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BLOCH'S CYCLE COMPLEX AND COHERENT DUALIZING COMPLEXES IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let X be a separated scheme of dimension d of finite type over a perfect field k of positive characteristic p. In this work, we show that Bloch's cycle complex $\mathbb{Z}^c_X$ of zero cycles mod p^n is quasi-isomorphic to the Cartier operator fixed part of a certain dualizing complex from coherent duality theory. From this we obtain new vanishing results for the higher Chow groups of zero cycles with mod p^n coefficients for singular varieties.

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INTRODUCTION

Let X be a separated scheme of dimension d of finite type over a perfect field k of positive characteristic p. In this work, we show that Bloch's cycle complex $\mathbb{Z}^c_X$ of zero cycles mod p^n is quasi-isomorphic to the Cartier operator fixed part of a certain dualizing complex from coherent duality theory. From this we obtain new

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vanishing results for the higher Chow groups of zero cycles with mod p^n coefficients for singular varieties.

As the first candidate for a motivic complex, Bloch introduced his cycle complex $\mathbb{Z}^c_X$ in [3] under the framework of Beilinson-Lichtenbaum. Let i be an integer, and $\Delta^i = \operatorname{Spec} k[T_0, \ldots, T_i]/(\sum T_j - 1)$. Here $\mathbb{Z}^c_X := z_0(-, -\bullet)$ is a complex of sheaves in the Zariski or the étale topology. The global section of its degree $(-i)$-term $z_0(X, i)$ is the free abelian group generated by dimension i-cycles in $X \times \Delta^i$ intersecting all faces properly and the differentials are the alternating sums of the cycle-theoretic intersection of the cycle with each face (cf. Section 2). Let $\pi : X \to \operatorname{Spec} k$ be the structure morphism of X. Let $W_n X := (|X|, W_n\mathcal{O}_X)$, where $|X|$ is the underlying topological space of X, and $W_n\mathcal{O}_X$ is the sheaf of length n truncated Witt vectors. Let $W_n\pi : W_n X \to \operatorname{Spec} W_n k$ be the morphism induced from π via functoriality. In this article, our aim is to arrive at a triangle

$$\mathbb{Z}^c_X/p^n \to (W_n\pi)^! W_n k \xrightarrow{C'-1} (W_n\pi)^! W_n k \xrightarrow{+1}$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$, in either the étale topology or the Zariski topology with an extra $k = \overline{k}$ assumption. Here $(-)^!$ is the extraordinary inverse image functor in the coherent setting as defined in [21, VII.3.4], [9, (3.3.6)], and $(W_n\pi)^! W_n k$ is a dualizing complex for coherent sheaves on $W_n X$. This is a generalization of the top degree case of [14, 8.3], which in particular implies the above triangle in the smooth case. Our work is clearly inspired by Kato's paper [29], but the proofs in this article do not use any results from *loc. cit.*

Let us briefly recall Kato's work in [29] and introduce our main object of interest, the complex $K_{n,X,log}$. According to Grothendieck's coherent duality theory, there exists an explicit Zariski complex $K_{n,X}$ of quasi-coherent sheaves representing the dualizing complex $(W_n\pi)^! W_n k$ (such a complex $K_{n,X}$ is called a residual complex, cf. [21, VI.3.1]). There is a natural Cartier operator $C' : K_{n,X} \to K_{n,X}$, which is compatible with the classical Cartier operator $C : W_n\Omega^d_X \to W_n\Omega^d_X$ in the smooth case via Ekedahl's quasi-isomorphism (see Theorem 1.9). Here $W_n\Omega^d_X$ denotes the degree $d := \dim X$ part of the p-typical de Rham-Witt complex. We define the complex $K_{n,X,log}$ to be the mapping cone of $C' - 1$. Kato considered in [29] the FRP counterpart, where FRP is the "flat and relatively perfect" topology (this is a topology with étale coverings and with the underlying category lying in between the small and the big étale site). He showed that $K_{n,X,log}$ serves in the FRP topology as a dualizing complex in a rather big triangulated subcategory of the derived category of $\mathbb{Z}/p^n$-sheaves, containing all coherent sheaves and the logarithmic de Rham-Witt sheaves [29, 0.1]. Kato also showed that in the smooth setting, $K_{n,X,log}$ is concentrated in one degree and its only non-zero cohomology sheaf is the top degree logarithmic de Rham-Witt sheaf [29, 3.4]. An analogue of the latter statement holds naturally on the small étale site. Rülling later observed that with a trick from p^{-1}-linear algebra, [29, 3.4] can be done on the Zariski site as well, as long as one assumes $k = \overline{k}$ (cf. Proposition 1.15). Comparing this with the Kato-Moser complex $\widetilde{\nu}_{n,X}$ (cf. Section 4), which is precisely the Gersten resolution of the logarithmic de Rham-Witt sheaf in the smooth setting, one gets an identification in the smooth setting $\widetilde{\nu}_{n,X} \simeq K_{n,X,log}$ in the Zariski topology. Similar as in [29, 4.2] (cf. Proposition 1.21), Rülling also built up the localization sequence for $K_{n,X,log}$ on the Zariski site in his unpublished notes (cf. Proposition 1.22). Compared with the localization sequence for $\mathbb{Z}^c_X$ [2, 1.1] and for $\widetilde{\nu}_{n,X}$ (which trivially holds in the Zariski topology), it is reasonable to expect a chain map relating these objects in general.

The aim of this article is to construct a quasi-isomorphism $\overline{\zeta}_{log} : \widetilde{\nu}_{n,X} \xrightarrow{\simeq} K_{n,X,log}$, for a possibly singular k-scheme X. When pre-composed with Zhong's quasi-isomorphism $\overline{\psi} : \mathbb{Z}^c_X/p^n \to \widetilde{\nu}_{n,X}$ [43, 2.16], we therefore obtain another perspective of Bloch's cycle complex with $\mathbb{Z}/p^n$-coefficients in terms of coherent dualizing complexes. More precisely, we prove the following result.

Theorem 0.1 (Theorem 5.10, Theorem 6.1)**.** *Let X be a separated scheme of finite type over a perfect field k of positive characteristic p. Then there exists a chain map*

$$\overline{\zeta}_{log,\text{ét}} : \widetilde{\nu}_{n,X,\text{ét}} \xrightarrow{\simeq} K_{n,X,log,\text{ét}},$$

which is a quasi-isomorphism. If moreover k is algebraically closed, then this chain map induces a quasi-isomorphism on the Zariski site

$$\overline{\zeta}_{log,\text{Zar}} : \widetilde{\nu}_{n,X,\text{Zar}} \xrightarrow{\simeq} K_{n,X,log,\text{Zar}}.$$

Composition with Zhong's quasi-isomorphism $\overline{\psi}$ yields the chain map

$$\overline{\zeta}_{log,\text{ét}} \circ \overline{\psi}_{\text{ét}} : \mathbb{Z}^c_{X,\text{ét}}/p^n \xrightarrow{\simeq} K_{n,X,log,\text{ét}}$$

which is a quasi-isomorphism. If moreover $k = \overline{k}$, then the composition

$$\overline{\zeta}_{log,\text{Zar}} \circ \overline{\psi}_{\text{Zar}} : \mathbb{Z}^c_{X,\text{Zar}}/p^n \xrightarrow{\simeq} K_{n,X,log,\text{Zar}}$$

is a quasi-isomorphism as well.

We explain more on the motivation behind the definition of $K_{n,X,log}$. For a smooth k-scheme X, the logarithmic de Rham-Witt sheaves can be defined in two ways: either as the subsheaves of $W_n\Omega^d_X$ generated by log forms or as the invariant part under the Cartier operator C. In the singular case, these two perspectives give two different (complexes of) sheaves. The first definition can also be done in the singular case, and this was studied by Morrow [34]. For the second definition one has to replace $W_n\Omega^d_X$ by a dualizing complex on W_nX: for this Grothendieck's duality theory yields a canonical and explicit choice, and this is what we have denoted by $K_{n,X}$. And then this method leads naturally to Kato's and also our construction of $K_{n,X,log}$. Now with our main theorem one knows that $\mathbb{Z}^c_X/p^n$ sits in a distinguished triangle

$$\mathbb{Z}^c_X/p^n \to K_{n,X} \xrightarrow{C'-1} K_{n,X} \xrightarrow{+1}$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$, in either the étale topology or the Zariski topology with the extra assumption $k = \overline{k}$. In particular, if X is Cohen-Macaulay of pure dimension d, then the triangle above becomes

$$\mathbb{Z}^c_X/p^n \to W_n\omega_X[d] \xrightarrow{C'-1} W_n\omega_X[d] \xrightarrow{+1},$$

where $W_n\omega_X$ is the only non-vanishing cohomology sheaf of $K_{n,X}$ (if $n = 1$, $W_1\omega_X = \omega_X$ is the usual dualizing sheaf on X), and $\mathbb{Z}^c_X/p^n$ is concentrated at degree $-d$ (cf. Proposition 8.1).

As corollaries, we arrive at some properties of the higher Chow groups of 0-cycles with p-primary torsion coefficients. (We have specialized several statements here in this introduction section. Please see the main text for more general statements.)

Corollary 0.2 (Proposition 8.2, Corollary 8.6, Corollary 8.3, Corollary 8.4, Corollary 8.9, Corollary 8.12, Corollary 8.14)**.** *Let X be a separated scheme of finite type over a perfect field k of characteristic $p > 0$ and $\pi : X \to k$ be the structure map of X.*

(1) *(Cartier invariance) Assume $k = \overline{k}$. Then*
$$\mathrm{CH}_0(X, q; \mathbb{Z}/p^n) = H^{-q}(W_nX, K_{n,X,\mathrm{Zar}})^{C'_{\mathrm{Zar}}-1}.$$
(2) *(Semisimplicity) Assume $k = \overline{k}$. Let X be proper over k. Then for any q,*
$$H^{-q}(W_nX, K_{n,X})_{\mathrm{ss}} = \mathrm{CH}_0(X, q; \mathbb{Z}/p^n) \otimes_{\mathbb{Z}/p^n} W_nk.$$
(We refer to Definition A.4 *and Remark* A.5(2) *for the definition of the semisimple part.)*

(3) *(Relation with p-torsion Poincaré duality) There is an isomorphism in $D^b(X_{\mathrm{\acute{e}t}}, \mathbb{Z}/p^n)$*
$$K_{n,X,log,\mathrm{\acute{e}t}} \simeq R\pi^!(\mathbb{Z}/p^n),$$
where $R\pi^!$ is the extraordinary inverse image functor defined in [40, Exposé XVIII, Thm 3.1.4]*.*

(4) *(Affine vanishing) Assume $k = \overline{k}$. Suppose X is affine and Cohen-Macaulay of pure dimension d. Then*
$$\mathrm{CH}_0(X, q, \mathbb{Z}/p^n) = 0$$
for $q \neq d$.

(5) *(Étale descent) Assume $k = \overline{k}$. Suppose X is Cohen-Macaulay of pure relative dimension d. Then*
$$R^i\epsilon_*(\mathbb{Z}^c_{X,\mathrm{\acute{e}t}}/p^n) = R^i\epsilon_*\widetilde{\nu}_{n,X,\mathrm{\acute{e}t}} = 0, \quad i \neq -d.$$
(6) *(Invariance under rational resolution) Assume $k = \overline{k}$. For a rational resolution of singularities $f : \widetilde{X} \to X$ (cf. Definition* 8.10*) of an integral k-scheme X of pure dimension, the trace map induces an isomorphism*
$$\mathrm{CH}_0(\widetilde{X}, q; \mathbb{Z}/p^n) \xrightarrow{\simeq} \mathrm{CH}_0(X, q; \mathbb{Z}/p^n)$$
for each q.

(7) *(Galois descent) Assume $k = \overline{k}$. Let $f : Y \to X$ be a finite étale Galois map with Galois group G. Then*
$$\mathrm{CH}_0(X, d; \mathbb{Z}/p^n) = \mathrm{CH}_0(Y, d; \mathbb{Z}/p^n)^G.$$

Now we give a more detailed description of the structure of this article.

The general setting is that X is a separated scheme of finite type over a perfect field k of positive characteristic p (except in Section 1.1, where a scheme refers to a noetherian scheme of finite Krull dimension). In Part 1, we review the basic properties of the chain complexes to appear. Section 1 is devoted to the properties of the complex $K_{n,X,log}$, the most important object of our studies. We study the Zariski version in Sections 1.2 - 1.5. Following an idea in [29], we define the Cartier operator C' for the residual complex $K_{n,X}$, and then define the complex $K_{n,X,log}$ to be the mapping cone of $C' - 1$ in Section 1.2. We compare our C' with the classical definition of the Cartier operator C for top degree de Rham-Witt sheaves in Section 1.3. The necessary computations are presented in Sections 1.3.2 and 1.3.3. The localization sequence is discussed in Section 1.4. The main ingredients are a surjectivity result for $C' - 1$, which needs the base field k to be algebraically closed, see Proposition 1.15 (see also Appendix A for a short discussion of the necessary semilinear algebra), the trace map of a nilpotent thickening (cf. Proposition 1.21), and the localization sequence (cf. Proposition 1.22). They were already observed by Rülling and are only re-presented here by the author. After a short discussion on functoriality in Section 1.5, we move to the étale case in Section 1.6. Most of the properties mentioned above continue to hold in a similar manner, moreover the surjectivity of $C_{\text{ét}} - 1 : W_n\Omega^d_{X,\text{ét}} \to W_n\Omega^d_{X,\text{ét}}$ over a smooth k-scheme X only requires k to be perfect. This enables us to build the quasi-isomorphism $\zeta_{log,\text{ét}}$ for any perfect field k in the next part. In the remaining Sections 2 - 4 of Part 1 we recall Bloch's cycle complex $\mathbb{Z}^c_X$, Kato's complex of Milnor K-theory $C^M_{X,t}$, and the Kato-Moser complex of logarithmic de Rham-Witt sheaves $\widetilde{\nu}_{n,X,t}$. There are no new results in these three short sections.

In Part 2 we construct the quasi-isomorphism $\overline{\zeta}_{log} : \widetilde{\nu}_{n,X} \xrightarrow{\simeq} K_{n,X,log}$ and study its properties in Section 5. We first construct a chain map $\zeta : C^M_X \to K_{n,X}$ and then show that it induces a chain map $\zeta_{log} : C^M_X \to K_{n,X,log}$. This map actually factors through the chain map $\overline{\zeta}_{log} : \widetilde{\nu}_{n,X} \to K_{n,X,log}$ via the Bloch-Gabber-Kato isomorphism [4, 2.8]. We prove that $\overline{\zeta}_{log}$ is a quasi-isomorphism for $t =$ ét, and also for $t =$ Zar with an extra $k = \overline{k}$ assumption. In Section 6, we review the main results of [43, §2] and compose Zhong's quasi-isomorphism $\overline{\psi} : \mathbb{Z}^c_X/p^n \to \widetilde{\nu}_{n,X}$ with $\overline{\zeta}_{log}$. This composite map enables us to use tools from the coherent duality theory in the calculation of certain higher Chow groups of 0-cycles.

In Part 3 we discuss the applications. Section 7 mainly serves as a preparatory section for Section 8. In Section 8 we arrive at several results for higher Chow groups of 0-cycles with p-primary torsion coefficients: affine vanishing, finiteness (reproof of a theorem of Geisser), étale descent, and invariance under rational resolutions.

Part 1. The complexes

1. Kato's complex $K_{n,X,log,t}$

1.1. Preliminaries: Residual complexes and Grothendieck's duality theory.

The general references for this topic are [21] and [9]. All schemes in this section will be assumed to be noetherian of finite Krull dimension.

1.1.1. *Residual complexes.* A *residual complex* (see [9, p.125] and [21, p.304]) on a scheme X is a complex K such that

- K is bounded as a complex,
- all the terms of K are quasi-coherent and injective $\mathcal{O}_X$-modules,
- the cohomology sheaves are coherent, and
- there is an isomorphism of $\mathcal{O}_X$-modules

$$\bigoplus_{q\in\mathbb{Z}} K^q \simeq \bigoplus_{x\in X} i_{x*}J(x),$$

  where $i_x : \operatorname{Spec}\mathcal{O}_{X,x} \to X$ is the canonical map and $J(x)$ is the quasi-coherent sheaf on $\operatorname{Spec}\mathcal{O}_{X,x}$ associated to an injective hull of $k(x)$ over $\mathcal{O}_{X,x}$ (i.e. the unique injective $\mathcal{O}_{X,x}$-module up to non-unique isomorphisms which contains $k(x)$ as a submodule and such that, for any $0 \neq a \in J(x)$, there exists an element $b \in \mathcal{O}_{X,x}$ with $0 \neq ba \in k(x)$.)

Given a residual complex K on X and a point $x \in X$, there is a unique integer $d_K(x)$, such that $i_{x*}J(x)$ is a direct summand of K^q, i.e.,

$$K^q \simeq \bigoplus_{d_K(x)=q} i_{x*}J(x).$$

The assignment $x \mapsto d_K(x)$ is called the *codimension function on X associated to K* (cf. [21, IV, 1.1(a)], [9, p.125]). We define the *associated filtration*

$$Z^\bullet(K) = \{x \in X \mid d_K(x) \geq p\}.$$

On each irreducible component of X, this filtration equals the shifted codimension filtration. By the *codimension filtration* of a scheme X we refer to the filtration $Z^\bullet$ with

$$Z^p = \{x \in X \mid \dim \mathcal{O}_{X,x} \geq p\}.$$

If $Z^\bullet$ is a filtration on X, we denote by $Z^\bullet[n]$ the shifted filtration with $Z^\bullet[n]^p = Z^{p+n}$.

Let $Z^\bullet$ be a filtration on X such that when restricted to each irreducible component, it is the shifted codimension filtration. For any bounded below complex $\mathcal{F}^\bullet$, choose a bounded below injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}^\bullet$. Denote by $\underline{\Gamma}_{Z^p}$ the sheafified local cohomology functor with support in Z^p, cf. [21, p223 5.]. Then one has a natural decreasing exhaustive filtration by subcomplexes of $\mathcal{I}^\bullet$:

$$\cdots \supset \underline{\Gamma}_{Z^p}(\mathcal{I}^\bullet) \supset \underline{\Gamma}_{Z^{p+1}}(\mathcal{I}^\bullet) \supset \cdots.$$

This filtration is stalkwise bounded below. Now consider the E_1-spectral sequence associated to this filtration

$$E_1^{p,q} \Rightarrow H^{p+q}(\mathcal{F}^\bullet).$$

The *Cousin complex* [9, p.105] $E_{Z^\bullet}(\mathcal{F}^\bullet)$ associated to $\mathcal{F}^\bullet$ is defined to be the 0-th line of the E_1-page, namely

$$E_{Z^\bullet}(\mathcal{F}^\bullet) := (E_1^{p,0} = \mathcal{H}^p_{Z^p/Z^{p+1}}(\mathcal{F}), d_1^{p,0}).$$

Here $\mathcal{H}^p_{Z^p/Z^{p+1}}(\mathcal{F}) := R^p\underline{\Gamma}_{Z^p/Z^{p+1}}(\mathcal{F})$ and $\underline{\Gamma}_{Z^p/Z^{p+1}}(\mathcal{F}) := \underline{\Gamma}_{Z^p}(\mathcal{F})/\underline{\Gamma}_{Z^{p+1}}(\mathcal{F})$ (cf. [21, p.225 Variation 7]). We will also use the shortened notation E for $E_{Z^\bullet}$ when the filtration $Z^\bullet$ is clear from the context. Note that $E_{Z^\bullet}(\mathcal{F}^\bullet)$ is indeed a Cousin complex in the sense of [9, p.105] by the canonical functorial isomorphism [21, p.226], [9, (3.1.4)]

$$\mathcal{H}^i_{Z^p/Z^{p+1}}(\mathcal{F}^\bullet) \xrightarrow{\simeq} \bigoplus_{x \in Z^p - Z^{p+1}} i_{x*}(H^i_x(\mathcal{F}^\bullet)),$$

where $i_x : \operatorname{Spec} \mathcal{O}_{X,x} \to X$ is the canonical map, and $H^i_x(\mathcal{F}^\bullet)$ is the local cohomology groups at x as defined in [21, p.225 Variation 8]. By slight abuse of notation we denote by the same notation $H^i_x(\mathcal{F}^\bullet)$ the quasi-coherent sheaf on $\operatorname{Spec} \mathcal{O}_{X,x}$ associated to this local cohomology group, and it is a sheaf supported on the closed point if it is non-zero.

Let X be a scheme and $Z^\bullet$ be a filtration on X which is a shift of the codimension filtration on each irreducible component of X. Denote by Q the natural functor from the category of complexes of $\mathcal{O}_X$-modules to the derived category of $\mathcal{O}_X$-modules. Then $E_{Z^\bullet}$ and Q induce quasi-inverses [9, 3.2.1]

$$(1.1.1) \qquad \begin{pmatrix} \text{dualizing complexes whose} \\ \text{associated filtration is } Z^\bullet \end{pmatrix} \underset{E_{Z^\bullet}}{\overset{Q}{\rightleftarrows}} \begin{pmatrix} \text{residual complexes whose} \\ \text{associated filtration is } Z^\bullet \end{pmatrix}.$$

For the definition of a dualizing complex (as an object in the derived category) we refer the reader to [9, p.118]. Since we have assumed that X is noetherian and of finite Krull dimension, there always exists a residual complex on X.

1.1.2. *The functor $f^{\triangle}$.* Let $f : X \to Y$ be a finite type morphism between noetherian schemes of finite Krull dimension and let K be a residual complex on Y with associated filtration $Z^\bullet := Z^\bullet(K)$ and codimension function d_K. Define the function $d_{f^\triangle K}$ on X to be [9, (3.2.4)]

$$d_{f^\triangle K}(x) := d_K(f(x)) - \operatorname{trdeg}(k(x)/k(f(x))$$

(so far the subscript $f^\triangle K$ is simply regarded as a formal symbol), and define $f^\triangle Z^\bullet$ accordingly

$$f^\triangle Z^\bullet = \{x \in X \mid d_{f^\triangle K}(x) \geq p\}.$$

Notice that if f has constant fiber dimension r, $f^\triangle Z^\bullet$ is simply $f^{-1}Z^\bullet[r]$.

Following [21, VI, 3.1], [9, 3.2.2], we list some properties of the functor $f^\triangle$ below.

Proposition 1.1. *There exists a functor*

$$f^\triangle : \begin{pmatrix} \text{residual complexes on } Y \\ \text{with filtration } Z^\bullet \end{pmatrix} \to \begin{pmatrix} \text{residual complexes on } X \\ \text{with filtration } f^\triangle Z^\bullet \end{pmatrix}$$

having the following properties (we assume all schemes are noetherian schemes of finite Krull dimension, and all morphisms are of finite type).

