is 1. For $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_{g-1})\in A^{g-1}$, we set

$$D_{\boldsymbol{\alpha}}:=\{x_0e_3+x_1\alpha_1+\cdots+x_{g-1}\alpha_{g-1}\mid x_0,x_1,\dots,x_{g-1}\in\mathbb{R}_{>0}\},$$
$$\check{D}^i_{\boldsymbol{\alpha}}:=\{u\in\mathbb{R}^I_+\mid\exists\delta>0,\ 0<\forall\delta'<\delta,\ u-\delta' e_i\in D_{\boldsymbol{\alpha}}\}\qquad(i=1,2),$$

and

$$f_{\boldsymbol{\alpha}}(t) := \operatorname{sgn}(e_3, \alpha_1, \dots, \alpha_{g-1}) \left\{ \sum_{\beta \in \check{D}^1_{\boldsymbol{\alpha}} \cap \mathcal{O}_{F+}} t^\beta - \sum_{\beta \in \check{D}^2_{\boldsymbol{\alpha}} \cap \mathcal{O}_{F+}} t^\beta \right\}.$$

While $f_{\boldsymbol{\alpha}}(t)$ is *a priori* an element of $\prod_{\beta \in \mathcal{O}_{F+}} \mathbb{Z} \cdot t^\beta$, it actually belongs to $\Gamma(U_{\boldsymbol{\alpha}}, \mathscr{O}_{\mathbb{T}})$. To see this, note first that the difference of two sets $\check{D}^1_{\boldsymbol{\alpha}}$ and $\check{D}^2_{\boldsymbol{\alpha}}$ concentrates on the proper faces of $D_{\boldsymbol{\alpha}}$, i.e., the cones spanned by proper subsets of $\{e_3, \alpha_1, \dots, \alpha_{g-1}\}$. Furthermore, the faces spanned by the proper subsets including e_3 contains no element of $\mathcal{O}_{F+}$ (if there exists such $\beta \in \mathcal{O}_{F+}$, the subspace generated by β and a proper subset of $\{\alpha_1, \dots \alpha_{g-1}\}$ contain e_3, which contradicts [31, Lemma 5.3]). Thus $f_{\boldsymbol{\alpha}}(t)$ is a linear combination of the series of the form $\sum_{\beta \in \sigma} t^\beta$ with the cones σ generated by subsets of $\{\alpha_1, \dots, \alpha_{g-1}\}$, which can be regarded as elements of $\Gamma(U_{\boldsymbol{\alpha}}, \mathscr{O}_{\mathbb{T}})$.

Since $\varepsilon \check{D}^i_{\boldsymbol{\alpha}} = \check{D}^i_{\varepsilon\boldsymbol{\alpha}}$, we see that $f := (f_{\boldsymbol{\alpha}})$ belongs to $C^{g-2}(\mathfrak{U}/\Delta, \mathscr{O}_{\mathbb{T}})$. Now let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g) \in A^g$ be given. Then, by applying (16) to $\alpha_0 = e_3$ and α_1, $\dots$, α_g, we obtain

$$\mathcal{G}^1_{\boldsymbol{\alpha}} - \mathcal{G}^2_{\boldsymbol{\alpha}} + \sum_{j=1}^{g} (-1)^j f_{\alpha_1, \dots, \check{\alpha}_j, \dots, \alpha_g} = 0,$$

which amounts to $(\mathcal{G}^1_{\boldsymbol{\alpha}}) - (\mathcal{G}^2_{\boldsymbol{\alpha}}) = df$. Therefore, the cocycles $(\mathcal{G}^1_{\boldsymbol{\alpha}})$ and $(\mathcal{G}^2_{\boldsymbol{\alpha}})$ are cohomologous. □