(1) *If f is finite, there is an isomorphism of complexes* [21, VI.3.1]

$$\psi_f : f^\triangle K \xrightarrow{\simeq} E_{f^{-1}Z^\bullet}(\overline{f}^* R\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)) \simeq \overline{f}^* \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K),$$

where $\overline{f}^* := f^{-1}(-) \otimes_{f^{-1}f_*\mathcal{O}_X} \mathcal{O}_X$ is the pullback functor associated to the map of ringed spaces $\overline{f} : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$. Since $\overline{f}$ is flat, the pullback functor $\overline{f}^*$ is exact. The last isomorphism is due to the fact that $\overline{f}^*\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K)$ is a residual complex with respect to filtration $f^{-1}Z^\bullet$ (see [21, VI.4.1], [9, (3.4.5)]*).*

(2) *If f is smooth and separated of relative dimension r, there is an isomorphism of complexes* [21, VI.3.1]

$$\varphi_f : f^\triangle K \xrightarrow{\simeq} E_{f^{-1}Z^\bullet[r]}(\Omega^r_{X/Y}[r] \otimes^L_{\mathcal{O}_X} Lf^*K) = E_{f^{-1}Z^\bullet[r]}(\Omega^r_{X/Y}[r] \otimes_{\mathcal{O}_X} f^*K).$$

The last equality is due to the flatness of f and local freeness of $\Omega^r_{X/Y}$.

If f is étale (or more generally residually stable, see (5) *below), this becomes*

$$\varphi_f : f^\triangle K \xrightarrow{\simeq} E_{f^{-1}Z^\bullet}(f^*K) \simeq f^*K.$$

The last isomorphism is due to [21, VI.5.3]*. In particular, if $f = j : X \hookrightarrow Y$ is an open immersion, $j^\triangle K = j^*K$ is a residual complex with respect to filtration $X \cap Z^\bullet$* [9, p.128].

(3) *If f is finite étale, the chain maps ψ_f, φ_f are compatible. Namely, for a given residual complex K on Y, there exists an isomorphism of complexes*

$\overline{f}^*\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K) \xrightarrow{\simeq} f^*K$ *as defined in* [9, (2.7.9)], *such that the following diagram of complexes commutes*

$$\begin{tikzcd} & \overline{f}^*\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K) \arrow[dd, "\simeq"] \\ f^{\triangle}(K) \arrow[ur, "\psi_f", "\simeq"'] \arrow[dr, "\simeq", "\varphi_f"'] & \\ & f^*K. \end{tikzcd}$$

(4) *(Composition) If $f : X \to Y$ and $g : Y \to Z$ are two such morphisms, there is a natural isomorphism of functors* [9, (3.2.3)]

$$c_{f,g} : (gf)^{\triangle} \xrightarrow{\simeq} f^{\triangle}g^{\triangle}.$$

(5) *(Residually stable base change) Following* [9, p.132], *we say a (not necessarily locally finite type) morphism $f : X \to Y$ between locally noetherian schemes is* residually stable *if*
- *f is flat,*
- *the fibers of f are discrete and for all $x \in X$, the extension $k(x)/k(f(x))$ is algebraic, and*
- *the fibers of f are Gorenstein schemes.*

As an example, an étale morphism is residually stable. For more properties of residually stable morphisms, see [21, VI, §5]. *Let f be a morphism of finite type, and u be a residually stable morphism. Let*

(1.1.2)
$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

be a cartesian diagram. Then there is a natural transformation between functors [21, VI.5.5]

$$d_{u,f} : f'^{\triangle}u^* \xrightarrow{\simeq} u'^*f^{\triangle}.$$

(6) *$f^{\triangle}$ is compatible with translation and tensoring with an invertible sheaf. More precisely, for an invertible sheaf $\mathcal{L}$ on Y and a locally constant $\mathbb{Z}$-valued function n on Y, one has canonical isomorphisms of complexes* [9, (3.3.9)]

$$f^{\triangle}(\mathcal{L}[n] \otimes K) \simeq (f^*K)[n] \otimes f^{\triangle}K \simeq (f^*\mathcal{L} \otimes f^{\triangle}K)[n].$$

More properties and compatibility diagrams can be found in [9, §3.3] and [21, VI, §3, §5].

1.1.3. *Trace map for residual complexes.*

Proposition 1.2. *Let $f : X \to Y$ be a* proper *morphism between noetherian schemes of finite Krull dimensions and let K be a residual complex on Y. Then there exists a map of complexes*

$$\mathrm{Tr}_f : f_*f^{\triangle}K \to K,$$

such that the following properties hold [9, §3.4].

(1) *If f is finite,* Tr_f *at a given residual complex K agrees with the following composite as a map of complexes* [9, (3.4.8)]*:*

$$\mathrm{Tr}_f : f_*f^{\triangle}K \underset{\simeq}{\xrightarrow{\psi_f}} \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, K) \xrightarrow{ev.\ at\ 1} K. \tag{1.1.3}$$

(2) *If $f : \mathbf{P}^d_Y \to Y$ is the natural projection, then the trace map Tr_f at K, as a map in the derived category $D^b_c(Y)$, agrees with the following composite* [9, p.151]

$$f_*f^{\triangle}K \underset{\simeq}{\xrightarrow{\varphi_f}} Rf_*(\Omega^n_{\mathbf{P}^n_X/X}[n]) \otimes_{\mathcal{O}_Y} K \to K.$$

The first map is induced from φ_f followed by the projection formula [9, (2.1.10)]*, and the second map is induced by base change from the following isomorphism of groups* [9, (2.3.3)]

$$\mathbb{Z} \xrightarrow{\simeq} H^d(\mathbf{P}^d_{\mathbb{Z}}, \Omega^d_{\mathbf{P}^d_{\mathbb{Z}}/\mathbb{Z}}) = \check{H}^d(\mathfrak{U}, \Omega^d_{\mathbf{P}^d_{\mathbb{Z}}/\mathbb{Z}}), \quad 1 \mapsto (-1)^{\frac{d(d+1)}{2}} \frac{dt_1 \wedge \cdots \wedge dt_d}{t_1 \dots t_d},$$

where $\mathfrak{U} = \{U_0, \dots, U_d\}$ is the standard covering of $\mathbf{P}^d_{\mathbb{Z}}$ and the t_i's are the coordinate functions on U_0.

(3) *(Functoriality* [9, 3.4.1(1)]*)* Tr_f *is functorial with respect to residual complexes with the same associated filtration.*

(4) *(Composition* [9, 3.4.1(2)]*) If $g : Y \to Z$ is another proper morphism, then*

$$\mathrm{Tr}_{gf} = \mathrm{Tr}_g \circ g_*(\mathrm{Tr}_f) \circ (gf)_* c_{f,g}.$$

(5) *(Residually stable base change* [21, VI.5.6]*) Notations are the same as in diagram* (1.1.2)*, and we assume f proper and u residually stable. Then the diagram*

$$\begin{tikzcd} u^*Rf_*f^{\triangle} \arrow[r, "u^*\mathrm{Tr}_f"] \arrow[d, "\simeq"] & u^* \\ Rf'_*u'^*f^{\triangle} \arrow[r, "Rf'_*(d_{u,f})", "\simeq"'] & Rf'_*f'^{\triangle}u^* \arrow[u, "\mathrm{Tr}_{f'}"'] \end{tikzcd}$$

commutes.

(6) Tr_f *is compatible with translation and tensoring with an invertible sheaf* [9, p.148].

(7) *(Grothendieck-Serre duality* [9, 3.4.4]*) If $f : X \to Y$ is proper, then for any $\mathcal{F} \in D^-_{\mathrm{qc}}(X)$, the composition*

$$Rf_*R\mathcal{H}om_X(\mathcal{F}, f^{\triangle}K) \to R\mathcal{H}om_Y(Rf_*\mathcal{F}, Rf_*f^{\triangle}K) \xrightarrow{\mathrm{Tr}_f} R\mathcal{H}om_Y(Rf_*\mathcal{F}, K)$$

is an isomorphism in $D^+_{\mathrm{qc}}(Y)$.

More properties and compatibility diagrams can be found in [9, §3.4] and [21, VI, §4-5; VII, §2].

1.2. **Definition of $K_{n,X,log}$.** Let k be a perfect field of characteristic p. Let W_nk be the ring of Witt vectors of length n of k. Notice that W_nk is an injective W_nk-module by Baer's criterion. So $\operatorname{Spec} W_nk$ is a Gorenstein scheme by [21, V. 9.1(ii)], and its structure sheaf placed at degree 0 is a residual complex (with codimension function being the zero function and the associated filtration being $Z^\bullet(W_nk) = \{Z^0(W_nk)\}$, where $Z^0(W_nk)$ is the set of the unique point in $\operatorname{Spec} W_nk$) by [21, p299 1.] and the categorical equivalence (1.1.1) (note that in this case the Cousin functor

$E_{Z^\bullet(W_nk)}$ applied to W_nk is still W_nk). This justifies the symbol $(W_nF_k)^\triangle$ to appear. To avoid possible confusion we will distinguish the source and target of the absolute Frobenius by using the symbols $k_1 = k_2 = k$. Absolute Frobenius is then written as $F_k : (\operatorname{Spec} k_1, k_1) \to (\operatorname{Spec} k_2, k_2)$, and the n-th Witt lift is written as $W_nF_k : (\operatorname{Spec} W_nk_1, W_nk_1) \to (\operatorname{Spec} W_nk_2, W_nk_2)$. There is a natural isomorphism of W_nk_1-modules (the last isomorphism is given by Proposition 1.1(1))

$$\begin{aligned}(1.2.1)\quad W_nk_1 &\xrightarrow{\simeq} \overline{W_nF_k}^* \operatorname{Hom}_{W_nk_2}((W_nF_k)_*(W_nk_1), W_nk_2) \simeq (W_nF_k)^\triangle(W_nk_2),\\ a &\mapsto a \otimes [(W_nF_k)_*1 \mapsto 1] \quad (= [(W_nF_k)_*a \mapsto 1]),\end{aligned}$$

where $\overline{W_nF_k} : (\operatorname{Spec} W_nk_1, W_nk_1) \to (\operatorname{Spec} W_nk_2, (W_nF_k)_*(W_nk_1))$ is the natural map of ringed spaces, and the Hom set is given the $(W_nF_k)_*(W_nk_1)$-module structure via the first place. In fact, it is clearly a bijection: identify the target with W_nk_2 via the evaluate-at-1 map, then one can see that the map (1.2.1) is identified with $a \mapsto (W_nF_k)^{-1}(a)$.

Let X be a separated scheme of finite type over k of dimension d with structure map $\pi : X \to k$. Since W_nk is a Gorenstein scheme as we recalled in the last paragraph,

$$K_{n,X} := (W_n\pi)^\triangle W_nk$$

is a residual complex on W_nX, associated to the codimension function $d_{K_{n,X}}$ with

$$d_{K_{n,X}}(x) = -\dim \overline{\{x\}},$$

and the filtration $Z^\bullet(K_{n,X}) = \{Z^p(K_{n,X})\}$ with

$$Z^p(K_{n,X}) = \{x \in X \mid \dim \overline{\{x\}} \le -p\}.$$

In particular, $K_{n,X}$ is a bounded complex of injective quasi-coherent $W_n\mathcal{O}_X$-modules with coherent cohomologies sitting in degrees

$$[-d, 0].$$

If $n = 1$, we write $K_X := K_{1,X}$. Now we turn to the definition of C'. Denote the level n Witt lift of the absolute Frobenius F_X by $W_nF_X : (W_nX_1, W_n\mathcal{O}_{X_1}) \to (W_nX_2, W_n\mathcal{O}_{X_2})$. The structure maps of W_nX_1, W_nX_2 are $W_n\pi_1, W_n\pi_2$ respectively. These schemes fit into a commutative diagram

$$\begin{tikzcd} W_nX_1 \arrow[r, "W_nF_X"] \arrow[d, "W_n\pi_1"'] & W_nX_2 \arrow[d, "W_n\pi_2"] \\ \operatorname{Spec} W_nk_1 \arrow[r, "W_nF_k"] & \operatorname{Spec} W_nk_2. \end{tikzcd}$$

Denote

$$K_{n,X_i} := (W_n\pi_i)^\triangle(W_nk_i), \quad i = 1, 2.$$

Via functoriality, one has a $W_n\mathcal{O}_{X_1}$-linear map

(1.2.2)

$$\begin{aligned}K_{n,X_1} = (W_n\pi_1)^\triangle(W_nk_1) &\xrightarrow[\simeq]{(W_n\pi_1)^\triangle(1.2.1)} (W_n\pi_1)^\triangle(W_nF_k)^\triangle(W_nk_2)\\ &\simeq (W_nF_X)^\triangle(W_n\pi_2)^\triangle(W_nk_2) \simeq (W_nF_X)^\triangle K_{n,X_2}.\end{aligned}$$

Here the isomorphism at the beginning of the second line is given by Proposition 1.1(4). Then via the adjunction with respect to the morphism W_nF_X, one has a $W_n\mathcal{O}_{X_2}$-linear map

(1.2.3)
$$C' := C'_n : (W_nF_X)_*K_{n,X_1} \xrightarrow[\simeq]{(W_nF_X)_*(1.2.2)} (W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X_2} \xrightarrow{\mathrm{Tr}_{W_nF_X}} K_{n,X_2},$$

where the last map is the trace map of W_nF_X for residual complexes. We call it the (level n) *Cartier operator* for residual complexes. We sometimes omit the $(W_nF_X)_*$-module structure of the source and write simply as $C' : K_{n,X} \to K_{n,X}$.

Now we come to the construction of $K_{n,X,log}$ (cf. [29, §3]). Define

$$K_{n,X,log} := \mathrm{Cone}(K_{n,X} \xrightarrow{C'-1} K_{n,X})[-1]. \tag{1.2.4}$$

This is a complex of abelian sheaves sitting in degrees

$$[-d, 1].$$

If $n = 1$, we set $K_{X,log} := K_{1,X,log}$. Writing more explicitly, $K_{n,X,log}$ is the following complex

$$(K^{-d}_{n,X} \oplus 0) \to (K^{-d+1}_{n,X} \oplus K^{-d}_{n,X}) \to \cdots \to (K^0_{n,X} \oplus K^{-1}_{n,X}) \to (0 \oplus K^0_{n,X}).$$

The differential of $K_{n,X,log}$ at degree i is given by

$$\begin{aligned} d_{log} = d_{n,log} : K^i_{n,X,log} &\to K^{i+1}_{n,X,log}, \\ (K^i_{n,X} \oplus K^{i-1}_{n,X}) &\to (K^{i+1}_{n,X} \oplus K^i_{n,X}), \\ (a, b) &\mapsto (d(a), -(C'-1)(a) - d(b)), \end{aligned}$$

where d is the differential in $K_{n,X}$. The sign conventions we adopt here for shifted complexes and the cone construction are the same as in [9, p6, p8]. And naturally, one has a distinguished triangle

$$K_{n,X,log} \to K_{n,X} \xrightarrow{C'-1} K_{n,X} \xrightarrow{+1} K_{n,X,log}[1]. \tag{1.2.5}$$

Explicitly, the first map is in degree i given by

$$\begin{aligned} K^i_{n,X,log} = K^i_{n,X} \oplus K^{i-1}_{n,X} &\to K^i_{n,X}, \\ (a, b) &\mapsto a. \end{aligned}$$

The "+1" map is given by

$$\begin{aligned} K^i_{n,X} &\to (K_{n,X,log}[1])^i = K^{i+1}_{n,X,log} = (K^{i+1}_{n,X} \oplus K^i_{n,X}), \\ b &\mapsto (0, b). \end{aligned}$$

Both maps are indeed maps of chain complexes.

1.3. Comparison of $W_n\Omega^d_{X,log}$ with $K_{n,X,log}$.

Recall the following result from the classical Grothendieck duality theory [21, IV.3.4], [9, 3.1.3] and Ekedahl [10, §1] (see also [7, proof of 1.10.3 and Rmk. 1.10.4]).

Proposition 1.3 (Ekedahl)**.** *If X is smooth and of pure dimension d over k, then there is a canonical quasi-isomorphism*

$$W_n\Omega^d_X[d] \xrightarrow{\simeq} K_{n,X}.$$

Remark 1.4. Suppose X is a separated scheme of finite type over k of dimension d. Denote by U the smooth locus of X, and suppose that the complement Z of U is of dimension e. Suppose moreover that U is non-empty and equidimensional (it is satisfied, for example, if X is integral). Then Ekedahl's quasi-isomorphism Proposition 1.3 gives a quasi-isomorphism of dualizing complexes

$$W_n\Omega^d_U[d] \xrightarrow{\simeq} K_{n,U}. \tag{1.3.1}$$

Note that the associated filtrations of quasi-isomorphic dualizing complexes are the same (cf. [21, V.3.4]). Let $Z^\bullet$ be the codimension filtration of U. As explained above, the associated filtration of $K_{n,U}$ is the shifted codimension filtration, i.e., $Z^\bullet[d]$. Apply the Cousin functor associated to the shifted codimension filtration $Z^\bullet[d]$ to the quasi-isomorphism (1.3.1) between dualizing complexes, we have an isomorphism of residual complexes

$$E_{Z^\bullet[d]}(W_n\Omega^d_U[d]) \xrightarrow{\simeq} K_{n,U}$$

with the same filtration $Z^\bullet[d]$ by (1.1.1). Since W_nj is an open immersion, we can canonically identify the residual complexes $(W_nj)^*K_{n,X} \simeq K_{n,U}$ by Proposition 1.1(2). Since $K_{n,X}$ is a residual complex and in particular is a Cousin complex (cf. [9, p. 105]), the adjunction map $K_{n,X} \to (W_nj)_*(W_nj)^*K_{n,X} \simeq (W_nj)_*K_{n,U}$ is an isomorphism at degrees $[-d,-e-1]$. Thus the induced chain map

$$K_{n,X} \to (W_nj)_*E_{Z^\bullet[d]}(W_n\Omega^d_U[d])$$

is an isomorphism at degrees $[-d,-e-1]$.

1.3.1. *Compatibility of C' with the classical Cartier operator C.* We review the absolute Cartier operator in the classical literature (see e.g. [5, Chapter 1 §3], [24, §0.2], [30, 7.2], [25, III §1]). Let X be a k-scheme. The *(absolute) inverse Cartier operator* γ_X of degree i on a scheme X is affine locally, say, on $\operatorname{Spec} A \subset X$, given additively by the following expression ($\mathcal{H}^i(-)$ denotes the cohomology sheaf of the complex)

$$\begin{aligned} \gamma_A : \qquad \Omega^i_{A/k} &\to \mathcal{H}^i(F_{A,*}\Omega^\bullet_{A/k}), \\ a\,da_1\dots da_i &\mapsto a^p a_1^{p-1}da_1\dots a_i^{p-1}da_i, \end{aligned} \tag{1.3.2}$$

where $a, a_1, \dots a_i \in A$. Here $\mathcal{H}^i(F_{A,*}\Omega^\bullet_{A/k})$ denotes the A-module structure on $\mathcal{H}^i(\Omega^\bullet_{A/k})$ via the absolute Frobenius $F_A : A \to A, a \mapsto a^p$ (note that $F_{A,*}\Omega^\bullet_{A/k}$ is a complex of A-modules in positive characteristics). For each degree i, γ_A thus defined is an A-linear map. These local maps patch together and give rise to a map of sheaves

$$\gamma_X : \Omega^i_X \to \mathcal{H}^i(F_{X,*}\Omega^\bullet_X) \tag{1.3.3}$$

which is $\mathcal{O}_X$-linear. If X is smooth of dimension d, γ_X is an isomorphism of $\mathcal{O}_X$-modules, which is called the *(absolute) Cartier isomorphism.* See [5, 1.3.4] for a proof (note that although the authors there assumed the base field to be algebraically closed, the proof of this theorem works for any perfect field k of positive characteristics).

This can be generalized to the de Rham-Witt case.

Lemma 1.5 (Cf. [28, 4.1.3])**.** *Denote by $W_n\Omega'^i_X$ the abelian sheaf $F(W_{n+1}\Omega^i_X)$ regarded as a $W_n\mathcal{O}_X$-submodule of $(W_nF_X)_*W_n\Omega^i_X$. If X is smooth of dimension d, the map*

$$\overline{F}: W_n\Omega^i_X \to W_n\Omega'^i_X/dV^{n-1}\Omega^{i-1}_X$$

induced by Frobenius $F: W_{n+1}\Omega^i_X \to R_*(W_nF_X)_*W_n\Omega^i_X$ is an isomorphism of $W_n\mathcal{O}_X$-modules.

In particular, if $i = d$,

$$\overline{F}: W_n\Omega^d_X \to (W_nF_X)_*W_n\Omega^d_X/dV^{n-1}\Omega^{d-1}_X$$

is an isomorphism of $W_n\mathcal{O}_X$-modules.

Proof. Since

$$\mathrm{Ker}(R: W_{n+1}\Omega^i \to W_n\Omega^i) = V^n\Omega^i + dV^n\Omega^{i-1},$$

$FV^n\Omega^i = 0$ and $FdV^n\Omega^{i-1} = dV^{n-1}\Omega^{i-1}$, $F: W_{n+1}\Omega^i \to W_n\Omega^i$ reduces to

$$\overline{F}: W_n\Omega^i \to W_n\Omega^i/dV^{n-1}.$$

The surjectivity is clear. We show the injectivity. Suppose $x \in W_{n+1}\Omega^i$, $y \in \Omega^{i-1}$, such that $F(x) = dV^{n-1}y$. Then $F(x - dV^ny) = 0$, which implies by [24, I (3.21.1.2)] that $x - dV^ny \in V^n\Omega^i$.

The second claim follows from the fact that $F: W_{n+1}\Omega^d \to R_*(W_nF_X)_*W_n\Omega^d$ is surjective on top degree d [24, I (3.21.1.1)], and therefore $W_n\Omega'^d = (W_nF_X)_*W_n\Omega^d$ as $W_n\mathcal{O}_X$-modules. □

Definition 1.6 ((Absolute) Cartier operator)**.** Let X be a smooth scheme of dimension d over k.

(1) The composition

$$C := C_X : Z^i(F_{X,*}\Omega^\bullet_X) \to \mathcal{H}^i(F_{X,*}\Omega^\bullet_X) \xrightarrow{(\gamma_X)^{-1}} \Omega^i_X \tag{1.3.4}$$

$$\big(\text{with } Z^i(F_{X,*}\Omega^\bullet_X) := \mathrm{Ker}(F_{X,*}\Omega^i_X \xrightarrow{d} F_{X,*}\Omega^{i+1}_X)\big)$$

is called the *(absolute) Cartier operator* of degree i, denoted by C or C_X.

(2) (cf. [28, 4.1.2, 4.1.4]) More generally, for $n \geq 1$, define the *(absolute) Cartier operator $C_n := C_{n,X}$ of level* n to be the composite

$$C_n : W_n\Omega'^i_X \twoheadrightarrow W_n\Omega'^i_X/dV^{n-1}\Omega^{i-1}_X \xrightarrow[\simeq]{\overline{F}^{-1}} W_n\Omega^i_X, \tag{1.3.5}$$

where $\overline{F}: W_n\Omega^i_X \xrightarrow{\simeq} W_n\Omega'^i_X/dV^{n-1}\Omega^{i-1}_X$ is the map in Lemma 1.5. If $i = d$ is the top degree we obtain the $W_n\mathcal{O}_X$-linear map

$$C_n : (W_nF_X)_*W_n\Omega^d_X \twoheadrightarrow (W_nF_X)_*W_n\Omega^d_X/dV^{n-1}\Omega^{d-1}_X \xrightarrow{\overline{F}^{-1}} W_n\Omega^d_X. \tag{1.3.6}$$

Remark 1.7.

(1) According to the explicit formula for F, we have $C = C_1$ [24, I 3.3]. For this reason we will simply write C for C_n sometimes.

(2) C_n (for all n) are compatible with étale pullbacks. Actually any de Rham-Witt system (e.g. $(W_n\Omega^\bullet_X, F, V, R, \underline{p}, d)$) is compatible with étale base change [7, 1.3.2].

(3) The n-th power of Frobenius F induces a map

$$\overline{F}^n : W_n\Omega^i_X \xrightarrow{\simeq} \mathcal{H}^i((W_nF_X)^n_*W_n\Omega^\bullet_X),$$

which is the same as [25, III (1.4.1)].

(4) Notice that on $\operatorname{Spec} W_nk$, $C_n : W_nk \to W_nk$ is simply the map $(W_nF_k)^{-1}$, because $F : W_{n+1}k \to W_nk$ equals $R \circ W_{n+1}F_k$ in characteristic p.
(5) We sometimes omit "$(W_nF_X)_*$" in the source. But one should always keep that in mind and be careful with the module structure.

Remark 1.8. Before we move on, we state a remark on étale schemes over W_nX.

(1) Notice that every étale W_nX-scheme is of the form $W_ng : W_nU \to W_nX$, where $g : U \to X$ is an étale X-scheme. In fact, there are two functors

$$F : \{\text{étale } W_nX\text{-schemes}\} \leftrightarrows \{\text{étale } X\text{-schemes}\} : G,$$
$$V \mapsto V \times_{W_nX} X,$$
$$W_nU \leftarrow\!\shortmid U.$$

The functor F is a categorical equivalence according to [19, Ch. IV, 18.1.2]. The functor G is well-defined (i.e. produces étale W_nX-schemes) and is a right inverse of F by [23, Thm. 1.25]. We want to show that there is a natural isomorphism $GF \simeq id$, and this is the consequence of the following purely categorical statement: If $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are two functors satisfying both F being a categorical equivalence and $FG \simeq id$, then G is a quasi-inverse of F, i.e., there exists a canonical natural isomorphism $GF \simeq id$. We leave this as an easy exercise for the reader.
(2) The square

$$\begin{tikzcd} W_nU \arrow[r, "W_nF_U"] \arrow[d, "W_ng"'] & W_nU \arrow[d, "W_ng"] \\ W_nX \arrow[r, "W_nF_X"] & W_nX. \end{tikzcd}$$

is a cartesian square. This is because for any étale map $g : U \to X$, the relative Frobenius $F_{U/X}$ is an isomorphism by [11, 10.3.1]. Thus $W_nF_{U/X}$ is an also isomorphism and the claim follows.

We shall now state the main result in this subsection, which seems to be known by experts (cf. proof of [29, 3.4]) but we cannot find a proof in the literature. To eliminate possible sign inconsistency of the Cartier operator with the Grothendieck trace map calculated via residue symbols [9, Appendix A], we give a proof by explicit calculations (see Section 1.3.2-Section 1.3.3). At the same time, this result justifies our notation for C': The classical Cartier operator C is simply the $(-d)$-th cohomology of our C' in the smooth case.

Theorem 1.9 (Compatibility of C' with C)**.** *Suppose that X is a smooth scheme of dimension d over a perfect field k of characteristic $p > 0$. Then the top degree classical Cartier operator*

$$C : (W_nF_X)_*W_n\Omega^d_{X/k} \to W_n\Omega^d_{X/k}$$

as defined in Definition 1.6 *agrees with the* $(-d)$*-th cohomology of the Cartier operator for residual complexes*

$$C' : (W_nF_X)_*W_n\Omega^d_{X/k} \to W_n\Omega^d_{X/k}$$

as defined in (1.2.3) *via Ekedahl's quasi-isomorphism Proposition* 1.3.

Proof. The Cartier operator is stable under étale base change, i.e., for any étale morphism $W_n g : W_n X \to W_n Y$ (which must be of this form according to Remark 1.8(1)), we have

$$C_X \simeq (W_n g)^* C_Y : (W_n F_X)_* W_n \Omega^d_X \to W_n \Omega^d_X.$$

We claim that the map C' defined in (1.2.3) is also compatible with étale base change. That is, whenever we have an étale morphism $W_n g : W_n X \to W_n Y$, there is a canonical isomorphism

$$C'_X \simeq (W_n g)^* C'_Y : (W_n F_X)_* K_{n,X} \to K_{n,X}.$$

First of all, the Grothendieck trace map $\mathrm{Tr}_{W_n F_X}$ for residual complexes is compatible with étale base change by Proposition 1.2(5), i.e.,

$$\mathrm{Tr}_{W_n F_X} \simeq g^* \mathrm{Tr}_{W_n F_Y} : (W_n F_X)_* (W_n F_X)^{\triangle} K_{n,X} \to K_{n,X}.$$

Secondly, because of the cartesian square in Remark 1.8(2) and the flat base change theorem

$$(W_n g)^* (W_n F_X)_* \simeq (W_n F_X)^* (W_n g)_*,$$

we are reduced to show that (1.2.2) is compatible with étale base change. And this is true, because we have

$$(W_n g)^* \simeq (W_n g)^{\triangle}$$

by Proposition 1.1(2), and the compatibility of $(-)^{\triangle}$ with composition by Proposition 1.1(4). This finishes the claim.

Note that the question is local on $W_n X$. Thus to prove the statement for smooth k-schemes X, using the compatibility of C and C' with respect to étale base change, it suffices to prove it for $X = \mathbf{A}^d_k$. That is, we need to check that the expression given in Lemma 1.14 for C' agrees with the expression for C given in Lemma 1.10. This is apparent. $\square$

1.3.2. *Proof of Theorem* 1.9*: C for the top Witt differentials on the affine space.* Let k be a perfect field of positive characteristic p. The aim of this subsection is to provide the formula for the Cartier operator on the top degree de Rham-Witt sheaf over the affine space (Lemma 1.10).

Consider the polynomial ring $k[X_1, \dots, X_d]$. Let $h : \{1, \dots, d\} \to \mathbb{N}[\frac{1}{p}] \setminus \{0\}$ be a function such that $\mathrm{Im}(p^{n-1}h) \subset \mathbb{N}$. Write $h_i := h(i)$. Let $\{i_1, \dots, i_d\}$ be a reordering of $\{1, \dots, d\}$, such that

$$v_p(h_{i_1}) \le v_p(h_{i_2}) \le \cdots \le v_p(h_{i_d}).$$

This order depends on h. Since $\{1, \dots, d\}$ is a finite set, we can also choose a uniform order for elements in $\mathrm{Supp}\, h$ and $\mathrm{Supp}\, p^a h$ for any integer a and any function h. If $\mathrm{Im}(h) \not\subset \mathbb{N}$, let $r \in [1, d]$ be the unique integer such that

$$v_p(h_{i_1}) \le \cdots \le v_p(h_{i_r}) < 0 \le v_p(h_{i_{r+1}}) \le \cdots \le v_p(h_{i_d}).$$

For all $j \in [1, d]$, write

$$v_j := v_p(h_{i_j}), \quad h'_j := h_{i_j} p^{-v_j}.$$

According to [33, 2.17], any element in $W_n \Omega^d_{k[X_1, \dots, X_d]}$ is uniquely written as a sum of (1.3.7) and (1.3.8):

- h is a function such that $\mathrm{Im}(h) \not\subset \mathbb{N}$, $\alpha \in W_{n+v_1}k$,

$$dV^{-v_1}(\alpha[X_1]^{h'_1}) \cdots dV^{-v_r}([X_r]^{h'_r}) \cdot F^{v_{r+1}}d[X_{r+1}]^{h'_{r+1}} \cdots F^{v_d}d[X_d]^{h'_d}, \tag{1.3.7}$$

and
- h is a function such that $\mathrm{Im}(h) \subset \mathbb{N}$, $\beta \in W_nk$,

$$\beta F^{v_1}d[X_1]^{h'_1} \cdots F^{v_d}d[X_d]^{h'_d}. \tag{1.3.8}$$

Lemma 1.10 (C_n on $\mathbf{A}^d$)**.** *The Cartier operator (cf. Definition* 1.6*)*

$$C := C_n : W_n\Omega^d_{k[X_1,\dots,X_d]} \to W_n\Omega^d_{k[X_1,\dots,X_d]}$$

is the map uniquely determined by the following assignment by taking products: for $j \in [1,d]$,

$$\begin{array}{c} dV^{-v_j}(\alpha[X_j]^{h'_j}_{n+v_j}) \\ (\alpha \in W_{n+v_j}k, v_j < 0) \end{array} \mapsto \begin{cases} dV^{1-v_j}(R(\alpha)[X_j]^{h'_j}_{n-1+v_j}), & -v_j \in [1, n-2]; \\ 0, & -v_j = n-1. \end{cases}$$

$$\begin{array}{c} \beta \\ (\beta \in W_nk) \end{array} \mapsto (W_nF_k)^{-1}(\beta),$$

$$\begin{array}{c} F^{v_j}d[X_j]^{h'_j}_{n+v_j} \\ (v_j \geq 0) \end{array} \mapsto \begin{cases} F^{v_j-1}d[X_j]^{h'_j}_{n+v_j-1}, & v_j \geq 1; \\ dV[X_j]^{h_{i_j}}_{n-1}, & v_j = 0. \end{cases}$$

Proof. For any $\gamma, \delta \in W_n\Omega^\bullet_{k[X_1,\dots,X_d]}$, $C(F(\gamma) \cdot F(\delta)) = C(F(\gamma)) \cdot C(F(\delta))$. Hence it suffices to check the formulae on each factor. For $\alpha \in W_{n+v_j}k, -v_j \in [1, n-2]$, the formulae $CF = R$, $d = FdV$ imply

$$\begin{aligned} C\Big(dV^{-v_j}(\alpha[X_j]^{h'_j}_{n+v_j})\Big) &= C\Big(FdV^{1-v_j}(\alpha[X_j]^{h'_j}_{n+v_j})\Big) \\ &= dV^{1-v_j}(R(\alpha)[X_j]^{h'_j}_{n-1+v_j}). \end{aligned}$$

For $\alpha \in k, -v_j = n-1$, the formulae $CF = R$, $d = FdV$ and $\mathrm{Ker}\, R = V^n + dV^n$ imply

$$\begin{aligned} C\Big(dV^{-v_j}(\alpha[X_j]^{h'_j}_{n+v_j})\Big) &= C\Big(FdV^n(\alpha X_j^{h_{i_j}})\Big) \\ &= R\Big(dV^n(\alpha X_j^{h_{i_j}})\Big) \\ &= 0. \end{aligned}$$

For $\beta \in W_nk$,

$$C(\beta) = (W_nF_k)^{-1}(\beta).$$

For $v_j \geq 1$, the formula $CF = R$ implies

$$\begin{aligned} C\Big(F^{v_j}d[X_j]^{h'_j}_{n+v_j}\Big) &= R\Big(F^{v_j-1}d[X_j]^{h'_j}_{n+v_j}\Big) \\ &= F^{v_j-1}d[X_j]^{h'_j}_{n+v_j-1}. \end{aligned}$$

For $v_j = 0$, the formulae $CF = R$ and $d = FdV$ imply

$$\begin{aligned} C\Big(F^{v_j} d[X_j]_{n+v_j}^{h'_j}\Big) &= C\Big(FdV[X_j]_n^{h_{i_j}}\Big) \\ &= R\Big(dV[X_j]_n^{h_{i_j}}\Big) \\ &= dV[X_j]_{n-1}^{h_{i_j}}. \end{aligned}$$

□

1.3.3. *Proof of Theorem* 1.9*: C' for the top Witt differentials on the affine space.* The aim of this section is to calculate C' for the top de Rham-Witt sheaves on the affine space (Lemma 1.14). To do this, one needs to first calculate the trace map of the canonical lift of the absolute Frobenius.

1.3.3.1. *Trace map of the canonical lift $\widetilde{F}_{\widetilde{X}}$ of absolute Frobenius F_X.* Before we start with the computation we recall some properties of the residue symbol from [9, §A]. Let $X \hookrightarrow P$ be a closed immersion of affine schemes with the sheaf of ideals generated by $t_1, \ldots, t_d \in \Gamma(P, \mathcal{O}_P)$. Let $P \to Y$ be a separated smooth morphism of affine schemes with pure relative dimension d. Suppose $X \to Y$ is finite flat. For any $\omega \in \Gamma(P, \omega_{P/Y})$, there is a well-defined element

$$\operatorname{Res}_{P/Y}\begin{bmatrix} \omega \\ t_1, \ldots, t_d \end{bmatrix} \in \Gamma(Y, \mathcal{O}_Y)$$

which is called the residue symbol (cf. [9, (A.1.4)]). It satisfies the following properties (we use the same numbering as in [9, §A.1]):

- Suppose $h : Y' \to Y$ is any morphism of schemes, and $P' = P \times_Y Y'$. Then

(R5) $$\operatorname{Res}_{P'/Y'}\begin{bmatrix} h^*\omega \\ h^*t_1, \ldots, h^*t_d \end{bmatrix} = h^* \operatorname{Res}_{P/Y}\begin{bmatrix} \omega \\ t_1, \ldots, t_d \end{bmatrix}.$$

- For any $\varphi \in \Gamma(P, \mathcal{O}_P)$,

(R6) $$\operatorname{Res}_{P/Y}\begin{bmatrix} \varphi \cdot dt_1 \ldots dt_d \\ t_1, \ldots, t_d \end{bmatrix} = \operatorname{Tr}_{X/Y}(\varphi|_X).$$

  Here the notation $\operatorname{Tr}_{X/Y}$ denotes the classical trace map associated to the finite locally free ring extension $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$.
- For $\eta \in \Gamma(P, \Omega^{n-1}_{P/Y})$, and $k_1, \ldots, k_d$ positive integers,

(R9) $$\operatorname{Res}_{P/Y}\begin{bmatrix} d\eta \\ t_1^{k_1}, \ldots, t_d^{k_d} \end{bmatrix} = \sum_{i=1}^{n} k_i \cdot \operatorname{Res}_{P/Y}\begin{bmatrix} dt_i \wedge \eta \\ t_1^{k_1}, \ldots, t_i^{k_i+1}, \ldots, t_d^{k_d} \end{bmatrix}.$$

Let k be a perfect field of positive characteristic p. Let $X = \mathbf{A}^d_k$, and denote by $\widetilde{X} := \operatorname{Spec} W_n(k)[X_1, \ldots, X_d]$ the canonical smooth lift of X over $W_n(k)$. To make the module structures in the following discussion explicit, we distinguish the source and the target of the absolute Frobenius of $\operatorname{Spec} k$ and write it as

$$F_k : \operatorname{Spec} k_1 \to \operatorname{Spec} k_2.$$

Similarly, write the absolute Frobenius on X as

$$F_X : X = \operatorname{Spec} k_1[X_1, \ldots, X_d] \to Y = \operatorname{Spec} k_2[Y_1, \ldots, Y_d].$$

There is a canonical lift of F_X over $\widetilde{X}$

$$\widetilde{F}_{\widetilde{X}} : \widetilde{X} = \operatorname{Spec} W_n(k_1)[X_1, \ldots, X_d] \to \widetilde{Y} := \operatorname{Spec} W_n(k_2)[Y_1, \ldots, Y_d],$$

given by

$$\begin{aligned}\widetilde{F}^*_{\widetilde{X}}: \quad \Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}}) = W_n(k_2)[Y_1, \dots, Y_d] &\to W_n(k_1)[X_1, \dots, X_d] = \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}),\\ W_n k_2 \ni \alpha &\mapsto W_n(F_k)(\alpha),\\ Y_i &\mapsto X_i^p.\end{aligned}$$

Let

$$\pi_X : X \to \operatorname{Spec} k_1, \quad \pi_Y : Y \to \operatorname{Spec} k_2, \quad \pi_{\widetilde{X}} : \widetilde{X} \to W_n k_1, \quad \pi_{\widetilde{Y}} : \widetilde{Y} \to W_n k_2$$

be the structure maps. The composition $\pi_{\widetilde{Y}} \circ \widetilde{F}_{\widetilde{X}} : \widetilde{X} \to \operatorname{Spec} W_n k_2$ gives $\widetilde{X}$ a $W_n k_2$-scheme structure, and the map $\widetilde{F}_{\widetilde{X}}$ is then a map of $W_n k_2$-schemes. Therefore the trace map

$$\operatorname{Tr}_{\widetilde{F}_{\widetilde{X}}} : \widetilde{F}_{\widetilde{X},*} \widetilde{F}^{\triangle}_{\widetilde{X}} K_{\widetilde{Y}} \to K_{\widetilde{Y}}$$

makes sense. Consider the following map of complexes

$$\begin{aligned}\widetilde{F}_{\widetilde{X},*} K_{\widetilde{X}} \simeq \widetilde{F}_{\widetilde{X},*} \pi^{\triangle}_{\widetilde{X}} W_n k_1 \xrightarrow[\sim]{\widetilde{F}_{\widetilde{X},*} \pi^{\triangle}_{\widetilde{X}}(1.2.1)} \widetilde{F}_{\widetilde{X},*} \pi^{\triangle}_{\widetilde{X}} W_n F_k^{\triangle} W_n k_2 \simeq \\ \widetilde{F}_{\widetilde{X},*} \widetilde{F}^{\triangle}_{\widetilde{X}} \pi^{\triangle}_{\widetilde{Y}} W_n k_2 \simeq \widetilde{F}_{\widetilde{X},*} \widetilde{F}^{\triangle}_{\widetilde{X}} K_{\widetilde{Y}} \xrightarrow{\operatorname{Tr}_{\widetilde{F}_{\widetilde{X}}}} K_{\widetilde{Y}}.\end{aligned}$$

Taking the $(-d)$-th cohomology, it induces a map

$$\widetilde{F}_{X,*} \Omega^d_{\widetilde{X}/W_n k_1} \to \Omega^d_{\widetilde{Y}/W_n k_2}. \tag{1.3.9}$$

Lemma 1.11. *The notations are the same as above. The map* (1.3.9) *has the following expression:*

$$\begin{aligned}\Omega^d_{\widetilde{X}/W_n k_1} &\xrightarrow{(1.3.9)} \Omega^d_{\widetilde{Y}/W_n k_2},\\ \alpha \boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}} d\boldsymbol{X} &\mapsto \begin{cases} (W_n F_k)^{-1}(\alpha) \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{Y}, & \text{if } \lambda_i = p-1 \text{ for all } i;\\ 0, & \text{if } \lambda_i \neq p-1 \text{ for some } i.\end{cases}\end{aligned} \tag{1.3.10}$$

Proof. Consider the closed immersion $i : \widetilde{X} \hookrightarrow \widetilde{P} = \mathbf{A}^d_{\widetilde{Y}}$ associated to the following homomorphism of rings:

$$\begin{aligned}\Gamma(\widetilde{P}, \mathcal{O}_{\widetilde{P}}) = W_n(k_2)[Y_1, \dots, Y_d, T_1, \dots, T_d] &\to W_n(k_1)[X_1, \dots, X_d] = \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}),\\ \alpha &\mapsto (W_n F_k)(\alpha), \quad \alpha \in W_n(k_2),\\ Y_i &\mapsto X_i^p, \quad i = 1, \dots, d,\\ T_i &\mapsto X_i, \quad i = 1, \dots, d.\end{aligned}$$

Its kernel is

$$I = (T_1^p - Y_1, \dots, T_d^p - Y_d).$$

Denote

$$t_i = T_i^p - Y_i, \quad i = 1, \dots, d.$$

Obviously the t_i's form a regular sequence in $\Gamma(\widetilde{P}, \mathcal{O}_{\widetilde{P}})$, and hence i is a regular immersion. Then one has a factorization of $\widetilde{F}_{\widetilde{X}}$:

(1.3.11)

$$\begin{tikzcd} \widetilde{X} = \operatorname{Spec} W_n(k_1)[X_1, \dots, X_d] \arrow[r, hook, "i"] \arrow[dr, "\widetilde{F}_{\widetilde{X}}"'] & \widetilde{P} = \operatorname{Spec} W_n(k_2)[Y_1, \dots, Y_d, T_1, \dots, T_d] \arrow[d, "\pi"] \\ & \widetilde{Y} = \operatorname{Spec} W_n(k_2)[Y_1, \dots, Y_d]. \end{tikzcd}$$

Regarding $\widetilde{X}$ as a $W_n k_2$-scheme via the composite map $\widetilde{F}_{\widetilde{X}} \circ \pi_{\widetilde{Y}}$, the diagram (1.3.11) is then a diagram in the category of $W_n k_2$-schemes.