Remark 4.4. We can get rid of the dependence of $\mathcal{G}$ on the ordering of I, described in Proposition 4.3 (2), as follows: For a $\mathbb{Z}$-basis $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g)$ of $\mathcal{O}_F$, we define a g-form ω on $\mathbb{T}$ by

$$\omega := \operatorname{sgn}(\boldsymbol{\alpha}) \frac{dt^{\alpha_1}}{t^{\alpha_1}} \wedge \cdots \wedge \frac{dt^{\alpha_g}}{t^{\alpha_g}}, \tag{17}$$

which is independent of the choice of the basis but changes by a sign under a permutation of the numbering τ_1, $\dots$, τ_g. Then, by using the isomorphism

$$\mathscr{O}_{\mathbb{T}} \longrightarrow \Omega^g_{\mathbb{T}};\ f \longmapsto f \cdot \omega,$$

we set

$$\widetilde{\mathcal{G}} := \mathcal{G} \cdot \omega \in H^{g-1}(U/\Delta, \Omega^g_{\mathbb{T}}).$$

This does not change under any permutation of τ_1, $\dots$, τ_g.

In the forthcoming article [3], we will consider a de Rham cohomology class with coefficients in the logarithm sheaf, which is similar to this $\widetilde{\mathcal{G}}$.

We will next define differential operators ∂_τ for $\tau \in I$ on equivariant cohomology. Since $t^\alpha = e^{2\pi i \operatorname{Tr}(\alpha z)}$ through (13) for any $\alpha \in \mathcal{O}_F$, we have

$$\frac{dt^\alpha}{t^\alpha} = \sum_{\tau \in I} 2\pi i \alpha^\tau dz_\tau. \tag{18}$$

Let α_1, $\dots$, α_g be a basis of $\mathcal{O}_F$. For any $\tau \in I$, we let ∂_τ be the differential operator

$$\partial_\tau := \sum_{j=1}^{g} \alpha_j^\tau t^{\alpha_j} \frac{\partial}{\partial t^{\alpha_j}}.$$

By (18), we see that ∂_τ corresponds to the differential operator $\frac{1}{2\pi i}\frac{\partial}{\partial z_\tau}$ through the uniformization (13), and hence is independent of the choice of the basis $\alpha_1, \dots, \alpha_g$. By Theorem 2.9 and Lemma 4.1, we have the following result.

Proposition 4.5. *Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_g) \in A^g$ and $\sigma = \sigma_{\boldsymbol{\alpha}}$. For any $\boldsymbol{k} = (k_\tau) \in \mathbb{N}^I$ and $\partial^{\boldsymbol{k}} := \prod_{\tau\in I} \partial_\tau^{k_\tau}$, we have*

$$\partial^{\boldsymbol{k}} \mathcal{G}_\sigma(\xi) = \zeta_\sigma(\xi, -\boldsymbol{k})$$

for any torsion point $\xi \in U_{\boldsymbol{\alpha}}$.

The differential operator ∂_τ gives a morphism of abelian sheaves

$$\partial_\tau \colon \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\boldsymbol{k}) \to \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\boldsymbol{k} - 1_\tau)$$

compatible with the action of Δ for any $\boldsymbol{k} \in \mathbb{Z}^I$. This induces a homomorphism

$$\partial_\tau \colon H^m(U \otimes \widetilde{F}/\Delta, \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\boldsymbol{k})) \to H^m(U \otimes \widetilde{F}/\Delta, \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\boldsymbol{k} - 1_\tau))$$

on equivariant cohomology.

Lemma 4.6. *The operators ∂_τ for $\tau \in I$ commute. Moreover, the composite*

$$\partial := \prod_{\tau\in I} \partial_\tau \colon \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}} \to \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(1, \dots, 1) = \mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}$$

is defined over $\mathbb{Z}$, that is, it is a base change to $\widetilde{F}$ of a morphism of abelian sheaves $\partial \colon \mathscr{O}_{\mathbb{T}} \to \mathscr{O}_{\mathbb{T}}$. In particular, ∂ induces a homomorphism

$$\partial \colon H^m(U/\Delta, \mathscr{O}_{\mathbb{T}}) \to H^m(U/\Delta, \mathscr{O}_{\mathbb{T}}). \tag{19}$$

Proof. The commutativity is clear from the definition. Next note that, by the definition, the equivariant sheaves $\mathscr{O}_{\mathbb{T}\otimes\widetilde{F}}(\mathbf{k})$ and the operators are actually defined over the ring of integers in $\widetilde{F}$. On the other hand, since the Galois group $\mathrm{Gal}(\widetilde{F}/\mathbb{Q})$ permutes the operators ∂_τ, the operator ∂ is invariant under this action. Hence our assertion follows. □

Our main result, which we prove in §5, concerns the specialization of the classes

$$\partial^k \mathcal{G} \in H^{g-1}\big(U/\Delta, \mathscr{O}_{\mathbb{T}}\big) \tag{20}$$

for $k \in \mathbb{N}$ at nontrivial torsion points of $\mathbb{T}$.