A general element in $\Gamma(\widetilde{X}, \Omega^d_{\widetilde{X}/W_n k_1})$ is a sum of expressions of the form

$$\alpha \boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}} d\boldsymbol{X}, \quad \alpha \in W_n k_1, \boldsymbol{\lambda} \in [0, p-1]^d, \boldsymbol{\mu} \in \mathbb{N}^d. \tag{1.3.12}$$

Here $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_d\}$, $\boldsymbol{\mu} = \{\mu_1, \dots, \mu_d\}$ are multi-indices, and $\boldsymbol{X}^{\boldsymbol{\lambda}} := X_1^{\lambda_1} \dots X_d^{\lambda_d}$ (similar for $\boldsymbol{Y}^{\boldsymbol{\mu}}$, $\boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}}$, etc.), $d\boldsymbol{X} := dX_1 \dots dX_d$ (similar for $d\boldsymbol{T}$, etc.). The element (1.3.12) in $\Gamma(\widetilde{X}, \Omega^d_{\widetilde{X}/W_n k_1})$ corresponds to

$$(W_n F_k)^{-1}(\alpha) \boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}} d\boldsymbol{X}, \quad \alpha \in W_n k_2, \boldsymbol{\lambda} \in [0, p-1]^d, \boldsymbol{\mu} \in \mathbb{N}^d, \tag{1.3.13}$$

in $\Gamma(\widetilde{X}, \Omega^d_{\widetilde{X}/W_n k_2})$ under $(-d)$-th cohomology of the map $\widetilde{F}_{\widetilde{X},*}\pi^{\triangle}_{\widetilde{X}}$(1.2.1), and

$$(W_n F_k)^{-1}(\alpha) \boldsymbol{T}^{\boldsymbol{\lambda}} \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{T}, \quad \alpha \in W_n k_2, \boldsymbol{\lambda} \in [0, p-1]^d, \boldsymbol{\mu} \in \mathbb{N}^d$$

is a lift of (1.3.13) to $\Gamma(\widetilde{P}, \Omega^d_{\widetilde{P}/W_n k_2})$. Write

$$\begin{aligned} \beta :&= dt_d \wedge \dots \wedge dt_1 \wedge (W_n F_k)^{-1}(\alpha) \boldsymbol{T}^{\boldsymbol{\lambda}} \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{T} \\ &= (-1)^d dY_d \wedge \dots \wedge dY_1 \wedge (W_n F_k)^{-1}(\alpha) \boldsymbol{T}^{\boldsymbol{\lambda}} \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{T} \end{aligned}$$

in $\Gamma(\widetilde{P}, \omega_{\widetilde{P}/W_n k_2})$, where $\omega_{\widetilde{P}/W_n k_2}$ denotes the dualizing sheaf with respect to the smooth morphism $\widetilde{P} \to W_n k_2$. Recall that there is a natural isomorphism [9, p.30 (a)]

$$\omega_{\widetilde{P}/W_n k_2} \simeq \omega_{\widetilde{P}/\widetilde{Y}} \otimes_{\mathcal{O}_{\widetilde{P}}} \pi^* \omega_{\widetilde{Y}/W_n k_2},$$

where $\omega_{\widetilde{P}/\widetilde{Y}}$ and $\omega_{\widetilde{Y}/W_n k_2}$ denote the dualizing sheaves with respect to the smooth morphisms $\pi : \widetilde{P} \to \widetilde{Y}$ and $\widetilde{Y} \to W_n k_2$. This isomorphism maps β to

$$(-1)^{\frac{d(3d+1)}{2}} (W_n F_k)^{-1}(\alpha) \boldsymbol{T}^{\boldsymbol{\lambda}} d\boldsymbol{T} \otimes \pi^* \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{Y}.$$

It is easily seen that $\widetilde{F}_{\widetilde{X}}$ is a finite flat morphism between smooth $W_n k_2$-schemes. Applying [6, Lemma A.3.3], one has

$$\operatorname{Tr}_{\widetilde{F}_{\widetilde{X}}}((W_n F_k)^{-1}(\alpha) \boldsymbol{X}^{\boldsymbol{\lambda}+p\boldsymbol{\mu}} d\boldsymbol{X}) = (W_n F_k)^{-1}(\alpha) \operatorname{Res}_{\widetilde{P}/\widetilde{Y}} \begin{bmatrix} \boldsymbol{T}^{\boldsymbol{\lambda}} d\boldsymbol{T} \\ t_1, \dots, t_d \end{bmatrix} \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{Y},$$

where $\operatorname{Res}_{\widetilde{P}/\widetilde{Y}} \begin{bmatrix} \boldsymbol{T}^{\boldsymbol{\lambda}} d\boldsymbol{T} \\ t_1, \dots, t_d \end{bmatrix} \in \Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}})$ is the residue symbol defined in [9, (A.1.4)], and $\operatorname{Tr}_{\widetilde{F}_{\widetilde{X}}}$ is the trace map on top differentials of the $W_n k_2$-morphism $\widetilde{F}_{\widetilde{X}}$ [9, (2.7.36)].

We consider the following cases:

- If $(\lambda_1,\dots,\lambda_d)\neq(p-1,\dots,p-1)$, $\boldsymbol{T}^{\boldsymbol{\lambda}}d\boldsymbol{T}=d\eta$ for some $\eta\in\Omega^{d-1}_{\widetilde{P}/\widetilde{Y}}$. Suppose without loss of generality that $\lambda_1\neq p-1$. Then we can take

$$\eta=\frac{1}{\lambda_1+1}T_1^{\lambda_1+1}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_2\dots dT_d.$$

Noticing that

$$dt_i=d(T_i^p-Y_i)=pT_i^{p-1}dT_i$$

in $\Omega^1_{\widetilde{P}/\widetilde{Y}}$, and that λ_1+mp+1 ($m\in\mathbb{Z}_{>0}$) is not divisible by p if λ_1+1 is so. Now we calculate

$$\begin{aligned}
&\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix}\boldsymbol{T}^{\boldsymbol{\lambda}}d\boldsymbol{T}\\ t_1,\dots,t_d\end{bmatrix}\\
&=\frac{1}{\lambda_1+1}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix}d(T_1^{\lambda_1+1}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_2\dots dT_d)\\ t_1,t_2,\dots,t_d\end{bmatrix}\\
&=\frac{p}{\lambda_1+1}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix}T_1^{\lambda_1+p}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_1dT_2\dots dT_d\\ t_1^2,t_2,\dots,t_d\end{bmatrix}\\
&=\frac{p}{(\lambda_1+1)(\lambda_1+p+1)}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix}d(T_1^{\lambda_1+p+1}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_2\dots dT_d)\\ t_1^2,t_2,\dots,t_d\end{bmatrix}\\
&=\frac{2p^2}{(\lambda_1+1)(\lambda_1+p+1)}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix}T_1^{\lambda_1+2p}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_1dT_2\dots dT_d\\ t_1^3,t_2,\dots,t_d\end{bmatrix}\\
&=\frac{2p^2}{\prod_{i=0}^2(\lambda_1+ip+1)}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix}d(T_1^{\lambda_1+2p+1}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_2\dots dT_d)\\ t_1^3,t_2,\dots,t_d\end{bmatrix}\\
&=\frac{6p^3}{\prod_{i=0}^2(\lambda_1+ip+1)}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix}T_1^{\lambda_1+3p}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_1dT_2\dots dT_d\\ t_1^4,t_2,\dots,t_d\end{bmatrix}\\
&=\dots\\
&=\frac{(\prod_{i=1}^n i)\cdot p^n}{\prod_{i=0}^{n-1}(\lambda_1+ip+1)}\operatorname{Res}_{\widetilde{P}/\widetilde{Y}}\begin{bmatrix}T_1^{\lambda_1+np}T_2^{\lambda_2}\dots T_d^{\lambda_d}dT_1dT_2\dots dT_d\\ t_1^{n+1},t_2,\dots,t_d\end{bmatrix}\\
&=0.
\end{aligned}$$

We have used (R9) on the second, the fourth, the sixth, and the eighth equality signs. The last equality is because $p^n=0$ in $\Gamma(\widetilde{Y},\mathcal{O}_{\widetilde{Y}})$.
- If $(\lambda_1,\dots,\lambda_d)=(p-1,\dots,p-1)$, consider

$$\begin{tikzcd}
X':=\operatorname{Spec}\frac{\mathbb{Z}[Y_1',\dots,Y_d',T_1',\dots,T_d']}{(T_1'^p-Y_1',\dots,T_d'^p-Y_d')} \arrow[r, hook] \arrow[dr, "f"'] & \operatorname{Spec}\mathbb{Z}[Y_1',\dots,Y_d',T_1',\dots,T_d']=:P' \arrow[d] \\
& \operatorname{Spec}\mathbb{Z}[Y_1',\dots,Y_d']=:Y'.
\end{tikzcd} \tag{1.3.14}$$

The map f is given by $f^*(Y_i')=Y_i'=T_i'^p$ in $\Gamma(X',\mathcal{O}_{X'})$. This is a finite locally free morphism of rank p^d. Consider the map $h:\widetilde{Y}\to Y'$ given by

$$\Gamma(Y',\mathcal{O}_{Y'})=\mathbb{Z}[Y_1',\dots,Y_d']\to W_n(k_2)[Y_1,\dots,Y_d]=\Gamma(\widetilde{Y},\mathcal{O}_{\widetilde{Y}}),$$
$$Y_i'\mapsto Y_i\text{ for all }i$$

that relates the two diagrams (1.3.14) and (1.3.11). In $\Gamma(Y', \mathcal{O}_{Y'})$, we have

$$
\begin{aligned}
p^d \cdot \mathrm{Res}_{P'/Y'} \begin{bmatrix} T_1'^{p-1} \dots T_d'^{p-1} dT_1' \dots dT_d' \\ T_1'^p - Y_1', \dots, T_d'^p - Y_d' \end{bmatrix} &= \mathrm{Res}_{P'/Y'} \begin{bmatrix} d(T_1'^p - Y_1') \dots d(T_d'^p - Y_d') \\ T_1'^p - Y_1', \dots, T_d'^p - Y_d' \end{bmatrix} \\
&\overset{\text{(R6)}}{=} \mathrm{Tr}_{X'/Y'}(1) \\
&= p^d.
\end{aligned}
$$

The notation $\mathrm{Tr}_{X'/Y'}$ denotes the classical trace map associated to the finite locally free ring extension $\Gamma(Y', \mathcal{O}_{Y'}) \to \Gamma(X', \mathcal{O}_{X'})$. As for the last equality, $\mathrm{Tr}_{X'/Y'}(1) = p^d$ because f is a finite locally free map of rank p^d. Since p^d is a nonzerodivisor in $\Gamma(Y', \mathcal{O}_{Y'})$, one deduces

$$
\mathrm{Res}_{P'/Y'} \begin{bmatrix} T_1'^{p-1} \dots T_d'^{p-1} dT_1' \dots dT_d' \\ T_1'^p - Y_1', \dots, T_d'^p - Y_d' \end{bmatrix} = 1.
$$

Set

$$
\boldsymbol{T}^{\boldsymbol{p-1}} = T_1^{p-1} \dots T_d^{p-1},
$$

which is the canonical lift of $\boldsymbol{X}^{\boldsymbol{\lambda}}$ via the map $i : \widetilde{X} \hookrightarrow \widetilde{P}$ in our current case. Pulling back to $\Gamma(\widetilde{Y}, \mathcal{O}_{\widetilde{Y}})$ via h, one has

$$
\mathrm{Res}_{\widetilde{P}/\widetilde{Y}} \begin{bmatrix} \boldsymbol{T}^{\boldsymbol{p-1}} d\boldsymbol{T} \\ t_1, \dots, t_d \end{bmatrix} \overset{\text{(R5)}}{=} h^* \, \mathrm{Res}_{P'/Y'} \begin{bmatrix} T_1'^{p-1} \dots T_d'^{p-1} dT_1' \dots dT_d' \\ T_1'^p - Y_1', \dots, T_d'^p - Y_d' \end{bmatrix} = 1. \tag{1.3.15}
$$

Altogether, we know that the map (1.3.9) takes the following expression

$$
\begin{aligned}
\Omega^d_{\widetilde{X}/W_n k_1} &\to \Omega^d_{\widetilde{Y}/W_n k_2}, \\
\alpha \boldsymbol{X}^{\boldsymbol{\lambda} + p\boldsymbol{\mu}} d\boldsymbol{X} &\mapsto \begin{cases} (W_n F_k)^{-1}(\alpha) \boldsymbol{Y}^{\boldsymbol{\mu}} d\boldsymbol{Y}, & \text{if } \lambda_i = p-1 \text{ for all } i; \\ 0, & \text{if } \lambda_i \neq p-1 \text{ for some } i. \end{cases}
\end{aligned}
$$

□

1.3.3.2. *C' for top Witt differentials.* Now we turn to the W_n-version. The aim of this subsection is to calculate C' for top Witt differentials on $\mathbf{A}^d_k$ (Lemma 1.14).

Let $f : X \to Y$ be a finite morphism between smooth, separated and equidimensional k-schemes of dimension d. Same as before, we denote by $\pi_X : X \to k$ and $\pi_Y : Y \to k$ the respective structure maps. The complexes $K_{n,X} := (W_n \pi_X)^{\triangle} W_n k, K_{n,Y} = (W_n \pi_Y)^{\triangle} W_n k$ are residual complexes on X and Y. Then we define the trace map

$$
\mathrm{Tr}_{W_n f} : (W_n f)_* (W_n \Omega^d_X) \to W_n \Omega^d_Y \tag{1.3.16}
$$

to be the $(-d)$-th cohomology map of the composition

$$
\mathrm{Tr}_{W_n f} : (W_n f)_* K_{n,X} \simeq \mathcal{H}om_{W_n \mathcal{O}_Y}((W_n f)_* W_n \mathcal{O}_X, K_{n,Y}) \xrightarrow{\text{ev. at } 1} K_{n,Y} \tag{1.3.17}
$$

via Ekedahl's isomorphism $W_n \Omega^d_X \simeq \mathcal{H}^{-d}(K_{n,X})$ in Proposition 1.3.

It suffices to compute the trace map locally on Y. Thus by possibly shrinking Y we can assume that Y and (therefore also X) is affine. In this case, there exist smooth affine $W_n k$-schemes $\widetilde{X}$ and $\widetilde{Y}$ which lift X and Y. Denote the structure morphisms of $\widetilde{X}, \widetilde{Y}$ by $\pi_{\widetilde{X}}$ and $\pi_{\widetilde{Y}}$, respectively. Then there exists a finite $W_n k$-morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ lifting $f : X \to Y$ by the formal smoothness property of $\widetilde{Y}$.

Consider the map of abelian sheaves [10, I (2.3)]

$$
\begin{aligned}
\varrho_Y^* : \quad & W_n\mathcal{O}_Y \xrightarrow[\simeq]{\vartheta_Y} \mathcal{H}^0(\Omega^\bullet_{\widetilde{Y}/W_nk}) \hookrightarrow \mathcal{O}_{\widetilde{Y}}, \\
& \sum_{i=0}^{n-1} V^i([a_i]) \mapsto \widetilde{a}_0^{p^n} + p\widetilde{a}_1^{p^{n-1}} + \cdots + p^{n-1}\widetilde{a}_{n-1}^p,
\end{aligned} \tag{1.3.18}
$$

where $a_i \in \mathcal{O}_Y$, and $\widetilde{a}_i \in \mathcal{O}_{\widetilde{Y}}$ are arbitrary liftings of a_i. The map ϑ_Y appearing above is the $i = 0$ case of the canonical isomorphism defined in [25, III. 1.5]

$$\vartheta_Y : W_n\Omega^i_Y \xrightarrow{\simeq} \mathcal{H}^i(\Omega^\bullet_{\widetilde{Y}/W_nk}). \tag{1.3.19}$$

Note that the map $\varrho_Y^* : W_n\mathcal{O}_Y \to \mathcal{O}_{\widetilde{Y}}$ is a morphism of sheaves of rings, and it induces a finite morphism $\varrho_Y : W_nY \to \widetilde{Y}$ (cf. [10, I, paragraph after (2.4)]). Altogether we have the following commutative diagram of schemes (cf. [10, I. (2.4)])

$$
\begin{tikzcd}
 & \widetilde{X} \arrow[rr, "\varrho_X"] \arrow[dd, "\widetilde{f}"] & & W_nX \arrow[dd, "W_nf"] \\
X \arrow[ur, hook, "\widetilde{i}_X"] \arrow[dd, "f"] & & X \arrow[ur, hook, "i_X"] \arrow[dd, "f"] & \\
 & \widetilde{Y} \arrow[rr, "\varrho_Y"] \arrow[dd, "\pi_{\widetilde{Y}}"] & & W_nY \arrow[dd, "W_n\pi_Y"] \\
Y \arrow[ur, hook, "\widetilde{i}_Y"] \arrow[dd, "\pi_Y"] & & Y \arrow[ur, hook, "i_Y"] \arrow[dd, "\pi_Y"] & \\
 & W_nk \arrow[rr, "\simeq", "W_n(F_k^n)"'] & & W_nk. \\
k \arrow[ur, hook] \arrow[rr, "\simeq", "F_k^n"'] & & k \arrow[ur, hook] &
\end{tikzcd}
$$

Lemma 1.12. *Set $K_{\widetilde{X}} = \pi^{\triangle}_{\widetilde{X}} W_nk$, and $K_{\widetilde{Y}} = \pi^{\triangle}_{\widetilde{Y}} W_nk$. The $(-d)$-th cohomology of the map $\mathrm{Tr}_{\widetilde{f}} : \widetilde{f}_* K_{\widetilde{X}} \to K_{\widetilde{Y}}$ gives a map $\widetilde{f}_*\Omega^d_{\widetilde{X}/W_nk} \to \Omega^d_{\widetilde{Y}/W_nk}$, which we again denote by $\mathrm{Tr}_{\widetilde{f}}$. Then by passing to quotients, this map $\mathrm{Tr}_{\widetilde{f}}$ induces a well-defined map*

$$\tau_{\widetilde{f}} : \mathcal{H}^d(\widetilde{f}_*\Omega^\bullet_{\widetilde{X}/W_nk}) \to \mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}/W_nk}).$$

Moreover, the map $\tau_{\widetilde{f}}$ is compatible with Tr_{W_nf} defined in (1.3.17), *i.e., the following diagram commutes:*

$$
\begin{tikzcd}
(W_nf)_*W_n\Omega^d_X \arrow[r, "\mathrm{Tr}_{W_nf}"] \arrow[d, "(W_nf)_*\vartheta_X"', "\simeq"] & W_n\Omega^d_Y \arrow[d, "\vartheta_Y"', "\simeq"] \\
(\varrho_Y\widetilde{f})_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{X}/W_nk}) \arrow[r, "(\varrho_Y)_*\tau_{\widetilde{f}}"] & (\varrho_Y)_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}/W_nk})
\end{tikzcd}
$$

Proof. We argue the other way around, namely we define the map $\tau_{\widetilde{f}}\colon \mathcal{H}^d(\widetilde{f}_*\Omega^\bullet_{\widetilde{X}/W_nk}) \to \mathcal{H}^d(\widetilde{\Omega}^\bullet_{\widetilde{Y}/W_nk})$ via $\mathrm{Tr}_{W_nf} : (W_nf)_*W_n\Omega^d_X \to W_n\Omega^d_Y$, and then show that this is the reduction of $\mathrm{Tr}_{\widetilde{f}} : \widetilde{f}_*\Omega^d_{\widetilde{X}/W_nk} \to \Omega^d_{\widetilde{Y}/W_nk}$.

First of all, via isomorphisms ϑ_X, ϑ_Y, the map $\mathrm{Tr}_{W_nf} : (W_nf)_*W_n\Omega^d_X \to W_n\Omega^d_Y$ defined in (1.3.17) induces a well-defined map $\tau_{\widetilde{f}} : \mathcal{H}^d(\widetilde{f}_*\Omega^\bullet_{\widetilde{X}/W_nk}) \to \mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}/W_nk})$. To show the compatibility with $\mathrm{Tr}_{\widetilde{f}}$, one needs the observation of Ekedahl that the composite map

$$
\begin{aligned}
t_Y : (\varrho_Y)_*\Omega^d_{\widetilde{Y}/W_nk}[d] \xrightarrow{\simeq} (\varrho_Y)_*K_{\widetilde{Y}} \simeq (\varrho_Y)_*\pi^\triangle_{\widetilde{Y}}W_nk \xrightarrow[\simeq]{(\varrho_Y)_*\pi^\triangle_{\widetilde{Y}}(1.2.1)^n} \\
(\varrho_Y)_*\pi^\triangle_{\widetilde{Y}}(W_nF^n_k)^\triangle W_nk \simeq (\varrho_Y)_*(\varrho_Y)^\triangle(W_n\pi_Y)^\triangle W_nk \xrightarrow{\mathrm{Tr}_{\varrho_Y}} K_{n,Y}
\end{aligned}
$$

factors through $\bar{t}_Y : (\varrho_Y)_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}/W_nk})[d] \to K_{n,Y}$ (cf. [10, §1 (2.6)]). Then he defined the map $W_n\Omega^d_Y[d] \to K_{n,Y}$ to be the composite

$$
s_Y : W_n\Omega^d_Y[d] \xrightarrow[\simeq]{\vartheta_Y} \mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}/W_nk})[d] \xrightarrow{\bar{t}_Y} K_{n,Y}. \tag{1.3.20}
$$

Now consider the following diagram of complexes of sheaves

$$
\begin{tikzcd}
 & (W_nf)_*K_{n,X} \arrow[rr, "\mathrm{Tr}_{W_nf}"] & & K_{n,Y} \\
(W_nf)_*W_n\Omega^d_X[d] \arrow[ur, "(W_nf)_*s_X"] \arrow[rr, "\mathrm{Tr}_{W_nf}"] \arrow[dd, "(W_nf)_*\vartheta_X"', "\simeq"] & & W_n\Omega^d_Y[d] \arrow[ur, "s_Y"] \arrow[dd, "\vartheta_Y"', "\simeq"] & \\
 & (\varrho_Y\widetilde{f})_*\Omega^d_{\widetilde{X}/W_nk}[d] \arrow[uu, "(W_nf)_*t_X"'] \arrow[rr, "(\varrho_Y)_*\,\mathrm{Tr}_{\widetilde{f}}"] \arrow[dl] & & (\varrho_Y)_*\Omega^d_{\widetilde{Y}/W_nk}[d]. \arrow[uu, "t_Y"'] \arrow[dl] \\
(\varrho_Y\widetilde{f})_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{X}/W_nk})[d] \arrow[uuur, "(W_nf)_*\bar{t}_X"] \arrow[rr, "(\varrho_Y)_*\tau_{\widetilde{f}}"] & & (\varrho_Y)_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}/W_nk})[d] \arrow[uuur, "\bar{t}_Y"'] &
\end{tikzcd}
$$

The unlabeled arrows are given by the natural quotient maps. The front commutes by the definition of $\tau_{\widetilde{f}}$. The top commutes by the definition of $\mathrm{Tr}_{W_nf} : (W_nf)_*W_n\Omega^d_X \to W_n\Omega^d_Y$. The triangles in the right (resp. the left) side commute due to the definition of $\bar{t}_Y$ and s_Y (resp. $\bar{t}_X$ and s_X). The back square commutes, because the trace map $\mathrm{Tr}_{\widetilde{f}}$ is functorial with respect to maps of residual complexes with the same associated filtration by Proposition 1.2(3). We want to show that the bottom square commutes. To this end, it suffices to show $(\varrho_Y)_*\,\mathrm{Tr}_{\widetilde{f}} : (\varrho_Y\widetilde{f})_*\Omega^d_{\widetilde{X}/W_nk} \to (\varrho_Y)_*\Omega^d_{\widetilde{Y}/W_nk}$ is compatible with $\mathrm{Tr}_{W_nf} : (W_nf)_*W_n\Omega^d_X \to W_n\Omega^d_Y$ via ϑ_X and ϑ_Y. Because the map $\mathrm{Tr}_{W_nf} : (W_nf)_*W_n\Omega^d_X \to W_n\Omega^d_Y$ is determined by the degree $-d$ part of the map $\mathrm{Tr}_{W_nf} : (W_nf)_*K_{n,X} \to K_{n,Y}$, we are reduced to showing compatibility of $(\varrho_Y)_*\,\mathrm{Tr}_{\widetilde{f}} : (\varrho_Y\widetilde{f})_*\Omega^d_{\widetilde{X}/W_nk} \to (\varrho_Y)_*\Omega^d_{\widetilde{Y}/W_nk}$ with $\mathrm{Tr}_{W_nf} : (W_nf)_*K_{n,X} \to K_{n,Y}$ via $(W_nf)_*(s_X \circ \vartheta_X^{-1})$ and $s_Y \circ \vartheta_Y^{-1}$. By the commutativity of the left and right squares, this is reduced to the commutativity of the square on the back, which is known. Therefore the bottom square commutes as a result. □

The notation $\tau_{\widetilde{f}}$ is only temporarily used in Lemma 1.12. Later we will denote $\tau_{\widetilde{f}}$ by $\mathrm{Tr}_{\widetilde{f}}$.

Lemma 1.13 ([1, 8.4(ii)])**.** *Let $W_n\Omega^\bullet_{\widetilde{Y}/W_nk}$ denote the relative de Rham-Witt complex defined by* [33]*. The rest of the notations are the same as above. There is a commutative diagram*

$$\begin{tikzcd} W_{n+1}\Omega^q_{\widetilde{Y}/W_nk} \arrow[r, "F^n"] \arrow[d] & Z^q(\Omega^\bullet_{\widetilde{Y}/W_nk}) \arrow[d] \\ W_n\Omega^q_Y \arrow[r, "\vartheta_Y"] & \mathcal{H}^q(\Omega^\bullet_{\widetilde{Y}/W_nk}). \end{tikzcd}$$

Recall that any element in $W_n\Omega^d_{k[X_1,\dots,X_d]}$ is uniquely written as a sum of (1.3.7) and (1.3.8).

Lemma 1.14. *Let*

$$C' = C'_n : W_n\Omega^d_{k[X_1,\dots,X_d]} \to W_n\Omega^d_{k[X_1,\dots,X_d]}$$

be the map given by the $-d$-th cohomology of the level n Cartier operator for residual complexes (cf. (1.2.3)*). Let $\alpha = \sum_{j=0}^{n+v_1-1} V^j[\alpha_j] \in W_{n+v_1}k$ with each $\alpha_j \in k$. Let $\beta = \sum_{j=0}^{n-1} V^j[\beta_j] \in W_nk$ with each $\beta_j \in k$.*

(1) *If $v_1 = 1-n$,*

$$C'\Big(dV^{-v_1}(\alpha[X_1]^{h'_1})\cdots dV^{-v_r}([X_r]^{h'_r})\cdot F^{v_{r+1}}d[X_{r+1}]^{h'_{r+1}}\cdots F^{v_d}d[X_d]^{h'_d}\Big) = 0.$$

(2) *If $1-n < v_1 < 0$, $v_{r+1} = \cdots = v_{r+s} = 0$ (s can be zero),*

$$\begin{aligned} &C'\Big(dV^{-v_1}(\alpha[X_1]^{h'_1})\cdots dV^{-v_r}([X_r]^{h'_r})\cdot F^{v_{r+1}}d[X_{r+1}]^{h'_{r+1}}\cdots F^{v_d}d[X_d]^{h'_d}\Big)\\ &\quad= dV^{1-v_1}(R(\widetilde{\alpha})[X_1]^{h'_1})\cdots dV^{1-v_r}[X_r]^{h'_r}\cdot dV[X_{r+1}]^{h'_{r+1}}\cdots dV[X_{r+s}]^{h'_{r+s}}\\ &\qquad\cdot F^{v_{r+s+1}-1}d[X_{r+s+1}]^{h'_{r+s+1}}\cdots F^{v_d-1}d[X_d]^{h'_d}. \end{aligned}$$

Here

$$\widetilde{\alpha} = \sum_{j=0}^{n+v_1-1} V^j[\widetilde{\alpha}_j]_{n+v_1+1-j} \in W_{n+v_1+1}(W_nk),$$

where $\widetilde{\alpha}_j := [\alpha_j]_n \in W_nk$ is the Teichmüller lift of $\alpha_j \in k$.

(3) *If $v_1 \geq 0$, $v_1 = \cdots = v_s = 0$ (s can be zero),*

$$\begin{aligned} &C'\Big(\beta F^{v_1}d[X_1]^{h'_1}\cdots F^{v_d}d[X_d]^{h'_d}\Big)\\ &= (W_nF_k)^{-1}(\beta)\cdot dV[X_1]^{h'_1}\cdots dV[X_s]^{h'_s}\cdot F^{v_{s+1}-1}d[X_{s+1}]^{h'_{s+1}}\cdots F^{v_d-1}d[X_d]^{h'_d}. \end{aligned}$$

Here

$$\widetilde{\beta} = \sum_{j=0}^{n-1} V^j[\widetilde{\beta}_j]_{n+1-j} \in W_{n+1}(W_nk),$$

where $\widetilde{\beta}_j := [\beta_j]_n \in W_nk$ is the Teichmüller lifts of $\beta_j \in k$.

Proof. Consider the map $W_nF_X : W_nX \to W_nX$ with $X := \mathbf{A}^d_k$. It is not a map of W_nk-schemes a priori, but after labeling the source by $W_nX := W_n\mathbf{A}^d_{k_1}$ and the target by $W_nY := W_n\mathbf{A}^d_{k_2}$, one can realize W_nF_X as a map of W_nk_2-schemes (the W_nk_2-scheme structure of W_nX is given by $W_nF_X \circ W_n\pi_Y$, where $\pi_Y : Y \to k_2$ denotes the structure morphism of the scheme Y). Write

$$\widetilde{X} = \mathbf{A}^d_{W_nk_1} = \operatorname{Spec} W_nk_1[X_1, \dots, X_d] \quad (\text{resp. } \widetilde{Y} = \mathbf{A}^d_{W_nk_2} = \operatorname{Spec} W_nk_2[X_1, \dots, X_d]),$$

and take the canonical lift $\widetilde{F}_{\widetilde{X}}$ of F_X as in Lemma 1.11. Consider
(1.3.21)

$$\begin{array}{ccccc}
(W_nF_X)_*W_n\Omega^d_{X/k_1} & \xrightarrow[\simeq]{(W_nF_X)_*(1.2.2)} & (W_nF_X)_*W_n\Omega^d_{X/k_2} & \xrightarrow{\mathrm{Tr}_{W_nF_X}} & W_n\Omega^d_{Y/k_2} \\
\downarrow{\scriptstyle (W_nF_X)_*\vartheta_X} & & \downarrow{\scriptstyle (W_nF_X)_*\vartheta_X} & & \downarrow{\scriptstyle \vartheta_Y} \\
(\varrho_Y\widetilde{F}_{\widetilde{X}})_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{X}/W_nk_1}) & \xrightarrow[\simeq]{(\varrho_Y\widetilde{F}_{\widetilde{X}})_*\pi^{\triangle}_{\widetilde{X}}(1.2.1)} & (\varrho_Y\widetilde{F}_{\widetilde{X}})_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{X}/W_nk_2}) & \xrightarrow{(\varrho_Y)_*(\mathrm{Tr}_{\widetilde{F}_{\widetilde{X}}})} & (\varrho_Y)_*\mathcal{H}^d(\Omega^\bullet_{\widetilde{Y}/W_nk_2}).
\end{array}$$

The composite map of the top row is C' (cf. (1.2.3) and Ekedahl's quasi-isomorphism Proposition 1.3). The composite of the bottom row is induced from $\varrho_{Y,*}(1.3.9)$. The right side commutes due to Lemma 1.12. The left side commutes by the naturality. Hence we can decompose C' in the following way:

$$C' = \vartheta_Y^{-1} \circ (1.3.9) \circ \vartheta_X : W_n\Omega^d_{X/k_1} \to W_n\Omega^d_{Y/k_2}.$$

Consider the first two cases. Suppose $v_1 < 0$ and suppose there are s many v_j's being zero,

$$v_1 \le \cdots \le v_r < 0 = \cdots = 0 < v_{r+s+1} \le \cdots \le v_d.$$

Note that in $W_{n+1}\Omega^1_{(W_nk)[X_1,\dots,X_d]/W_nk}$:

$$\begin{aligned}
(1.3.22)\qquad F^{n+v_1}d(\widetilde{\alpha}[X_1]^{h_1'}) &= F^{n+v_1}d\Big([X_1]^{h_1'} \cdot \sum_{j=0}^{n+v_1-1} V^j[\widetilde{\alpha}_j]\Big) \\
&= F^{n+v_1}d\Big(\sum_{j=0}^{n+v_1-1} V^j[\widetilde{\alpha}_j X_1^{h_1'p^j}]\Big) \\
&= \sum_{j=0}^{n+v_1-1} F^{n+v_1-j}d[\widetilde{\alpha}_j X_1^{h_1'p^j}] \\
&= \sum_{j=0}^{n+v_1-1} (\widetilde{\alpha}_j X_1^{h_1'p^j})^{p^{n+v_1-j}-1} d(\widetilde{\alpha}_j X_1^{h_1'p^j}) \\
&= \sum_{j=0}^{n+v_1-1} h_1'p^j \cdot \widetilde{\alpha}_j^{p^{n+v_1-j}} \cdot X_1^{h_1'p^{n+v_1}-1} dX_1.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
(1.3.23)\qquad \sum_{j=0}^{n+v_1-1} p^j h_1' \cdots h_d' \cdot (W_nF_k)^{-1}(\widetilde{\alpha}_j^{p^{n+v_1-j}}) X_1^{h_1'p^{n+v_1-1}-1} dX_1& \\
= F^n dV^{1-v_1}(R(\widetilde{\alpha})[X_1]^{h_1}).&
\end{aligned}$$

Here R is the restriction map $R: W_{n+v_1+1}(W_nk) \to W_{n+v_1}(W_nk)$. Now according to the formula (1.3.10) and Lemma 1.13, we carry out the following calculations.

(1.3.24)

$$\begin{aligned}
&C'\Big(dV^{-v_1}(\alpha[X_1]^{h'_1})\cdots dV^{-v_r}([X_r]^{h'_r})\cdot F^{v_{r+1}}d[X_{r+1}]^{h'_{r+1}}\cdots F^{v_d}d[X_d]^{h'_d}\Big)\\
&= \vartheta_Y^{-1}\circ(1.3.9)\circ F^n\Big(dV^{-v_1}(\widetilde{\alpha}[X_1]^{h'_1})\cdots dV^{-v_r}([X_r]^{h'_r})\\
&\quad \cdot F^{v_{r+1}}d[X_{r+1}]^{h'_{r+1}}\cdots F^{v_d}d[X_d]^{h'_d}\Big)\\
&= \vartheta_Y^{-1}\circ(1.3.9)\Big(F^{n+v_1}d(\widetilde{\alpha}[X_1]^{h'_1})\cdots F^{n+v_r}d([X_r]^{h'_r})\\
&\quad \cdot F^{n+v_{r+1}}d[X_{r+1}]^{h'_{r+1}}\cdots F^{n+v_d}d[X_d]^{h'_d}\Big)\\
&= \vartheta_Y^{-1}\circ(1.3.9)\Big(\Big(\sum_{j=0}^{n+v_1-1} h'_1p^j\cdot\widetilde{\alpha}_j^{p^{n+v_1-j}}\cdot X_1^{h'_1p^{n+v_1}-1}dX_1\Big)\cdot\Big(h'_2\cdot X_2^{h'_2p^{n+v_2}-1}dX_2\Big)\cdot\\
&\quad \cdots\Big(h'_d\cdot X_d^{h'_dp^{n+v_d}-1}dX_d\Big)\Big) \quad \text{(by (1.3.22))}\\
&= \vartheta_Y^{-1}\Big(\sum_{j=0}^{n+v_1-1} p^jh'_1\cdots h'_d\cdot(W_nF_k)^{-1}(\widetilde{\alpha}_j^{p^{n+v_1-j}})X_1^{h'_1p^{n+v_1-1}-1}dX_1\\
&\quad \cdots X_d^{h'_dp^{n+v_d-1}-1}dX_d\Big)\\
&= \vartheta_Y^{-1}\Big(F^ndV^{1-v_1}(R(\widetilde{\alpha})[X_1]^{h'_1})\\
&\quad \cdots F^ndV^{1-v_r}[X_r]^{h'_r}\cdot F^ndV[X_{r+1}]^{h'_{r+1}}\cdots F^ndV[X_{r+s}]^{h'_{r+s}}\\
&\quad \cdot F^{n+v_{r+s+1}-1}d[X_{r+s+1}]^{h'_{r+s+1}}\cdots F^{n+v_d-1}d[X_d]^{h'_d}\Big) \quad \text{(by (1.3.23))}.
\end{aligned}$$

If $v_1 = 1-n$, (1.3.24) $= 0$ because

$$F^ndV^{1-v_1}(R(\widetilde{\alpha})[X_1]^{h'_1}) = d(R(\widetilde{\alpha})[X_1]^{h'_1}) = 0$$

in $\mathcal{H}^1(\Omega^\bullet_{(W_nk)[X_1,\ldots,X_d]/W_nk})$. If $v_1 \neq 1-n$ (hence $v_j > 1-n$ for all j),

$$\begin{aligned}
(1.3.24) = {}& dV^{1-v_1}(R(\widetilde{\alpha})[X_1]^{h'_1})\cdots dV^{1-v_r}[X_r]^{h'_r}\\
&\cdot dV[X_{r+1}]^{h'_{r+1}}\cdots dV[X_{r+s}]^{h'_{r+s}}\\
&\cdot F^{v_{r+s+1}-1}d[X_{r+s+1}]^{h'_{r+s+1}}\cdots F^{v_d-1}d[X_d]^{h'_d}
\end{aligned}$$

and this is the same as what our lemma claims.

Now we check the third case. If all the $v_j \geq 0$, suppose the first s v_j's are zero,

$$0 = \cdots = 0 < v_{s+1} \leq \cdots \leq v_d.$$

Note that in W_nk,

(1.3.25)

$$
\begin{aligned}
&(W_nF_k)^{-1}(F^n(\widetilde{\beta})) = (W_nF_k)^{-1}(F^n(\sum_{j=0}^{n-1} V^j[\widetilde{\beta}_j]_{n+1-j}))\\
&= (W_nF_k)^{-1}(\sum_{j=0}^{n-1} F^{n-j}[\widetilde{\beta}_j]_{n+1-j})) = (W_nF_k)^{-1}(\sum_{j=0}^{n-1} \widetilde{\beta}_j^{p^{n-j}}) = \sum_{j=0}^{n-1} \widetilde{\beta}_j^{p^{n-j-1}}\\
&= F^n\Big(\sum_{j=0}^{n-1} V^j[(W_nF_k)^{-1}\widetilde{\beta}_j]_{n+1-j}\Big) = F^n\Big((W_{n+1}(W_nF_k))^{-1}(\beta)\Big).
\end{aligned}
$$

We carry out the computation

$$
\begin{aligned}
&C'\Big(\beta F^{v_1}d[X_1]^{h_1'}\cdots F^{v_d}d[X_d]^{h_d'}\Big)\\
&= \vartheta_Y^{-1}\circ(1.3.9)\circ F^n\Big(\widetilde{\beta}\cdot F^{v_1}d[X_1]^{h_1'}\cdots F^{v_d}d[X_d]^{h_d'}\Big)\\
&= \vartheta_Y^{-1}\circ(1.3.9)\Big(F^n(\widetilde{\beta})\cdot F^{n+v_1}d[X_1]^{h_1'}\cdots F^{n+v_d}d[X_d]^{h_d'}\Big)\\
&= \vartheta_Y^{-1}\circ(1.3.9)\Big(F^n(\widetilde{\beta})\cdot h_1'\cdots h_d'X_1^{h_1'p^{n+v_1}-1}dX_1\cdots F^{n+v_d}d[X_d]^{h_d'}\Big)\\
&= \vartheta_Y^{-1}\Big((W_nF_k)^{-1}(F^n(\widetilde{\beta}))\cdot h_1'X_1^{h_1'p^{n+v_1-1}-1}dX_1\cdots h_d'X_d^{h_d'p^{n+v_d-1}-1}dX_d\Big)\\
&= \vartheta_Y^{-1}\Big(F^n\Big((W_{n+1}(W_nF_k))^{-1}(\beta)\Big)\cdot F^ndV[X_1]^{h_1'}\cdots F^ndV[X_s]^{h_s'}\cdot\\
&\qquad F^{n+v_{s+1}-1}d[X_{s+1}]^{h_{s+1}'}\cdots F^{n+v_d-1}d[X_d]^{h_d'}\Big)\quad (by\ (1.3.25))\\
&= (W_nF_k)^{-1}(\beta)\cdot dV[X_1]^{h_1'}\cdots dV[X_s]^{h_s'}\cdot\\
&\qquad F^{v_{s+1}-1}d[X_{s+1}]^{h_{s+1}'}\cdots F^{v_d-1}d[X_d]^{h_d'}.
\end{aligned}
$$

In the last equality we have used that

$$\vartheta_Y^{-1}\Big(F^n\Big((W_{n+1}(W_nF_k))^{-1}(\beta)\Big)\Big) = (W_nF_k)^{-1}(\beta).$$

We hence proved the lemma. □

1.3.4. *Criterion for surjectivity of $C'-1$.* Proposition 1.15 is proven in the smooth case by Illusie-Raynaud-Suwa [39, 2.1]. The proof presented here is due to Rülling.

Proposition 1.15 (Raynaud-Illusie-Suwa)**.** *Let $k=\overline{k}$ be an algebraically closed field of characteristic $p>0$ and let X be a separated scheme of finite type over k. Then for every i, $C'-1$ induces a surjective map on global cohomology groups*

$$H^i(W_nX, K_{n,X}) := R^i\Gamma(W_nX, K_{n,X}) \xrightarrow{C'-1} H^i(W_nX, K_{n,X}).$$

In particular,

$$R^i\Gamma(W_nX, K_{n,X,log}) \simeq H^i(W_nX, K_{n,X})^{C'-1}.$$

Proof. Take a Nagata compactification of X, i.e., an open immersion

$$j: X\hookrightarrow \overline{X}$$

such that $\overline{X}$ is proper over k. The boundary $\overline{X}\setminus X$ is a closed subscheme in $\overline{X}$. By blowing up in $\overline{X}$ one can assume $\overline{X}\setminus X$ is the closed subscheme associated to an effective Cartier divisor D on $\overline{X}$. We can thus assume j is an affine morphism. Therefore

$$W_n j : W_n X \hookrightarrow W_n \overline{X}$$

is also an affine morphism.

For any quasi-coherent sheaf $\mathcal{M}$ on $W_n\overline{X}$, the difference between $\mathcal{M}$ and $(W_nj)_*(W_nj)^*\mathcal{M}$ is precisely those sections that have poles (of any order) at $\operatorname{Supp} D = W_n\overline{X} \setminus W_nX$. Suppose that the effective Cartier divisor D is represented by $(U_i, f_i)_i$, where $\{U_i\}_i$ is an affine cover of $\overline{X}$, and $f_i \in \Gamma(U_i, \mathcal{O}_X)$. Recall that $\mathcal{O}_{\overline{X}}(mD)$ denotes the line bundle on $\overline{X}$ which is the inverse (as line bundles) of the m-th power of the ideal sheaf of $\overline{X} \setminus X \hookrightarrow \overline{X}$. Locally, one has an isomorphism

$$\mathcal{O}_{\overline{X}}(mD) \mid_{U_i} \simeq \mathcal{O}_{U_i} \cdot \frac{1}{f_i^m}$$

for each i. Denote by $W_n\mathcal{O}_{\overline{X}}(mD)$ the line bundle on $W_n\overline{X}$ such that

$$W_n\mathcal{O}_{\overline{X}}(mD) \mid_{U_i} \simeq W_n\mathcal{O}_{U_i} \cdot \frac{1}{[f_i]^m},$$

where $[-] = [-]_n$ denotes the Teichmüller lift. Denote

$$\mathcal{M}(mD) := \mathcal{M} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD).$$

The natural map

(1.3.26)

$$\mathcal{M}(*D) := \operatorname{colim}_m \mathcal{M}(mD) \xrightarrow{\simeq} (W_nj)_*(W_nj)^*(\mathcal{M}(mD)) = (W_nj)_*(W_nj)^*\mathcal{M}$$

is an isomorphism of sheaves. Here the inductive system on the left hand side is given by the natural map

$$\mathcal{M}(mD) := \mathcal{M} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD) \to \mathcal{M} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}((m+1)D)$$

induced from the inclusion $W_n\mathcal{O}_{\overline{X}}(mD) \hookrightarrow W_n\mathcal{O}_{\overline{X}}((m+1)D)$, i.e., locally on U_i, this inclusion is the map

$$\begin{aligned} W_n\mathcal{O}_{\overline{X}}(mD) \mid_{U_i} &\hookrightarrow W_n\mathcal{O}_{\overline{X}}((m+1)D) \mid_{U_i}, \\ \frac{a}{[f_i]^m} &\mapsto \frac{a[f_i]}{[f_i]^{m+1}}. \end{aligned}$$

where $a \in W_n\mathcal{O}_{U_i}$. As a result,

(1.3.27)

$$\begin{aligned} H^i(W_nX, (W_nj)^*\mathcal{M}) &= H^i(R\Gamma(W_n\overline{X}, R(W_nj)_*(W_nj)^*\mathcal{M})) \\ &= H^i(R\Gamma(W_n\overline{X}, (W_nj)_*(W_nj)^*\mathcal{M})) && (W_nj \text{ is affine}) \\ &= H^i(R\Gamma(W_n\overline{X}, \operatorname{colim}_m \mathcal{M}(mD)) && (1.3.26) \\ &= \operatorname{colim}_m H^i(W_n\overline{X}, \mathcal{M}(mD)). \end{aligned}$$

Apply this to the bounded complex $K_{n,\overline{X}}$ of injective quasi-coherent $W_n\mathcal{O}_{\overline{X}}$-modules. Taking into account $K_{n,X} \simeq (W_nj)^*K_{n,\overline{X}}$ by Proposition 1.1(2), (1.3.26) gives an isomorphism of complexes

$$K_{n,\overline{X}}(*D) := \operatorname{colim}_m K_{n,\overline{X}}(mD) \xrightarrow{\simeq} (W_nj)_*K_{n,X}, \tag{1.3.28}$$

and (1.3.27) gives an isomorphism of W_nk-modules

$$\operatorname{colim}_m H^i(W_n\overline{X}, K_{n,\overline{X}}(mD)) = H^i(W_nX, K_{n,X}).$$

Via the projection formula [21, II.5.6] and tensoring

$$C' : (W_n F_X)_* K_{n,\overline{X}} \to K_{n,\overline{X}}$$

with $W_n\mathcal{O}_{\overline{X}}(mD)$, one gets a map

$$\begin{aligned}(W_nF_X)_*(K_{n,\overline{X}} &\otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(pmD)) \\ &\simeq (W_nF_X)_*(K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} (W_nF_X)^* W_n\mathcal{O}_{\overline{X}}(mD)) \\ &\simeq ((W_nF_X)_*K_{n,\overline{X}}) \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD) \\ &\xrightarrow{C'\otimes \mathrm{id}_{W_n\mathcal{O}_{\overline{X}}(mD)}} K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD).\end{aligned}$$

Precomposing with the natural map

$$(W_nF_X)_*(K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD)) \to (W_nF_X)_*(K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(pmD)),$$

and taking the global section cohomologies, one gets

$$C' : H^i(W_n\overline{X}, K_{n,\overline{X}}(mD)) \to H^i(W_n\overline{X}, K_{n,\overline{X}}(mD)).$$

To show the surjectivity of

$$C' - 1 : H^i(W_nX, K_{n,X}) \to H^i(W_nX, K_{n,X}),$$

it suffices to show the surjectivity for

$$C' - 1 : H^i(W_n\overline{X}, K_{n,\overline{X}}(mD)) \to H^i(W_n\overline{X}, K_{n,\overline{X}}(mD)).$$

Because $\mathcal{H}^q(K_{n,\overline{X}})$ are coherent sheaves on the proper scheme $\overline{X}$ for all q,

$$\mathcal{H}^q(K_{n,\overline{X}} \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD)) = \mathcal{H}^q(K_{n,\overline{X}}) \otimes_{W_n\mathcal{O}_{\overline{X}}} W_n\mathcal{O}_{\overline{X}}(mD)$$

are also coherent, therefore the local-to-global spectral sequence implies that

$$M := H^i(W_n\overline{X}, K_{n,\overline{X}}(mD))$$

is a finite W_nk-module. Now M is equipped with an endomorphism C' which acts p^{-1}-linearly (cf. Definition A.4). The proposition is then a direct consequence of Proposition A.6. □

Proposition 1.16 is a corollary of [39, Lemma 2.1]. We restate it here as a convenient reference.