5. Specialization to Torsion Points

For any nontrivial torsion point ξ of $\mathbb{T}$, let $\Delta_\xi \subset \Delta$ be the isotropic subgroup of ξ. Then we may view $\xi := \operatorname{Spec} \mathbb{Q}(\xi)$ as a Δ_ξ-scheme with a trivial action of Δ_ξ. Then the natural inclusion $\xi \to U$ for $U := \mathbb{T} \setminus \{1\}$ is compatible with the inclusion $\Delta_\xi \subset \Delta$, hence the pull-back (9) induces the specialization map

$$\xi^* \colon H^m(U/\Delta, \mathscr{O}_{\mathbb{T}}) \to H^m(\xi/\Delta_\xi, \mathscr{O}_\xi).$$

The purpose of this section is to prove our main theorem, given as follows.

Theorem 5.1. *Let ξ be a nontrivial torsion point of $\mathbb{T}$, and let k be an integer ≥ 0. If we let $\mathcal{G}$ be the Shintani generating class defined in Proposition 4.2, and if we let $\partial^k \mathcal{G}(\xi) \in H^{g-1}(\xi/\Delta_\xi, \mathscr{O}_\xi)$ be image by the specialization map ξ^* of the class $\partial^k \mathcal{G}$ defined in (20), then we have*

$$\partial^k \mathcal{G}(\xi) = \mathcal{L}(\xi\Delta, -k)$$

through the isomorphism $H^{g-1}(\xi/\Delta_\xi, \mathscr{O}_\xi) \cong \mathbb{Q}(\xi)$ given in Proposition 5.5.

We will prove Theorem 5.1 at the end of this section. The specialization map can be expressed explicitly in terms of cocycles as follows. We let $V_\alpha := U_\alpha \cap \xi$ for any $\alpha \in A$. Then $\mathfrak{V} := \{V_\alpha\}_{\alpha \in A}$ is an affine open covering of ξ. For any integer $q \geq 0$ and $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_q) \in A^{q+1}$, we let $V_{\boldsymbol{\alpha}} := V_{\alpha_0} \cap \cdots \cap V_{\alpha_q}$ and

$$C^q\bigl(\mathfrak{V}/\Delta_\xi, \mathscr{O}_\xi\bigr) := \Bigl(\prod_{\boldsymbol{\alpha} \in A^{q+1}}^{\text{alt}} \Gamma(V_{\boldsymbol{\alpha}}, \mathscr{O}_\xi)\Bigr)^{\Delta_\xi}. \tag{21}$$

Here, note that $\Gamma(V_{\boldsymbol{\alpha}}, \mathscr{O}_\xi) = \mathbb{Q}(\xi)$ if $V_{\boldsymbol{\alpha}} \neq \emptyset$ and $\Gamma(V_{\boldsymbol{\alpha}}, \mathscr{O}_\xi) = \{0\}$ otherwise. The same argument as that of Corollary 3.6 shows that we have

$$H^m(\xi/\Delta_\xi, \mathscr{O}_\xi) \cong H^m\bigl(C^\bullet\bigl(\mathfrak{V}/\Delta_\xi, \mathscr{O}_\xi\bigr)\bigr). \tag{22}$$