Proposition 1.16 (Raynaud-Illusie-Suwa)**.** *Assume $k = \overline{k}$. If X is separated smooth over k of pure dimension d,*

$$C - 1 : W_n\Omega^d_X \to W_n\Omega^d_X$$

is surjective.

Proof. Apply affine locally the H^{-d}-case of Proposition 1.15. Then Ekedahl's quasi-isomorphism $W_n\Omega^d_X[d] \simeq K_{n,X}$ from Proposition 1.3 together with compatibility of C' and C from Theorem 1.9 gives the claim. □

Remark 1.17. If X is Cohen-Macaulay of pure dimension d, W_nX is also Cohen-Macaulay by Serre's S_k-criterion [18, (5.7.3)(i)] of the same pure dimension, and thus the complex $K_{n,X}$ is concentrated at degree $-d$ for all n [9, 3.5.1]. Denote by $W_n\omega_X$ the only non-zero cohomology sheaf of $K_{n,X}$ in this case. Then the same reasoning as in Proposition 1.16 shows that if $k = \overline{k}$ and X is Cohen-Macaulay over k of pure dimension, the map

$$C' - 1 : W_n\omega_X \to W_n\omega_X$$

is surjective.

1.3.5. *Comparison between* $W_n\Omega^d_{X,log}$ *and* $K_{n,X,log}$. Let X be a k-scheme. Denote by $d\log$ the following map of abelian étale sheaves

$$\begin{aligned} d\log : (\mathcal{O}^*_{X,\text{ét}})^{\otimes q} &\to W_n\Omega^q_{X,\text{ét}}, \\ a_1\otimes\cdots\otimes a_q &\mapsto d\log[a_1]_n \dots d\log[a_q]_n, \end{aligned}$$

where $a_1,\dots,a_q \in \mathcal{O}^*_{X,\text{ét}}$, $[-]_n : \mathcal{O}_{X,\text{ét}} \to W_n\mathcal{O}_{X,\text{ét}}$ denotes the Teichmüller lift, and $d\log[a_i]_n := \frac{d[a_i]_n}{[a_i]_n}$. We will denote its sheaf theoretic image by $W_n\Omega^q_{X,log,\text{ét}}$ and call it the étale sheaf of log forms. We denote by $W_n\Omega^q_{X,log} := W_n\Omega^q_{X,log,\text{Zar}} := \epsilon_* W_n\Omega^q_{X,log,\text{ét}}$, and call it the Zariski sheaf of log forms.

Lemma 1.18 ([8, lemme 2], [16, 1.6(ii)]). *Let X be a separated smooth k-scheme. Then we have the following left exact sequences*

$$0 \to W_n\Omega^q_{X,log} \to W_n\Omega^q_X \xrightarrow{1-\overline{F}} W_n\Omega^q_X/dV^{n-1}, \tag{1.3.29}$$

$$0 \to W_n\Omega^q_{X,log} \to W_n\Omega'^q_X \xrightarrow{C-1} W_n\Omega^q_X, \tag{1.3.30}$$

where $W_n\Omega'^q_X := F(W_{n+1}\Omega^q_X)$. The right hand maps are also surjective if $t = $ ét.

Proposition 1.19 collects what we have done so far.

Proposition 1.19 (Cf. [29, Prop. 4.2]). *Let X be a separated smooth scheme of pure dimension d over a perfect field k. Then*

(1) *we have $\mathcal{H}^{-d}(K_{n,X,log}) = W_n\Omega^d_{X,log}$, and $\mathcal{H}^i(K_{n,X,log}) = 0$ for all $i \neq -d, -d+1$.*

(2) *If $k = \overline{k}$, the natural map*

$$W_n\Omega^d_{X,log}[d] \to K_{n,X,log}$$

is a quasi-isomorphism of complexes of abelian sheaves.

Proof. (1) Since C is compatible with C' by Theorem 1.9, the natural map $\mathrm{Cone}(W_n\Omega^d_X[d] \xrightarrow{C-1} W_n\Omega^d_X[d])[-1] \to K_{n,X,log}$ is a quasi-isomorphism by the five lemma and the Ekedahl quasi-isomorphism Proposition 1.3. The claim thus follows from the exact sequence (1.3.30).

(2) Proposition 1.16+(1) above. □

1.4. **Localization triangle associated to** $K_{n,X,log}$.

1.4.1. *Definition of* $\mathrm{Tr}_{W_nf,log}$.

Proposition 1.20 (Proper pushforward, cf. [29, (3.2.3)]). *Let $f : X \to Y$ be a proper map between separated schemes of finite type over k. Then so is $W_nf : W_nX \to W_nY$, and we have a map*

$$\mathrm{Tr}_{W_nf,log} : (W_nf)_*K_{n,X,log} \to K_{n,Y,log}$$

of complexes that fits into the following commutative diagram of complexes, where the two rows are distinguished triangles in $D^b(W_nX, \mathbb{Z}/p^n)$

$$\begin{tikzcd}
(W_nf)_*K_{n,X,log} \arrow[r] \arrow[d, "\mathrm{Tr}_{W_nf,log}"] & (W_nf)_*K_{n,X} \arrow[r, "C'-1"] \arrow[d, "\mathrm{Tr}_{W_nf}"] & (W_nf)_*K_{n,X} \arrow[r] \arrow[d, "\mathrm{Tr}_{W_nf}"] & {} \\
K_{n,Y,log} \arrow[r] & K_{n,Y} \arrow[r, "C'-1"] & K_{n,Y} \arrow[r] & {}.
\end{tikzcd}$$

Moreover $\mathrm{Tr}_{W_nf,log}$ *is compatible with compositions and open restrictions.*

This is the covariant functoriality of $K_{n,X,log}$ with respect to proper morphisms. Thus we also denote $\mathrm{Tr}_{W_nf,log}$ by f_*.

Proof. It suffices to show the following diagrams commute.

$$\begin{tikzcd}
(W_nF_Y)_*(W_nf)_*K_{n,X} \arrow[r, "(W_nF_Y)_*(W_nf)_*(1.2.2)", "\simeq"'] \arrow[d, "(W_nF_Y)_*\,\mathrm{Tr}_{W_nf}"] & (W_nF_Y)_*(W_nf)_*(W_nF_X)^{\triangle}K_{n,X} \arrow[d, "(W_nF_Y)_*\,\mathrm{Tr}_{W_nf}"] \\
(W_nF_Y)_*K_{n,Y} \arrow[r, "(W_nF_Y)_*(1.2.2)", "\simeq"'] & (W_nF_Y)_*(W_nF_Y)^{\triangle}K_{n,Y},
\end{tikzcd}$$

$$\begin{tikzcd}
(W_nF_Y)_*(W_nf)_*(W_nF_X)^{\triangle}K_{n,X} \arrow[r, "\simeq"] \arrow[d, "(W_nF_Y)_*\,\mathrm{Tr}_{W_nf}"] & (W_nf)_*(W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X} \arrow[r, "(W_nf)_*\,\mathrm{Tr}_{W_nF_X}"] & (W_nf)_*K_{n,X} \arrow[d, "\mathrm{Tr}_{W_nf}"] \\
(W_nF_Y)_*(W_nF_Y)^{\triangle}K_{n,Y} \arrow[rr, "\mathrm{Tr}_{W_nF_Y}"] & & K_{n,Y},
\end{tikzcd}$$

where Tr_{W_nf} on the right of the first diagram and the left of the second diagram denote the trace map of the residual complex $(W_nF_Y)^{\triangle}K_{n,Y}$:

$$\mathrm{Tr}_{W_nf} : (W_nf)_*(W_nF_X)^{\triangle}K_{n,X} \simeq (W_nf)_*(W_nf)^{\triangle}(W_nF_Y)^{\triangle}K_{n,Y} \to (W_nF_Y)^{\triangle}K_{n,Y}.$$

The commutativity of the first diagram is due to the functoriality of the trace map with respect to residual complexes with the same associated filtration (Proposition 1.2(3)). The commutativity of the second is because of the compatibility of the trace map with compositions of morphisms (Proposition 1.2(4)). □

1.4.2. $\mathrm{Tr}_{W_nf,log}$ *in the case of a nilpotent immersion.*

Proposition 1.21 (Rülling. Cf. [29, 4.2])**.** *Let* $i : X_0 \hookrightarrow X$ *be a nilpotent immersion (thus so is* $W_ni : W_n(X_0) \to W_nX$*). Then the natural map*

$$\mathrm{Tr}_{W_ni,log} : (W_ni)_*K_{n,X_0,log} \to K_{n,X,log}$$

is a quasi-isomorphism.

Proof. Put $I_n := \mathrm{Ker}(W_n\mathcal{O}_X \to (W_ni)_*W_n\mathcal{O}_{X_0})$. Applying $\mathcal{H}om_{W_n\mathcal{O}_X}(-, K_{n,X})$ to the sequence of $W_n\mathcal{O}_X$-modules

$$0 \to I_n \to W_n\mathcal{O}_X \to (W_ni)_*W_n\mathcal{O}_{X_0} \to 0, \tag{1.4.1}$$

we get again a short exact sequence of complexes of $W_n\mathcal{O}_X$-modules

$$0 \to (W_ni)_*K_{n,X_0} \xrightarrow{\mathrm{Tr}_{W_ni}} K_{n,X} \to Q_n := \mathcal{H}om_{W_n\mathcal{O}_X}(I_n, K_{n,X}) \to 0.$$

The first map is clearly $\mathrm{Tr}_{W_n i}$ by duality. The restriction of the map $(W_nF_X)^* : W_n\mathcal{O}_X \to (W_nF_X)_*W_n\mathcal{O}_X$ to I_n gives a map

$$\begin{aligned}(W_nF_X)^*\mid_{I_n}: \quad I_n &\to (W_nF_X)_*I_n,\\ \sum_{i=0}^{n-1} V([a_i]) &\mapsto \sum_{i=0}^{n-1} V([a_i^p]).\end{aligned}$$

Define

$$\begin{aligned}(1.4.2)\quad C'_{I_n} : (W_nF_X)_*Q_n &= (W_nF_X)_*\mathcal{H}om_{W_n\mathcal{O}_X}(I_n, K_{n,X})\\ &\to \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*I_n, (W_nF_X)_*K_{n,X})\\ &\xrightarrow[\simeq]{(W_nF_X)_*(1.2.2)\circ} \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*I_n, (W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X})\\ &\xrightarrow{\mathrm{Tr}_{W_nF_X}\circ} \mathcal{H}om_{W_n\mathcal{O}_X}((W_nF_X)_*I_n, K_{n,X})\\ &\xrightarrow{((W_nF_X^*)|_{I_n})^\vee} \mathcal{H}om_{W_n\mathcal{O}_X}(I_n, K_{n,X}) = Q_n.\end{aligned}$$

According to the definition of C' in (1.2.3), C' is compatible with C'_{I_n}. Thus one has the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (W_nF_X)_*(W_ni)_*K_{n,X_0} \arrow[r, "(W_nF_X)_*\mathrm{Tr}_{W_ni}"] \arrow[d, "C'"] & (W_nF_X)_*K_{n,X} \arrow[r] \arrow[d, "C'"] & (W_nF_X)_*Q_n \arrow[r] \arrow[d, "C'_{I_n}"] & 0\\
0 \arrow[r] & (W_ni)_*K_{n,X_0} \arrow[r, "\mathrm{Tr}_{W_ni}"] & K_{n,X} \arrow[r] & Q_n \arrow[r] & 0.
\end{tikzcd}$$

Replacing C' by $C'-1$, and C'_{I_n} by $C'_{I_n}-1$, we arrive at the two lower rows of the following diagram. Denote

$$Q_{n,log} := \mathrm{Cone}(Q_n \xrightarrow{C'_{I_n}-1} Q_n)[-1].$$

Taking into account the shifted cones of $C'-1$ and $C'_{I_n}-1$, we get the first row of the following diagram which is naturally a short exact sequence. Now we have the whole commutative diagram of complexes, where all the three rows are exact, and all the three columns are distinguished triangles in the derived category:

$$\begin{tikzcd}
0 \arrow[r] & (W_ni)_*K_{n,X_0,log} \arrow[r, "\mathrm{Tr}_{W_ni,log}"] \arrow[d] & K_{n,X,log} \arrow[r] \arrow[d] & Q_{n,log} \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & (W_ni)_*K_{n,X_0} \arrow[r, "\mathrm{Tr}_{W_ni}"] \arrow[d, "C'-1"] & K_{n,X} \arrow[r] \arrow[d, "C'-1"] & Q_n \arrow[r] \arrow[d, "C'_{I_n}-1"] & 0\\
0 \arrow[r] & (W_ni)_*K_{n,X_0} \arrow[r, "\mathrm{Tr}_{W_ni}"] \arrow[d, "+1"] & K_{n,X} \arrow[r] \arrow[d, "+1"] & Q_n \arrow[r] \arrow[d, "+1"] & 0.\\
 & {} & {} & {} &
\end{tikzcd}$$

We want to show that $\mathrm{Tr}_{W_ni,log}$ is a quasi-isomorphism. By the exactness of the first row, it suffices to show that $Q_{n,log}$ is an acyclic complex. Because the right column is a distinguished triangle, it suffices to show that $C'_{I_n}-1 : Q_n \to Q_n$ is a quasi-isomorphism. Actually it is even an isomorphism of complexes: since

$(W_nF_X)^* |_{I_n}: I_n \to (W_nF_X)_*I_n$ is nilpotent (because $I_1 = \mathrm{Ker}(\mathcal{O}_X \to i_*\mathcal{O}_{X_0})$ is a finitely generated nilpotent ideal of $\mathcal{O}_X$), the map $C'_{I_n} : Q_n \to Q_n$ is therefore nilpotent (because one can alter the order of the three labeled maps in (1.4.2) in the obvious sense), and $C'_{I_n} - 1$ is therefore an isomorphism of complexes. □

1.4.3. *Localization triangles associated to $K_{n,X,log}$.* Let $i : Z \hookrightarrow X$ be a closed immersion with $j : U \hookrightarrow X$ being its open complement. Recall

$$\underline{\Gamma}_Z(\mathcal{F}) := \mathrm{Ker}(\mathcal{F} \to j_*j^{-1}\mathcal{F}) \tag{1.4.3}$$

for any abelian sheaf $\mathcal{F}$. Denote its i-th derived functor by $\mathcal{H}^i_Z(\mathcal{F})$. Notice that

- $\Gamma_{Z'}(\mathcal{F}) = \Gamma_Z(\mathcal{F})$ for any nilpotent thickening Z' of Z (e.g. $Z' = W_nZ$),
- $\mathcal{F} \to j_*j^{-1}\mathcal{F}$ is surjective whenever $\mathcal{F}$ is flasque, and
- flasque sheaves are $\underline{\Gamma}_Z$-acyclic [22, 1.10] and f_*-acyclic for any morphism f.

Therefore, for any complex of flasque sheaves $\mathcal{F}^\bullet$ of $\mathbb{Z}/p^n$-modules on W_nX,

$$0 \to \underline{\Gamma}_Z(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet \to (W_nj)_*(\mathcal{F}^\bullet|_{W_nU}) \to 0$$

is a short exact sequence of complexes. Thus the induced triangle

$$\underline{\Gamma}_Z(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet \to (W_nj)_*(\mathcal{F}^\bullet|_{W_nU}) \xrightarrow{+1} \tag{1.4.4}$$

is a distinguished triangle in $D^b(W_nX, \mathbb{Z}/p^n)$, whenever $\mathcal{F}^\bullet$ is a flasque complex with bounded cohomologies. In particular, since $K_{n,X,log}$ is a bounded complex of flasque sheaves, this is true for $\mathcal{F}^\bullet = K_{n,X,log}$.

Proposition 1.22 is proven in the smooth case by Gros-Milne-Suwa [39, 2.6]. The proof presented here comes from an unpublished manuscript of Rülling.

Proposition 1.22 (Rülling)**.** *Let $i : Z \hookrightarrow X$ be a closed immersion with $j : U \hookrightarrow X$ its open complement. Then*

(1) *(Purity) The map*

$$(W_ni)_*K_{n,Z,log} = \underline{\Gamma}_Z((W_ni)_*K_{n,Z,log}) \xrightarrow{\mathrm{Tr}_{W_ni,log}} \underline{\Gamma}_Z(K_{n,X,log})$$

is a quasi-isomorphism of complexes of sheaves.

(2) *(Localization triangle) The following*

$$(W_ni)_*K_{n,Z,log} \xrightarrow{\mathrm{Tr}_{W_ni,log}} K_{n,X,log} \to (W_nj)_*K_{n,U,log} \xrightarrow{+1} \tag{1.4.5}$$

is a distinguished triangle in $D^b(W_nX, \mathbb{Z}/p^n)$.

Note that we are working on the Zariski site and abelian sheaves on W_nX can be identified with abelian sheaves on X canonically. Thus we can replace $(W_ni)_*K_{n,Z,log}$ by $i_*K_{n,Z,log}$, and $(W_nj)_*K_{n,U,log}$ by $j_*K_{n,U,log}$ freely.

Proof. (1) Let I_n be the ideal sheaf associated to the closed immersion $W_ni : W_nZ \hookrightarrow W_nX$, and let $Z_{n,m}$ be the closed subscheme of W_nX determined by m-th power ideal I_n^m. In particular, $Z_{n,1} = W_nZ$. Denote by $i_{n,m} : Z_{n,m} \hookrightarrow W_nX$ and by $j_{n,m} : W_nZ \hookrightarrow Z_{n,m}$ the associated closed immersions. In this way, for each m, one has a decomposition of W_ni as maps of W_nk-schemes:

$$\begin{tikzcd} W_nZ \arrow[r, hook, "j_{n,m}"] \arrow[drr, "W_n\pi_Z"'] & Z_{n,m} \arrow[r, hook, "i_{n,m}"] \arrow[dr, "\pi_{Z_{n,m}}"] & W_nX \arrow[d, "W_n\pi_X"] \\ & & W_nk. \end{tikzcd}$$

Denote $K_{Z_{n,m}} := (\pi_{Z_{n,m}})^{\triangle}(W_nk)$, where $\pi_{Z_{n,m}} : Z_{n,m} \to W_nk$ is the structure morphism. We have a canonical isomorphism

$$i_{n,m,*}\mathcal{H}^i(K_{Z_{n,m}}) \simeq \mathcal{E}xt^i_{W_n\mathcal{O}_X}(i_{n,m,*}\mathcal{O}_{Z_{n,m}}, K_{n,X}) \tag{1.4.6}$$

by Proposition 1.1(4) and Proposition 1.1(1) associated to the closed immersion $i_{n,m}$. The trace maps associated to the closed immersions

$$Z_{n,m} \hookrightarrow Z_{n,m+1}$$

for different m make the left hand side of (1.4.6) an inductive system. The right hand side also lies in an inductive system when m varies: the canonical surjections

$$i_{n,m+1,*}\mathcal{O}_{Z_{n,m+1}} \to i_{n,m,*}\mathcal{O}_{Z_{n,m}}$$

induce the maps

$$\mathcal{H}om_{W_n\mathcal{O}_X}(i_{n,m,*}\mathcal{O}_{Z_{n,m}}, K_{n,X}) \to \mathcal{H}om_{W_n\mathcal{O}_X}(i_{n,m+1,*}\mathcal{O}_{Z_{n,m+1}}, K_{n,X}) \tag{1.4.7}$$

whose i-th cohomologies are the connecting homomorphisms of the inductive system. By duality, the map (1.4.7) is the trace map associated to the closed immersion $Z_{n,m} \hookrightarrow Z_{n,m+1}$, and thus is compatible with the inductive system on the left hand side of (1.4.6).