We let A_ξ be the subset of elements $\alpha \in A$ satisfying $\xi \in U_\alpha$. This is equivalent to the condition that $\xi(\alpha) \neq 1$. We will next prove in Lemma 5.2 that the cochain complex $C^\bullet\bigl(\mathfrak{V}/\Delta_\xi, \mathscr{O}_\xi\bigr)$ of (21) is isomorphic to the dual of the chain complex $C_\bullet(A_\xi)$ defined as follows. For any integer $q \geq 0$, we let

$$C_q(A_\xi) := \bigoplus_{\boldsymbol{\alpha} \in A_\xi^{q+1}}^{\text{alt}} \mathbb{Z}\boldsymbol{\alpha}$$

be the quotient of $\bigoplus_{\boldsymbol{\alpha} \in A_\xi^{q+1}} \mathbb{Z}\boldsymbol{\alpha}$ by the submodule generated by

$$\{\rho(\boldsymbol{\alpha}) - \operatorname{sgn}(\rho)\boldsymbol{\alpha} \mid \boldsymbol{\alpha} \in A_\xi^{q+1}, \rho \in \mathfrak{S}_{q+1}\} \cup \{\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_q) \mid \alpha_i = \alpha_j \text{ for some } i \neq j\}.$$

We denote by $\langle\boldsymbol{\alpha}\rangle$ the class represented by $\boldsymbol{\alpha}$ in $C_q(A_\xi)$. We see that $C_q(A_\xi)$ has a natural action of Δ_ξ and is a free $\mathbb{Z}[\Delta_\xi]$-module. In fact, a basis of $C_q(A_\xi)$ may be constructed in a similar way to the construction of B_0 in the proof of Proposition 3.4. Then $C_\bullet(A_\xi)$ is a complex of $\mathbb{Z}[\Delta_\xi]$-modules with respect to the standard differential operator $d_q \colon C_q(A_\xi) \to C_{q-1}(A_\xi)$ given by

$$d_q(\langle\alpha_0, \ldots, \alpha_q\rangle) := \sum_{j=0}^{q} (-1)^j \langle\alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_q\rangle$$

for any $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_q) \in A_\xi^{q+1}$. If we let $d_0 \colon C_0(A_\xi) \to \mathbb{Z}$ be the homomorphism defined by $d_0(\langle\alpha\rangle) := 1$ for any $\alpha \in A_\xi$, then $C_\bullet(A_\xi)$ is a free resolution of $\mathbb{Z}$ with trivial Δ_ξ-action. We have the following.

Lemma 5.2. *There exists a natural isomorphism of complexes*

$$C^\bullet(\mathfrak{V}/\Delta_\xi, \mathscr{O}_\xi) \xrightarrow{\cong} \operatorname{Hom}_{\Delta_\xi}(C_\bullet(A_\xi), \mathbb{Q}(\xi)).$$

Proof. The natural isomorphism

$$\prod_{\boldsymbol{\alpha} \in A^{q+1}} \Gamma(V_{\boldsymbol{\alpha}}, \mathscr{O}_\xi) = \prod_{\boldsymbol{\alpha} \in A_\xi^{q+1}} \mathbb{Q}(\xi) \cong \operatorname{Hom}_{\mathbb{Z}}\left(\bigoplus_{\boldsymbol{\alpha} \in A_\xi^{q+1}} \mathbb{Z}\boldsymbol{\alpha}, \mathbb{Q}(\xi)\right)$$

induces an isomorphism between the submodules

$$C^q(\mathfrak{V}/\Delta_\xi, \mathscr{O}_\xi) = \Bigl(\prod_{\boldsymbol{\alpha} \in A^{q+1}}^{\text{alt}} \Gamma(V_{\boldsymbol{\alpha}}, \mathscr{O}_\xi)\Bigr)^{\Delta_\xi} \subset \prod_{\boldsymbol{\alpha} \in A^{q+1}} \Gamma(V_{\boldsymbol{\alpha}}, \mathscr{O}_\xi)$$

and

$$\mathrm{Hom}_{\Delta_\xi}(C_q(A_\xi),\mathbb{Q}(\xi)) \subset \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_{\boldsymbol{\alpha}\in A_\xi^{q+1}} \mathbb{Z}\boldsymbol{\alpha},\mathbb{Q}(\xi)\right).$$

Moreover, this isomorphism is compatible with the differential. □

We will next use a Shintani decomposition (see Definition 2.10) to construct a complex which is quasi-isomorphic to the complex $C_\bullet\big(A_\xi\big)$.