Consider the trace map associated to the closed immersion $i_{n,m} : Z_{n,m} \hookrightarrow W_nX$, i.e., the evaluation-at-1 map

$$\mathcal{H}om_{W_n\mathcal{O}_X}(i_{n,m,*}\mathcal{O}_{Z_{n,m}}, K_{n,X}) \to K_{n,X}.$$

Its image naturally lies in $\Gamma_{W_nZ}(K_{n,X})$. It induces an isomorphism on cohomology sheaves after taking the colimit on m

$$\operatorname{colim}_m \mathcal{E}xt^i_{W_n\mathcal{O}_X}(i_{n,m,*}\mathcal{O}_{Z_{n,m}}, K_{n,X}) \xrightarrow[\simeq]{\mathrm{ev}_1} \mathcal{H}^i_Z(K_{n,X})$$

by [21, V.4.3].

Now we consider

$$\begin{aligned} \operatorname{colim}_m i_{n,m,*}\mathcal{H}^i(K_{Z_{n,m}}) &\simeq \operatorname{colim}_m \mathcal{E}xt^i_{W_n\mathcal{O}_X}(i_{n,m,*}\mathcal{O}_{Z_{n,m}}, K_{n,X}) \\ &\xrightarrow[\simeq]{\mathrm{ev}_1} \mathcal{H}^i_Z(K_{n,X}). \end{aligned} \tag{1.4.8}$$

The composite map of (1.4.8) is $\operatorname{colim}_m \mathrm{Tr}_{i_{n,m}}$. On the other hand, consider the log trace associated to the closed immersion $i_{n,m}$ (cf. Proposition 1.20)

(1.4.9)

$$\begin{aligned} \mathrm{Tr}_{i_{n,m},log} : \mathcal{H}^i(i_{n,m,*}K_{Z_{n,m},log}) &= \mathcal{H}^i(\underline{\Gamma}_Z(i_{n,m,*}K_{Z_{n,m},log})) \\ &\to \mathcal{H}^i(\underline{\Gamma}_Z(K_{n,X,log})) = \mathcal{H}^i_Z(K_{n,X,log}). \end{aligned}$$

The maps (1.4.8), (1.4.9) give the vertical maps in the following diagram (due to formatting reason we omit $i_{n,m,*}$ from every term of the first row) which are automatically compatible by Proposition 1.20:

$$\begin{tikzcd}
\mathcal{H}^{i-1}(K_{Z_{n,m}}) \arrow[r, "C'-1"] \arrow[d, "\mathrm{Tr}_{i_{n,m}}"'] & \mathcal{H}^{i-1}(K_{Z_{n,m}}) \arrow[r] \arrow[d, "\mathrm{Tr}_{i_{n,m}}"'] & \mathcal{H}^{i}(K_{Z_{n,m},log}) \arrow[r] \arrow[d, "\mathrm{Tr}_{i_{n,m},log}"'] & \mathcal{H}^{i}(K_{Z_{n,m}}) \arrow[r, "C'-1"] \arrow[d, "\mathrm{Tr}_{i_{n,m}}"'] & \mathcal{H}^{i}(K_{Z_{n,m}}) \arrow[d, "\mathrm{Tr}_{i_{n,m}}"'] \\
\mathcal{H}^{i-1}_Z(K_{n,X}) \arrow[r, "C'-1"] & \mathcal{H}^{i-1}_Z(K_{n,X}) \arrow[r] & \mathcal{H}^{i}_Z(K_{n,X,log}) \arrow[r] & \mathcal{H}^{i}_Z(K_{n,X}) \arrow[r, "C'-1"] & \mathcal{H}^{i}_Z(K_{n,X}).
\end{tikzcd}$$

Taking the colimit with respect to m, the five lemma immediately gives that $\mathrm{colim}_m \mathrm{Tr}_{i_{n,m},log}$ is an isomorphism. Then $\mathrm{Tr}_{W_n i,log}$, which is the composition of

$$(W_n i)_*\mathcal{H}^i(K_{n,Z,log}) \xrightarrow[\text{Proposition 1.21},\simeq]{\mathrm{colim}_m \mathrm{Tr}_{j_{n,m},log}} \mathrm{colim}_m\, i_{n,m,*}\mathcal{H}^i(K_{Z_{n,m},log})$$
$$\xrightarrow[\simeq]{\mathrm{colim}_m \mathrm{Tr}_{i_{n,m},log}} \mathcal{H}^i_Z(K_{n,X,log}),$$

is an isomorphism. This proves the statement.

(2) Since $\underline{\Gamma}_Z(K_{n,X,log}) \to K_{n,X,log} \to (W_n j)_*K_{n,U,log} \xrightarrow{+1}$ is a distinguished triangle, the second part follows from the first part. $\square$

1.5. **Functoriality.** The pushforward functoriality of $K_{n,X,log}$ has been done in Proposition 1.20 for proper f. Now we define the pullback map for an étale morphism f. Since $W_n f$ is then also étale, we have an isomorphism of functors $(W_nf)^* \simeq (W_nf)^{\triangle}$ by Proposition 1.1(2). Define a chain map of complexes of $W_n\mathcal{O}_Y$-modules

(1.5.1)
$$f^* : K_{n,Y} \xrightarrow{\mathrm{adj}} (W_nf)_*(W_nf)^*K_{n,Y} \simeq (W_nf)_*(W_nf)^{\triangle}K_{n,Y} \simeq (W_nf)_*K_{n,X}.$$

Here adj stands for the adjunction map of the identity map of $(W_nf)^*K_{n,Y}$.

Proposition 1.23 (Étale pullback). *Suppose $f : X \to Y$ is an étale morphism. Then*

$$f^* : K_{n,Y,log} \to (W_nf)_*K_{n,X,log},$$

defined by termwise applying (1.5.1)*, is a chain map between complexes of abelian sheaves.*

Proof. It suffices to prove that C' is compatible with f^* defined above. Consider the following diagram in the category of complexes of $W_n\mathcal{O}_Y$-modules

$$\begin{tikzcd}
(W_nF_Y)_*K_{n,Y} \arrow[r, "(1.2.2)", "\simeq"'] \arrow[d, "\mathrm{adj}"] \arrow[dr, phantom, "(a)"] & (W_nF_Y)_*(W_nF_Y)^{\triangle}K_{n,Y} \arrow[r, "\mathrm{Tr}_{W_nF_Y}"] \arrow[d, "\mathrm{adj}"] \arrow[dr, phantom, "(d)"] & K_{n,Y} \arrow[d, "\mathrm{adj}"] \\
(W_nf)_*(W_nf)^*(W_nF_Y)_*K_{n,Y} \arrow[r, "(1.2.2)", "\simeq"'] \arrow[d, "\alpha"', "\simeq"] \arrow[dr, phantom, "(b)"] & (W_nf)_*(W_nf)^*(W_nF_Y)_*(W_nF_Y)^{\triangle}K_{n,Y} \arrow[r, "\mathrm{Tr}_{W_nF_Y}"] \arrow[d, "\beta"', "\simeq"] & (W_nf)_*(W_nf)^*K_{n,Y} \arrow[dd, "\simeq"] \\
(W_nf)_*(W_nF_X)_*(W_nf)^*K_{n,Y} \arrow[r, "(1.2.2)", "\simeq"'] \arrow[d, "\simeq"] \arrow[dr, phantom, "(c)"] & (W_nf)_*(W_nF_X)_*(W_nf)^*(W_nF_Y)^{\triangle}K_{n,Y} \arrow[d, "\simeq"] \arrow[r, phantom, "(e)"] & \\
(W_nf)_*(W_nF_X)_*K_{n,X} \arrow[r, "(1.2.2)", "\simeq"'] & (W_nf)_*(W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X} \arrow[r, "\mathrm{Tr}_{W_nF_X}"] & (W_nf)_*K_{n,X}.
\end{tikzcd}$$

In this diagram we use shortened notations for the maps due to formatting reasons, e.g. we write (1.2.2) instead of $(W_nf)_*(W_nF_X)_*(1.2.2)$, etc. The maps labeled

α and β are base change maps, and they are isomorphisms because W_nf is flat (actually W_nf is étale because f is étale) [21, II.5.12]. The composites of the maps on the very left and very right are $(W_nF_Y)_*(f^*)$ and f^* (where f^* is as defined in (1.5.1)). The composites of the maps on the very top and very bottom are C'_Y and $(W_nf)_*C'_X$. Diagrams (a), (b), (c), (d) commute due to naturality. Diagram (e) commutes, because we have a cartesian square

$$\begin{tikzcd}
W_nX \arrow[r, "W_nF_X"] \arrow[d, "W_nf"'] & W_nX \arrow[d, "W_nf"] \\
W_nY \arrow[r, "W_nF_Y"] & W_nY
\end{tikzcd}$$

by Remark 1.8(2), and then the base change formula of the Grothendieck trace map as given in Proposition 1.2(5) gives the result. □

Lemma 1.24. *Consider the following cartesian diagram*

$$\begin{tikzcd}
W \arrow[r, "f'"] \arrow[d, "g'"] & Z \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

with g being proper, and f being étale. Then we have a commutative diagram of residual complexes

$$\begin{tikzcd}
(W_ng)_*K_{n,Z} \arrow[r, "f'^*"] \arrow[d, "\mathrm{Tr}_{W_ng}"] & (W_ng)_*(W_nf')_*K_{n,W} \arrow[r, "\simeq"] & (W_nf)_*(W_ng')_*K_{n,W} \arrow[d, "\mathrm{Tr}_{W_ng'}"] \\
K_{n,Y} \arrow[rr, "f^*"] & & (W_nf)_*K_{n,X}.
\end{tikzcd}$$

Proof. We decompose the diagram into the following two diagrams and show their commutativity one by one. First we consider

$$\begin{tikzcd}
(W_ng)_*K_{n,Z} \arrow[r, "\mathrm{adj}"] \arrow[dddd, "\mathrm{Tr}_{W_ng}"] & (W_ng)_*(W_nf')_*(W_nf')^*K_{n,Z} \arrow[d, "\simeq"] \\
& (W_nf)_*(W_ng')_*(W_nf')^*(W_ng)^{\triangle}K_{n,Y} \\
& (W_nf)_*(W_nf)^*(W_ng)_*(W_ng)^{\triangle}K_{n,Y} \arrow[u, "\simeq"', "\alpha"] \arrow[d, "\mathrm{Tr}_{W_ng}"] \\
& \\
K_{n,Y} \arrow[r, "\mathrm{adj}"] & (W_nf)_*(W_nf)^*K_{n,Y}.
\end{tikzcd}$$

Here α denotes the base change map, it is an isomorphism because W_nf is flat [21, II.5.12]. This diagram commutes by the naturality. Next we consider

$$\begin{tikzcd}
(W_ng)_*(W_nf')_*(W_nf')^*K_{n,Z} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & (W_ng)_*(W_nf')_*K_{n,W} \arrow[d, "\simeq"] \\
(W_nf)_*(W_ng')_*(W_nf')^*(W_ng)^{\triangle}K_{n,Y} \arrow[r, "\simeq"] & (W_nf)_*(W_ng')_*K_{n,W} \arrow[dd, "\mathrm{Tr}_{W_ng'}"] \\
(W_nf)_*(W_nf)^*(W_ng)_*(W_ng)^{\triangle}K_{n,Y} \arrow[u, "\alpha", "\simeq"'] \arrow[d, "\mathrm{Tr}_{W_ng}"] & \\
(W_nf)_*(W_nf)^*K_{n,Y} \arrow[r, "\simeq"] & (W_nf)_*K_{n,X}.
\end{tikzcd}$$

The top part commutes by the naturality. The bottom part commutes by the base change formula of the Grothendieck trace maps with respect to étale morphisms (Proposition 1.2(5)). □

Since both f^* for log complexes in Proposition 1.23 and $g_* := \mathrm{Tr}_{W_ng,log}$ are defined termwise, we arrive immediately at the following compatibility as a consequence of Lemma 1.24.

Proposition 1.25. *Notations are the same as Lemma* 1.24*. One has a commutative diagram of complexes*

$$\begin{tikzcd}
(W_ng)_*K_{n,Z,log} \arrow[r, "f'^*"] \arrow[d, "g_*"] & (W_ng)_*(W_nf')_*K_{n,W,log} \arrow[r, "\simeq"] & (W_nf)_*(W_ng')_*K_{n,W,log} \arrow[d, "g'_*"] \\
K_{n,Y,log} \arrow[rr, "f^*"] & & (W_nf)_*K_{n,X,log}.
\end{tikzcd}$$

1.6. **Étale counterpart** $K_{n,X,log,\text{ét}}$**.** Let X be a separated scheme of finite type over k of dimension d. In this subsection we will use $t = \mathrm{Zar}, \text{ét}$ to distinguish objects, morphisms on different sites. If t is omitted, it means $t = \mathrm{Zar}$ unless otherwise stated.

Denote the structure sheaf on the small étale site $(W_nX)_{\text{ét}}$ by $W_n\mathcal{O}_{X,\text{ét}}$. Denote by

$$(\epsilon_*, \epsilon^*) : ((W_nX)_{\text{ét}}, W_n\mathcal{O}_{X,\text{ét}}) \to ((W_nX)_{\mathrm{Zar}}, W_n\mathcal{O}_X)$$

the module-theoretic functors. Recall that every étale W_nX-scheme is of the form $W_ng : W_nU \to W_nX$, where $g : U \to X$ is an étale X-scheme by Remark 1.8(1). Now let $\mathcal{F}$ be a $W_n\mathcal{O}_{X,\text{ét}}$-module on $(W_nX)_{\text{ét}}$. Consider the following map (cf. [29, p. 264])

$$\tau : (W_nF_X)_*\mathcal{F} \to \mathcal{F}, \tag{1.6.1}$$

which is defined to be

$$\begin{aligned}((W_nF_X)_*\mathcal{F})(W_nU \xrightarrow{W_ng} W_nX) &= \mathcal{F}(W_nX \times_{W_nF_X, W_nX} W_nU \xrightarrow{pr_1} W_nX)\\ &\xrightarrow[\simeq]{W_nF^*_{U/X}} \mathcal{F}(W_nU \xrightarrow{W_ng} W_nX)\end{aligned}$$

for any étale map $W_ng : W_nU \to W_nX$ (here we use pr_1 to denote the first projection map of the fiber product). This is an automorphism of $\mathcal{F}$ as an abelian étale sheaf, but it changes the $W_n\mathcal{O}_{X,\text{ét}}$-module structure of $\mathcal{F}$.

Define

$$K_{n,X,\text{ét}} := \epsilon^* K_{n,X}$$

to be the complex of étale $W_n\mathcal{O}_{X,\text{ét}}$-modules associated to the Zariski complex $K_{n,X}$ of $W_n\mathcal{O}_X$-modules. This is still a complex of quasi-coherent sheaves with coherent cohomologies. For a proper map $f : X \to Y$ of k-schemes, define

$$\mathrm{Tr}_{W_nf,\text{ét}} : (W_nf)_*K_{n,X,\text{ét}} = \epsilon^*((W_nf)_*K_{n,X}) \xrightarrow{\epsilon^* \mathrm{Tr}_{W_nf}} K_{n,Y,\text{ét}}$$

to be the étale map of $W_n\mathcal{O}_{Y,\text{ét}}$-modules associated to the Zariski map $\mathrm{Tr}_{W_nf} : K_{n,X} \to K_{n,X}$ of $W_n\mathcal{O}_X$-modules. Define the Cartier operator $C'_{\text{ét}}$ for étale complexes to be the composite

$$C'_{\text{ét}} : K_{n,X,\text{ét}} \underset{\simeq}{\xrightarrow{\tau^{-1}}} (W_nF_X)_*K_{n,X,\text{ét}} = \epsilon^*((W_nF_X)_*K_{n,X}) \xrightarrow{\epsilon^*(1.2.3)} K_{n,X,\text{ét}}.$$

Define

$$K_{n,X,log,\text{ét}} := \mathrm{Cone}(K_{n,X,\text{ét}} \xrightarrow{C'_{\text{ét}}-1} K_{n,X,\text{ét}})[-1].$$

We also have the sheaf-level Cartier operator. Let X be a smooth k-scheme. Recall that by definition, $C_{\text{ét}}$ is the composition of the inverse of (1.6.1) with the module-theoretic etalization of the $W_n\mathcal{O}_X$-linear map (1.3.6) (it has appeared in Lemma 1.18 before):

$$C_{\text{ét}} : W_n\Omega^d_{X,\text{ét}} \underset{\simeq}{\xrightarrow{\tau^{-1}}} (W_nF_X)_*W_n\Omega^d_{X,\text{ét}} = \epsilon^*((W_nF_X)_*W_n\Omega^d_X) \xrightarrow{\epsilon^*(1.3.6)} W_n\Omega^d_{X,\text{ét}}.$$

Proposition 1.26 (cf. Theorem 1.9)**.** *$C'_{\text{ét}}$ is the natural extension of C' to the small étale site, i.e.,*

$$\epsilon_*C'_{\text{ét}} = C' : K_{n,X} \to K_{n,X}.$$

If X is smooth, $C_{\text{ét}}$ is the natural extension of C to the small étale site

$$\epsilon_*C_{\text{ét}} = C : W_n\Omega^d_X \to W_n\Omega^d_X.$$

And one has compatibility

$$C_{\text{ét}} = \mathcal{H}^{-d}(C'_{\text{ét}}).$$

Proof. The first two claims are clear. The last claim follows from the compatibility of C and C' in the Zariski case (Theorem 1.9). ☐

Proposition 1.27 (cf. Proposition 1.15)**.** *Let X be a separated scheme of finite type over k with $k = \overline{k}$. Then*

$$H^i(W_nX, K_{n,X,\text{ét}}) := R^i\Gamma(W_nX, K_{n,X,\text{ét}}) \xrightarrow{C'_{\text{ét}}-1} H^i(W_nX, K_{n,X,\text{ét}})$$

is surjective for every i. In particular,

$$R^i\Gamma(W_nX, K_{n,X,log,\text{ét}}) \simeq H^i(W_nX, K_{n,X,\text{ét}})^{C'_{\text{ét}}-1}.$$

Proof. The quasi-coherent descent from the étale site to the Zariski site gives

$$R\Gamma((W_nX)_{\text{ét}}, K_{n,X,\text{ét}}) = R\Gamma((W_nX)_{\text{Zar}}, K_{n,X,\text{Zar}}).$$

Taking the i-th cohomology groups, the desired surjectivity then follows from the compatibility of C' and $C'_{\text{ét}}$ (Proposition 1.26) and the Zariski case (Proposition 1.15). ☐

In the étale topology and for any perfect field k, the surjectivity of

$$C_{\text{ét}} - 1 : W_n\Omega^d_{X,\text{ét}} \to W_n\Omega^d_{X,\text{ét}}$$

is known without the need of Proposition 1.27 (cf. Lemma 1.18). For the same reasoning as in Proposition 1.19, we have

Proposition 1.28 (cf. Proposition 1.19)**.** *Assume X is smooth of pure dimension d over a perfect field k. Then the natural map*

$$W_n\Omega^d_{X,log,\text{ét}}[d] \to K_{n,X,log,\text{ét}}$$

is a quasi-isomorphism of complexes of abelian sheaves.

We go back to the general non-smooth case. The proper pushforward property in the étale setting is very similar to the Zariski case.

Proposition 1.29 (Proper pushforward, cf. Proposition 1.20)**.** *For $f : X \to Y$ proper, we have a well-defined map of complexes of étale sheaves*

$$\mathrm{Tr}_{W_nf,log,\text{ét}} : (W_nf)_*K_{n,X,log,\text{ét}} \to K_{n,X,log,\text{ét}} \tag{1.6.2}$$

given by applying $\mathrm{Tr}_{W_nf,\text{ét}}$ termwise.

Proof. The map τ^{-1} is clearly functorial with respect to any map of abelian sheaves. The rest of the proof goes exactly as in Proposition 1.20. □

Proposition 1.30 (cf. Proposition 1.21)**.** *Let $i : X_0 \hookrightarrow X$ be a nilpotent immersion. Then the natural map*

$$\mathrm{Tr}_{W_ni,log,\text{ét}} : (W_ni)_*K_{n,X_0,log,\text{ét}} \to K_{n,X,log,\text{ét}}$$

is a quasi-isomorphism.

Proof. This is a direct consequence of the functoriality of the map τ^{-1} and Proposition 1.21. □

Let $i : Z \hookrightarrow X$ be a closed immersion with $j : U \hookrightarrow X$ being the open complement as before. Define

$$\underline{\Gamma}_Z(\mathcal{F}) := \mathrm{Ker}(\mathcal{F} \to j_*j^{-1}\mathcal{F})$$

for any étale abelian sheaf $\mathcal{F}$ on X, just as in the Zariski case (cf. (1.4.3)). Replacing Z (resp. X) by a nilpotent thickening will define the same functor as $\underline{\Gamma}_Z(-)$, because the definition of the functor $\underline{\Gamma}_Z$ only depends on the pair (X, U). Recall that if $\mathcal{F} = \mathcal{I}$ is an injective $\mathbb{Z}/p^n$-sheaf,

$$0 \to \underline{\Gamma}_Z(\mathcal{I}) \to \mathcal{I} \to j_*j^{-1}\mathcal{I} \to 0$$

is exact. In fact, because $j_!\mathbb{Z}/p^n$ is a subsheaf of the constant sheaf $\mathbb{Z}/p^n$ on X, the map $\mathrm{Hom}_X(\mathbb{Z}/p^n, \mathcal{I}) \to \mathrm{Hom}_X(j_!\mathbb{Z}/p^n, \mathcal{I})$ is surjective. Since $\mathrm{Hom}_X(j_!\mathbb{Z}/p^n, \mathcal{I}) = \mathrm{Hom}_U(\mathbb{Z}/p^n, j^{-1}\mathcal{I}) = \mathrm{Hom}_X(\mathbb{Z}/p^n, j_*j^{-1}\mathcal{I})$, the map

$$\mathrm{Hom}_X(\mathbb{Z}/p^n, \mathcal{I}) \to \mathrm{Hom}_X(\mathbb{Z}/p^n, j_*j^{-1}\mathcal{I})$$

is surjective, and hence we have the claim. This implies that for any complex $\mathcal{F}^\bullet$ of étale $\mathbb{Z}/p^n$-sheaves with bounded cohomologies,

$$R\underline{\Gamma}_Z(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet \to j_*j^{-1}\mathcal{F}^\bullet \xrightarrow{+1} \tag{1.6.3}$$

is a distinguished triangle in $D^b(X, \mathbb{Z}/p^n)$ (cf. (1.4.4)).