Lemma 5.3. *Let ξ be as above. There exists a Shintani decomposition Φ such that any $\sigma\in\Phi$ is of the form $\sigma_{\boldsymbol{\alpha}}=\sigma$ for some $\boldsymbol{\alpha}\in A_\xi^{q+1}$.*

Proof. Let Φ be a Shintani decomposition. We will deform Φ to construct a Shintani decomposition satisfying our assertion. Let Λ be a finite subset of A such that $\{\sigma_\alpha \mid \alpha\in\Lambda\}$ represents the quotient set $\Delta_\xi\backslash\Phi_1$. If $\xi(\alpha)\neq 1$ for any $\alpha\in\Lambda$, then Φ satisfies our assertion since $\alpha\in A_\xi$ if and only if $\xi(\alpha)\neq 1$.

Suppose that there exists $\alpha\in\Lambda$ such that $\xi(\alpha)=1$. Since ξ is a nontrivial character on $\mathcal{O}_F$, there exists $\beta\in\mathcal{O}_{F+}$ such that $\xi(\beta)\neq 1$. Then for any integer N, we have $\xi(N\alpha+\beta)\neq 1$. Let Φ' be the set of cones obtained by deforming $\sigma=\sigma_\alpha$ to $\sigma' := \sigma_{N\alpha+\beta}$ and $\varepsilon\sigma$ to $\varepsilon\sigma'$ for any $\varepsilon\in\Delta_\xi$. By taking N sufficiently large, the amount of deformation can be made arbitrarily small so that Φ' remains a fan. By repeating this process, we obtain a Shintani decomposition satisfying the desired condition. □

In what follows, we fix a Shintani decomposition Φ satisfying the condition of Lemma 5.3. Let $N\colon \mathbb{R}_+^I\to\mathbb{R}_+$ be the norm map defined by $N((a_\tau)) := \prod_{\tau\in I} a_\tau$, and we let

$$\mathbb{R}_1^I := \{(a_\tau)\in\mathbb{R}_+^I \mid N((a_\tau))=1\}$$

be the subset of $\mathbb{R}_+^I$ of norm one. For any $\sigma\in\Phi_{q+1}$, the intersection $\sigma\cap\mathbb{R}_1^I$ is a subset of $\mathbb{R}_1^I$ which is homeomorphic to a simplex of dimension q, and the set $\{\sigma\cap\mathbb{R}_1^I \mid \sigma\in\Phi_+\}$ for $\Phi_+ := \bigcup_{q\geq 0}\Phi_{q+1}$ gives a simplicial decomposition of the topological space $\mathbb{R}_1^I$.

Recall that we have fixed a numbering of elements in I so that $I=\{\tau_1,\dots,\tau_g\}$. For any $\sigma\in\Phi_{q+1}$, we denote by $\langle\sigma\rangle$ the class $\mathrm{sgn}(\boldsymbol{\alpha})\langle\boldsymbol{\alpha}\rangle$ in $C_q(A_\xi)$, where $\boldsymbol{\alpha}\in A_\xi^{q+1}$ is a generator of σ. This is well-defined since such a generator $\boldsymbol{\alpha}$ is unique up to permutation, and both $\mathrm{sgn}(\boldsymbol{\alpha})$ and $\langle\sigma\rangle$ change by the signature of the permutation. We then have the following.

Lemma 5.4. *For any integer $q\geq 0$, we let $C_q(\Phi)$ be the $\mathbb{Z}[\Delta_\xi]$-submodule of $C_q(A_\xi)$ generated by $\langle\sigma\rangle$ for all $\sigma\in\Phi_{q+1}$. Then $C_\bullet(\Phi)$ is a subcomplex of $C_\bullet(A_\xi)$ which also gives a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Delta_\xi]$-module. In particular, the natural inclusion induces a quasi-isomorphism of complexes*

$$C_\bullet(\Phi)\xrightarrow{\mathrm{qis}} C_\bullet(A_\xi)$$

compatible with the action of Δ_ξ.

Proof. Note that $C_q(\Phi)$ for any integer $q\geq 0$ is a free $\mathbb{Z}[\Delta_\xi]$-module generated by representatives of the quotient $\Delta_\xi\backslash\Phi_{q+1}$. By construction, $C_\bullet(\Phi)$ can be identified with the chain complex associated to the simplicial decomposition $\{\sigma\cap\mathbb{R}_1^I \mid \sigma\in\Phi_+\}$ of the topological space $\mathbb{R}_1^I$, hence we see that the complex $C_\bullet(\Phi)$ is exact and gives a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Delta_\xi]$-module. Our assertion follows from the fact that $C_\bullet(A_\xi)$ also gives a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Delta_\xi]$-module. □

Let
$$L\colon \mathbb{R}^I_+ \to \mathbb{R}^g$$
be the homeomorphism defined by $(x_\tau) \mapsto (\log x_{\tau_i})$. If we let $W := \{(y_{\tau_i}) \in \mathbb{R}^g \mid \sum_{i=1}^g y_{\tau_i} = 0\}$, then W is an $\mathbb{R}$-linear subspace of $\mathbb{R}^g$ of dimension $g-1$, and L gives a homeomorphism $\mathbb{R}^I_1 \cong W \cong \mathbb{R}^{g-1}$. For $\Delta_\xi \subset F$, the Dirichlet unit theorem (see for example [25, Theorem 1 p.61]) shows that the discrete subset $L(\Delta_\xi) \subset W$ is a free $\mathbb{Z}$-module of rank $g-1$, hence we have
$$\mathscr{T}_\xi := \Delta_\xi \backslash \mathbb{R}^I_1 \cong \mathbb{R}^{g-1}/\mathbb{Z}^{g-1}.$$

We consider the coinvariant
$$C_q(\Delta_\xi\backslash\Phi) := C_q(\Phi)_{\Delta_\xi}$$
of $C_q(\Phi)$ with respect to the action of Δ_ξ, that is, the quotient of $C_q(\Phi)$ by the subgroup generated by $\langle\sigma\rangle - \langle\varepsilon\sigma\rangle$ for $\sigma \in \Phi_{q+1}$ and $\varepsilon \in \Delta_\xi$. For any $\sigma \in \Phi_{q+1}$, we denote by $\overline{\sigma}$ the image of σ in the quotient $\Delta_\xi\backslash\Phi_{q+1}$, and we denote by $\langle\overline{\sigma}\rangle$ the image of $\langle\sigma\rangle$ in $C_q(\Delta_\xi\backslash\Phi)$, which depends only on the class $\overline{\sigma} \in \Delta_\xi\backslash\Phi_{q+1}$. Then the set $\{\Delta_\xi\backslash(\Delta_\xi\sigma \cap \mathbb{R}^I_1) \mid \overline{\sigma} \in \Delta_\xi\backslash\Phi_+\}$ of subsets of $\mathscr{T}_\xi$ gives a simplicial decomposition of $\mathscr{T}_\xi$ and $C_\bullet(\Delta_\xi\backslash\Phi)$ may be identified with the associated chain complex. Hence we have
$$H_m(C_\bullet(\Delta_\xi\backslash\Phi)) = H_m(\mathscr{T}_\xi,\mathbb{Z}), \qquad H^m\Big(\mathrm{Hom}_\mathbb{Z}\big(C_\bullet(\Delta_\xi\backslash\Phi),\mathbb{Z}\big)\Big) = H^m(\mathscr{T}_\xi,\mathbb{Z}).$$
Since $\mathscr{T}_\xi \cong \mathbb{R}^{g-1}/\mathbb{Z}^{g-1}$, the homology groups $H_m(\mathscr{T}_\xi,\mathbb{Z})$ for integers m are free abelian groups, and the pairing
$$H_m(\mathscr{T}_\xi,\mathbb{Z}) \times H^m(\mathscr{T}_\xi,\mathbb{Z}) \to \mathbb{Z}, \tag{23}$$
obtained by associating to a cycle $u \in C_m(\Delta_\xi\backslash\Phi)$ and a cocycle
$$\varphi \in \mathrm{Hom}_\mathbb{Z}\big(C_m(\Delta_\xi\backslash\Phi),\mathbb{Z}\big)$$
the element $\varphi(u) \in \mathbb{Z}$, is perfect (see for example [23, Theorem 45.8]).

The generator of the cohomology group
$$H_{g-1}(\mathscr{T}_\xi,\mathbb{Z}) = H_{g-1}\big(C_\bullet(\Delta_\xi\backslash\Phi)\big) \cong \mathbb{Z}$$
is given by the fundamental class, represented by
$$\sum_{\overline{\sigma}\in\Delta_\xi\backslash\Phi_g} \langle\overline{\sigma}\rangle \in C_{g-1}(\Delta_\xi\backslash\Phi), \tag{24}$$
and the canonical isomorphism
$$H^{g-1}(\mathscr{T}_\xi,\mathbb{Q}(\xi)) = H^{g-1}\Big(\mathrm{Hom}_\mathbb{Z}\big(C_\bullet(\Delta_\xi\backslash\Phi),\mathbb{Q}(\xi)\big)\Big) \cong \mathbb{Q}(\xi) \tag{25}$$
induced by the fundamental class (24) via the pairing (23) is given explicitly in terms of cocycles by mapping any $\varphi \in \mathrm{Hom}_\mathbb{Z}(C_{g-1}(\Delta_\xi\backslash\Phi),\mathbb{Q}(\xi))$ to the element $\sum_{\overline{\sigma}\in\Delta_\xi\backslash\Phi_g}\varphi(\langle\overline{\sigma}\rangle) \in \mathbb{Q}(\xi)$.

Proposition 5.5. *Let $\eta \in H^{g-1}(\xi/\Delta_\xi,\mathscr{O}_\xi)$ be represented by a cocycle*
$$(\eta_{\boldsymbol{\alpha}}) \in C^{g-1}(\mathfrak{V}/\Delta_\xi,\mathscr{O}_\xi) = \Bigg(\prod^{\mathrm{alt}}_{\boldsymbol{\alpha}\in A^g_\xi}\mathbb{Q}(\xi)\Bigg)^{\Delta_\xi}.$$

For any cone $\sigma \in \Phi_g$, let $\eta_\sigma := \operatorname{sgn}(\boldsymbol{\alpha})\eta_{\boldsymbol{\alpha}}$ for any $\boldsymbol{\alpha} \in A_\xi^g$ such that $\sigma_{\boldsymbol{\alpha}} = \sigma$. Then the homomorphism mapping the cocycle $(\eta_{\boldsymbol{\alpha}})$ to $\sum_{\overline{\sigma} \in \Delta_\xi \backslash \Phi_g} \eta_\sigma$ induces a canonical isomorphism

$$H^{g-1}(\xi/\Delta_\xi, \mathscr{O}_\xi) \cong \mathbb{Q}(\xi). \tag{26}$$

Proof. Since $C_q(\Phi)$ and $C_q(A_\xi)$ are free $\mathbb{Z}[\Delta_\xi]$-modules, the quasi-isomorphism $C_\bullet(\Phi) \xrightarrow{\text{qis}} C_\bullet(A_\xi)$ of Lemma 5.4 induces the quasi-isomorphism

$$\operatorname{Hom}_{\Delta_\xi}\big(C_\bullet(A_\xi), \mathbb{Q}(\xi)\big) \xrightarrow{\text{qis}} \operatorname{Hom}_{\Delta_\xi}\big(C_\bullet(\Phi), \mathbb{Q}(\xi)\big).$$

Combining this fact with Lemma 5.2 and (22), we see that

$$H^{g-1}(\xi/\Delta_\xi, \mathscr{O}_\xi) \cong H^{g-1}\big(\operatorname{Hom}_{\Delta_\xi}\big(C_\bullet(\Phi), \mathbb{Q}(\xi)\big)\big).$$

Since we have $\operatorname{Hom}_{\Delta_\xi}\big(C_\bullet(\Phi), \mathbb{Q}(\xi)\big) = \operatorname{Hom}_{\mathbb{Z}}\big(C_\bullet(\Delta_\xi \backslash \Phi), \mathbb{Q}(\xi)\big)$, our assertion follows from (25). □

We will now prove Theorem 5.1.

Proof of Theorem 5.1. By construction and Lemma 4.6, the class $\partial^k \mathcal{G}(\xi)$ is a class defined over $\mathbb{Q}(\xi)$ represented by the cocycle $(\partial^k \mathcal{G}_{\boldsymbol{\alpha}}(\xi)) \in C^{g-1}(\mathfrak{V}/\Delta_\xi, \mathscr{O}_\xi)$. By Proposition 5.5 and Proposition 4.5, the class $\partial^k \mathcal{G}(\xi)$ maps through(26) to

$$\sum_{\sigma \in \Delta_\xi \backslash \Phi_g} \partial^k \mathcal{G}_\sigma(\xi) = \sum_{\sigma \in \Delta_\xi \backslash \Phi_g} \zeta_\sigma(\xi, (-k, \dots, -k)).$$

Our assertion now follows from (7). □

Corollary 5.6. *Assume that the narrow class number of F is one, and let $\chi\colon \mathrm{Cl}_F^+(\mathfrak{f}) \to \mathbb{C}^\times$ be a finite primitive Hecke character of F of conductor $\mathfrak{f} \neq (1)$. If we let $U[\mathfrak{f}] := \mathbb{T}[\mathfrak{f}] \setminus \{1\}$, then we have*

$$L(\chi, -k) = \sum_{\xi \in U[\mathfrak{f}]/\Delta} c_\chi(\xi) \partial^k \mathcal{G}(\xi)$$

for any integer $k \geq 0$.

Proof. The result follows from Theorem 5.1 and Proposition 2.3. □

The significance of this result is that the special values at negative integers of *any* finite Hecke character of F may be expressed as a linear combination of special values of the derivatives of a single canonical cohomology class, the Shintani class $\mathcal{G}$ in $H^{g-1}(U/\Delta, \mathscr{O}_{\mathbb{T}})$.

Remark 5.7. As stated in Proposition 4.3, the Shintani generating class $\mathcal{G}$ depends on the ordering of I by a sign. On the other hand, the isomorphism (26) also depends on the ordering of I by a sign, through the definition $\eta_\sigma = \operatorname{sgn}(\boldsymbol{\alpha})\eta_{\boldsymbol{\alpha}}$. These dependences cancel with each other in Theorem 5.1, and thus we obtain the values in $\mathbb{Q}(\xi)$ independent of the ordering of I, namely, the Lerch zeta values $\mathcal{L}(\xi\Delta, -k)$.

In fact, we can modify the formulation to get rid of these dependences as follows. In Remark 4.4, we constructed $\widetilde{\mathcal{G}} \in H^{g-1}(U/\Delta, \Omega_{\mathbb{T}}^g)$ which is independent of the ordering of I. For a nontrivial torsion point ξ, we have an isomorphism

$$\mathscr{O}_\xi \longrightarrow \xi^* \Omega_{\mathbb{T}}^g;\ f \longmapsto f \cdot \xi^* \omega,$$

where $\xi^*\omega$ is the pull-back of the g-form ω given in (17). By composing the induced isomorphism $H^{g-1}(\xi/\Delta_\xi, \mathscr{O}_\xi) \cong H^{g-1}(\xi/\Delta_\xi, \xi^*\Omega^g_{\mathbb{T}})$ with (26), we obtain another isomorphism

$$H^{g-1}(\xi/\Delta_\xi, \xi^*\Omega^g_{\mathbb{T}}) \cong \mathbb{Q}(\xi).$$

This does not depend on the ordering of I, since both ω and the isomorphism (26) change by a sign under a permutation of the ordering. Therefore, by using $\widetilde{\mathcal{G}}$ and this isomorphism, Theorem 5.1 can be reformulated in a fashion invariant under permutations of the ordering of I.

We note that our choice of the fundamental class of $H_{g-1}(\mathcal{T}_\xi, \mathbb{Z})$, which is represeneted by (24), also depends on the ordering of I by a sign through the definition $\langle\sigma\rangle = \operatorname{sgn}(\boldsymbol{\alpha})\langle\boldsymbol{\alpha}\rangle$ (see the paragraph before Lemma 5.4). By using the g-form ω, the resulting isomorphism (25) is modified to

$$H^{g-1}(\mathcal{T}_\xi, \mathbb{Q}(\xi)) \otimes \xi^*\Omega^g_{\mathbb{T}} \cong \mathbb{Q}(\xi),$$

which is invariant under permutations of I.

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