Proposition 1.31 (cf. Proposition 1.22)**.** *Let $i : Z \hookrightarrow X$ be a closed immersion with open complement $j : U \hookrightarrow X$, as before. Then*

(1) *(Purity) We can identify canonically the functors*

$$(W_ni)_* = R\underline{\Gamma}_Z \circ (W_ni)_* : D^b((W_nZ)_{\text{ét}}, \mathbb{Z}/p^n) \to D^b((W_nX)_{\text{ét}}, \mathbb{Z}/p^n).$$

The composition of this canonical identification with the trace map

$$(W_ni)_*K_{n,Z,log,\text{ét}} = R\underline{\Gamma}_Z((W_ni)_*K_{n,Z,log,\text{ét}}) \xrightarrow{\mathrm{Tr}_{W_ni,log,\text{ét}}} R\underline{\Gamma}_Z(K_{n,X,log,\text{ét}})$$

is a quasi-isomorphism of complexes of étale $\mathbb{Z}/p^n$-sheaves.

(2) *(Localization triangle)*

$$(W_ni)_*K_{n,Z,log,\text{ét}} \xrightarrow{\mathrm{Tr}_{W_ni,log,\text{ét}}} K_{n,X,log,\text{ét}} \to (W_nj)_*K_{n,U,log,\text{ét}} \xrightarrow{+1}$$

is a distinguished triangle in $D^b((W_nX)_{\text{ét}}, \mathbb{Z}/p^n)$.

Proof. (1) One only needs to show that $(W_ni)_* = R\underline{\Gamma}_Z \circ (W_ni)_*$, and then the rest of the proof is the same as in Proposition 1.22(1). Let $\mathcal{I}$ be an injective étale $\mathbb{Z}/p^n$-sheaf on W_nZ. Since $\mathrm{Hom}_{W_nX}(-, (W_ni)_*\mathcal{I}) = \mathrm{Hom}_{W_nZ}((W_ni)^{-1}(-), \mathcal{I})$ and $(W_ni)^{-1}$ is exact, we know $(W_ni)_*\mathcal{I}$ is an injective abelian sheaf on $(W_nX)_{\text{ét}}$. This implies that $R(\underline{\Gamma}_Z \circ (W_ni)_*) = R\underline{\Gamma}_Z \circ (W_ni)_*$ by the Leray spectral sequence, and thus $(W_ni)_* = R(W_ni)_* = R(\underline{\Gamma}_Z \circ (W_ni)_*) = R\underline{\Gamma}_Z \circ (W_ni)_*$.

(2) Note that $(W_nj)_*K_{n,U,log,\text{ét}} = R(W_nj)_*K_{n,U,log,\text{ét}}$. In fact, the terms of $K_{n,U,log,\text{ét}}$ are quasi-coherent $W_n\mathcal{O}_{X,\text{ét}}$-modules which are $(W_nj)_*$-acyclic in the étale topology (because $R^if_*(\epsilon^*\mathcal{F}) = \epsilon^*(R^if_*\mathcal{F})$ for any quasi-coherent Zariski sheaf $\mathcal{F}$ and any quasi-compact quasi-separated morphism f [38, Tag 071N]). Now the first part and the distinguished triangle (1.6.3) imply the claim. $\square$

2. Bloch's cycle complex $\mathbb{Z}^c_{X,t}$

Let X be a separated scheme of finite type over k of dimension d. Let

$$\Delta^i = \mathrm{Spec}\, k[T_0, \ldots, T_i]/(\sum T_j - 1).$$

Define $z_0(X, i)$ to be the free abelian group generated by closed integral subschemes $Z \subset X \times \Delta^i$ that intersect all faces properly and $\dim Z = i$. We say two closed subschemes Z_1, Z_2 of a scheme Y *intersect properly* if for every irreducible component W of the schematic intersection $Z_1 \cap Z_2 := Z_1 \times_Y Z_2$, one has

$$\dim W \leq \dim Z_1 + \dim Z_2 - \dim Y \tag{2.0.1}$$

(cf. [15, A.1]). A subvariety of $X \times \Delta^i$ is called a *face* if it is determined by some $T_{j_1} = T_{j_2} = \cdots = T_{j_s} = 0$ $(0 \leq j_1 < \cdots < j_s \leq i)$. Note that a face is Zariski locally determined by a regular sequence of $X \times \Delta^i$. Therefore the given inequality condition (2.0.1) in the definition of $z_0(X, i)$ is equivalent to the equality condition [15, (53)].

The above definition defines a sheaf $z_0(-, i)$ in both the Zariski and the étale topology on X (see [3, p.270] and [12, Lemma 3.1]). Define the complex of sheaves

$$\to z_0(-, i) \xrightarrow{d} z_0(-, i-1) \to \ldots z_0(-, 0) \to 0$$

with differential map

$$d(Z) = \sum_j (-1)^j [Z \cap V(T_j)].$$

Here we mean by $V(T_j)$ the closed integral subscheme determined by T_j and by $[Z \cap V(T_j)]$ the linear combination of the reduced irreducible components of the scheme theoretic intersection $Z \cap V(T_j)$ with coefficients being intersection multiplicities. $z_0(X, \bullet)$ is then a homological complex concentrated in degree $[0, \infty)$. Labeling cohomologically, we set

$$(\mathbb{Z}^c_X)^i = z_0(-, -i).$$

This complex is non-zero in degrees

$$(-\infty, 0].$$

Define the higher Chow group

$$\mathrm{CH}_0(X, i) := H_i(z_0(X, \bullet)) = H^{-i}(\mathbb{Z}^c_X(X))$$

for any i. The higher Chow groups with coefficients in an abelian group A will be denoted

$$\mathrm{CH}_0(X, i; A) := H^{-i}(\mathbb{Z}^c_X(X) \otimes_{\mathbb{Z}} A).$$

The complex $\mathbb{Z}^c_{X,t}$, with either $t = \mathrm{Zar}$ or $t = \text{ét}$, has the following functoriality properties (cf. [3, Prop. 1.3]). If $f : X \to Y$ is a proper morphism, then there is a chain map $f_* : f_*\mathbb{Z}^c_X \to \mathbb{Z}^c_Y$ by the pushforward of cycles. If $f : X \to Y$ is a quasi-finite flat morphism, then there is a chain map $f^* : \mathbb{Z}^c_Y \to f_*\mathbb{Z}^c_X$ by the pullback of cycles.

3. Kato's complex of Milnor K-theory $C^M_{X,t}$

Recall that given a field L, the q-th Milnor K-group $K^M_q(L)$ of L is defined to be the q-th graded piece of the graded commutative ring

$$\bigoplus_{q \geq 0} K^M_q(L) = \frac{\bigoplus_{q\geq 0}(L^*)^{\otimes q}}{(a \otimes (1-a) \mid a, 1-a \in L^*)},$$

where $(a \otimes (1-a) \mid a, 1-a \in L^*)$ denotes the two-sided ideal of the graded commutative ring $\bigoplus_{q\geq 0}(L^*)^{\otimes q}$ generated by elements of the form $a \otimes (1-a)$ with $a, 1-a \in L^*$. The image of an element $a_1 \otimes \cdots \otimes a_q \in (L^*)^{\otimes q}$ in $K^M_q(L)$ is denoted by $\{a_1, \ldots, a_q\}$.

If L is a discrete valuation field with valuation v and residue field $k(v)$, the group homomorphism

$$\partial_v : K^M_q(L) \to K^M_{q-1}(k(v)), \quad \partial_v(\{\pi_v, u_1, \ldots, u_{q-1}\}) = \{\overline{u}_1, \ldots, \overline{u}_{q-1}\}$$

is called the map of the *tame symbol*. Here π_v is a local parameter with respect to v, $u_1, \ldots u_{q-1}$ are units in the valuation ring of v, and $\overline{u}_1, \ldots, \overline{u}_{q-1}$ are the images of $u_1, \ldots u_{q-1}$ in the residue field $k(v)$. This is consistent with the sign convention in [37, p.328].

For every natural number q and every finite field extension L'/L, there exists a unique group homomorphism

$$\mathrm{Nm}_{L'/L} : K^M_q(L') \to K^M_q(L)$$

such that

(1) For any field extensions $L \subset L' \subset L''$, one has $\mathrm{Nm}_{L/L} = id$ and $\mathrm{Nm}_{L'/L} \circ \mathrm{Nm}_{L''/L'} = \mathrm{Nm}_{L''/L}$;
(2) Let $L(T)$ be the function field of $\mathbf{A}^1_k$. For every $x \in K_q^M(L(T))$ one has

$$\sum_v \mathrm{Nm}_{L(v)/L}(\partial_v(x)) = 0,$$

where v runs over all discrete valuations of $L(T)$, and $L(v)$ denotes the residue field at valuation v.

The map $\mathrm{Nm}_{L'/L}$ is called the *norm map* associated to the finite field extension L'/L.

Recall the definition of a Milnor K-sheaf on a point $X = \operatorname{Spec} L$, where L is any field. $\mathcal{K}^M_{\operatorname{Spec} L,q,\mathrm{Zar}}$ is the constant sheaf associated to the abelian group $K^M_q(L)$ (without the assumption that L is an infinite field, cf. [31, Prop. 10(4)]), and $\mathcal{K}^M_{\operatorname{Spec} L,q,\text{ét}}$ is the étale sheaf associated to the presheaf

$$L' \mapsto K^M_q(L'); \quad L'/L \text{ finite separable.}$$

Choose a separable closure L^{sep} of L. Then the geometric stalk at the geometric point $\operatorname{Spec} L^{\mathrm{sep}}$ over $\operatorname{Spec} L$ is $\operatorname{colim}_{L\subset L'\subset L^{\mathrm{sep}}} K^M_q(L')$, which is equal to $K^M_q(L^{\mathrm{sep}})$ because the filtered colimit commutes with the tensor product and the quotient. Now by Galois descent of the étale sheaf condition, the sheaf $\mathcal{K}^M_{\operatorname{Spec} L,q,\text{ét}}$ is precisely

$$L' \mapsto K^M_q(L^{\mathrm{sep}})^{\mathrm{Gal}(L^{\mathrm{sep}}/L')}; \quad L'/L \text{ finite separable.}$$

Here the Galois action is given on each factor.

Let X be a separated scheme of finite type over k of dimension d. Now with the topology $t = \mathrm{Zar}$ or $t =$ ét, we have the corresponding Gersten complex of Milnor K-theory, denote by $C^M_{X,t}$ (the differentials d^M will be introduced below):

$$\bigoplus_{x\in X_{(d)}} \iota_{x*}\mathcal{K}^M_{x,d,t} \xrightarrow{d^M} \dots \xrightarrow{d^M} \bigoplus_{x\in X_{(1)}} \iota_{x*}\mathcal{K}^M_{x,1,t} \xrightarrow{d^M} \bigoplus_{x\in X_{(0)}} \iota_{x*}\mathcal{K}^M_{x,0,t}, \tag{3.0.1}$$

where $\iota_x : \operatorname{Spec} k(x) \hookrightarrow X$ the natural inclusion map. As part of the convention,

$$(C^M_{X,t})^i = \bigoplus_{x\in X_{(-i)}} \iota_{x,*}\mathcal{K}^M_{x,-i,t}.$$

In other words, (3.0.1) sits in degrees

$$[-d, 0].$$

It remains to introduce the differential maps.

If $t = \mathrm{Zar}$, the differential map d^M in (3.0.1) is defined in the following way. Let $x \in X_{(q)}$ be a dimension q point, and $\rho : X' \to \overline{\{x\}}$ be the normalization of $\overline{\{x\}}$ with generic point x'. Define

$$(d^M)^x_y : K^M_q(x) = K^M_q(x') \xrightarrow{\sum \partial^{x'}_{y'}} \bigoplus_{y'|y} K^M_{q-1}(y') \xrightarrow{\sum \mathrm{Nm}_{y'/y}} K^M_{q-1}(y).$$

Here we have used the shortened symbol $K^M_q(x) := K^M_q(k(x))$. The notation $y'|y$ means that $y' \in X'^{(1)}$ is in the fiber of y.

$$\partial^{x'}_{y'} : K^M_q(x') \to K^M_{q-1}(y') \tag{3.0.2}$$

is the Milnor tame symbol of the discrete valuation field $k(x')$ with valuation defined by y'. And

$$\mathrm{Nm}_{y'/y} : K^M_{q-1}(y') \to K^M_{q-1}(y) \tag{3.0.3}$$

is the Milnor norm map of the finite field extension $k(y) \subset k(y')$. The differential d^M of this complex is given by

$$d^M := \sum_{x\in X_{(q)}} \sum_{y\in X_{(q-1)}\cap\overline{\{x\}}} (d^M)^x_y : \bigoplus_{x\in X_{(q)}} K^M_q(x) \to \bigoplus_{y\in X_{(q-1)}} K^M_{q-1}(y).$$

If $t =$ ét, set $x \in X_{(q)}$, $y \in X_{(q-1)} \cap \overline{\{x\}}$. Denote by $\rho : X' \to \overline{\{x\}}$ the normalization map and denote by x' the generic point of X'. One can canonically identify the étale abelian sheaves $\mathcal{K}^M_{x,q,\text{ét}}$ and $\rho_*\mathcal{K}^M_{x',q,\text{ét}}$ on $\overline{\{x\}}$ (here $\mathcal{K}^M_{x,q,\text{ét}}$ on $\overline{\{x\}}$ means the pushforward of the sheaf $\mathcal{K}^M_{x,q,\text{ét}}$ on the point $\operatorname{Spec} k(x)$ via $\operatorname{Spec} k(x) \to \overline{\{x\}}$), and thus identify $\iota_{x,*}\mathcal{K}^M_{x,q,\text{ét}}$ and $\iota_{x,*}\rho_*\mathcal{K}^M_{x',q,\text{ét}}$ on X. Let $y' \in X'^{(1)}$ such that $\rho(y') = y$. Then the componentwise differential map

$$(d^M)^x_y : \iota_{x,*}\mathcal{K}^M_{x,q,\text{ét}} \to \iota_{y,*}\mathcal{K}^M_{y,q-1,\text{ét}}$$

is defined to be the composition

$$(d^M)^x_y = \iota_{y,*}(\mathrm{Nm}) \circ \rho_*(\partial).$$

Here $\partial := \sum_{y'\in X'^{(1)}\cap\rho^{-1}(y)} \partial^{x'}_{y'}$, where

$$\partial^{x'}_{y'} : \iota_{x',*}\mathcal{K}^M_{x',q,\text{ét}} \to \iota_{y',*}\mathcal{K}^M_{y',q-1,\text{ét}} \tag{3.0.4}$$

on X' is defined to be the sheafification of the tame symbol on the presheaf level. Indeed, the tame symbol is a map of étale presheaves by [37, R3a]. And $\mathrm{Nm} := \sum_{y'\in X'^{(1)}\cap\rho^{-1}(y)} \mathrm{Nm}_{y'/y}$, where

$$\mathrm{Nm}_{y'/y} : \rho_*\mathcal{K}^M_{y',q-1,\text{ét}} \to \mathcal{K}^M_{y,q-1,\text{ét}} \tag{3.0.5}$$

on y is defined to be the sheafification of the norm map on the presheaf level. The norm map is a map of étale presheaves by [37, R1c].

The complex $C^M_{X,t}$, either $t =$ Zar or $t =$ ét, is covariant for proper morphisms and contravariant for quasi-finite flat morphisms [37, (4.6)(1)(2)]. The pushforward map associated to a proper morphism is induced by the Milnor norm map, and the pullback map associated to a quasi-finite flat morphism is induced by the pullback map of the structure sheaves.

4. Kato-Moser's complex of logarithmic de Rham-Witt sheaves $\tilde{\nu}_{n,X,t}$

Kato first defined the Gersten complex of the logarithmic de Rham-Witt sheaves in [27, §1]. Moser in [35, (1.3)-(1.5)] sheafified Kato's construction on the étale site and studied its dualizing properties. We will adopt here the sign conventions in [37].

Let Y be a k-scheme. Let $q \in \mathbb{N}$ be an integer. Recall that in Section 1.3.5, we have defined $W_n\Omega^q_{Y,log,t}$, with either $t =$ Zar or $t =$ ét, to be the abelian subsheaf of $W_n\Omega^q_{Y,t}$ étale locally generated by log forms. We will freely use $W_n\Omega^q_{L,log,t}$ for $W_n\Omega^q_{\operatorname{Spec} L,log,t}$ below.

Now let X be a separated scheme of finite type over k of dimension d. Define the Gersten complex $\widetilde{\nu}_{n,X,t}$, in the topology $t = \mathrm{Zar}$ or $t = \text{ét}$, to be the complex of t-sheaves isomorphic to $C^M_{X,t}/p^n$ via the Bloch-Gabber-Kato isomorphism [4, 2.8]:

$$(4.0.1)\quad 0 \to \bigoplus_{x\in X_{(d)}} \iota_{x,*}W_n\Omega^d_{k(x),log,t} \to \cdots \to \bigoplus_{x\in X_{(1)}} \iota_{x*}W_n\Omega^1_{k(x),log} \\ \to \bigoplus_{x\in X_{(0)}} \iota_{x,*}W_n\Omega^0_{k(x),log,t} \to 0.$$

Here $\iota_x : \operatorname{Spec} k(x) \to X$ is the natural map. We will still denote by ∂ the reduction of the tame symbol $\partial \bmod p^n$ (cf. (3.0.2), (3.0.4)), but denote by tr the reduction of Milnor's norm Nm mod p^n (cf. (3.0.3), (3.0.5)). The reason for the later notation will be clear from Lemma 5.3. As part of the convention,

$$\widetilde{\nu}^i_{n,X,t} = \bigoplus_{x\in X_{(-i)}} \iota_{x*}W_n\Omega^{-i}_{k(x),log,t},$$

i.e. $\widetilde{\nu}_{n,X}$ is concentrated in degrees

$$[-d, 0].$$

Proposition 4.1. *Let $i : Z \hookrightarrow X$ be a closed immersion with $j : U \hookrightarrow X$ its open complement. We have the following short exact sequence for $t = \mathrm{Zar}$:*

$$0 \longrightarrow i_*\widetilde{\nu}_{n,Z,\mathrm{Zar}} \longrightarrow \widetilde{\nu}_{n,X,\mathrm{Zar}} \longrightarrow j_*\widetilde{\nu}_{n,U,\mathrm{Zar}} \longrightarrow 0.$$

For $t = \text{ét}$, one has the localization triangle

$$i_*\widetilde{\nu}_{n,Z,\text{ét}} \to \widetilde{\nu}_{n,X,\text{ét}} \to Rj_*\widetilde{\nu}_{n,U,\text{ét}} \xrightarrow{+1} .$$

Proof. $\widetilde{\nu}_{n,X,\mathrm{Zar}}$ is a complex of flasque sheaves (therefore $Rj_*(\widetilde{\nu}_{n,X,\mathrm{Zar}}) = j_*\widetilde{\nu}_{n,X,\mathrm{Zar}}$), and one has the sequence being short exact in this case. If $t = \text{ét}$, the purity theorem holds [35, Corollary on p.130], i.e., $i_*\widetilde{\nu}_{n,Z,\text{ét}} = \underline{\Gamma}_Z(\widetilde{\nu}_{n,X,\text{ét}}) \xrightarrow{\simeq} R\underline{\Gamma}_Z(\widetilde{\nu}_{n,X,\text{ét}})$. We are done with the help of the distinguished triangle (1.6.3) in the étale topology. ☐

Functoriality of $\widetilde{\nu}_{n,X,t}$ is the same as that of $C^M_{X,t}$ via $d\log$. We omit the statement.

Part 2. The maps

5. Construction of the chain map $\zeta_{n,X,log,t} : C^M_{X,t} \to K_{n,X,log,t}$

5.1. **Construction of the chain map $\zeta_{n,X,t} : C^M_{X,t} \to K_{n,X,t}$.** Let $x \in X_{(q)}$ be a dimension q point. $\iota_x : \operatorname{Spec} k(x) \to X$ is the canonical map and $i_x : \overline{\{x\}} \hookrightarrow X$ the closed immersion. At degree $i = -q$, and over a point x, we define the degree i map to be $\zeta^i_{n,X,t} := \sum_{x\in X_{(q)}} \zeta^i_{n,x,t}$, with

$$(5.1.1)\quad \zeta^i_{n,x,t} : (W_n\iota_x)_*\mathcal{K}^M_{x,q,t} \xrightarrow{d\log} (W_n\iota_x)_*W_n\Omega^q_{k(x),log,t} \subset (W_n\iota_x)_*W_n\Omega^q_{k(x),t} \\ = (W_ni_x)_*K^i_{n,\overline{\{x\}},t} \xrightarrow{(-1)^i \operatorname{Tr}_{W_ni_x}} K^i_{n,X,t}.$$

We will use freely the notation $\zeta^i_{n,X,t}$ with some of its subscript or superscript dropped.

It is worth noticing that all the maps of étale sheaves involved here are given by the sheafification of the respective Zariski maps on the étale presheaf level. So to check commutativity of a composition of such maps between étale sheaves, it suffices to check on the $t = \mathrm{Zar}$ level. Keeping the convention as before, we usually omit the subscript Zar if we are working with the Zariski topology.

Proposition 5.1. *Let X be a separated scheme of finite type over k with k being a perfect field of characteristic $p > 0$. For $t = \mathrm{Zar}$ and $t = \text{ét}$, the map*

$$\zeta_{n,X,t} : C^M_{X,t} \to K_{n,X,t},$$

as defined termwise in (5.1.1), *is a chain map of complexes of sheaves on the site $(W_nX)_t$.*

Note that we have a canonical identification $(W_nX)_t = X_t$ for both $t = \mathrm{Zar}$ and $t = \text{ét}$. We use $(W_nX)_t$ just for the convenience of describing the $W_n\mathcal{O}_X$-structure of residual complexes appearing later.

Proof. To check $\zeta_{n,X,t}$ is a map of complexes, it suffices to check that the diagram

$$\begin{tikzcd}
(C^M_{X,t})^i \arrow[r, "d^M_X"] \arrow[d, "\zeta^i_{n,X,t}"] & (C^M_{X,t})^{i+1} \arrow[d, "\zeta^{i+1}_{n,X,t}"] \\
(K_{n,X,t})^i \arrow[r, "d_X"] & (K^{i+1}_{n,X,t})^{i+1}
\end{tikzcd}$$

commutes for $t = \mathrm{Zar}$. To this end, it suffices to show: for each $x \in X_{(q)}$, and $y \in X_{(q-1)}$ which is a specialization of x, the diagram

(5.1.2)
$$\begin{tikzcd}
(W_n\iota_x)_*\mathcal{K}^M_{x,q} \arrow[r, "(d^M_X)^x_y"] \arrow[dd, "\zeta_{n,x}"] & (W_n\iota_y)_*\mathcal{K}^M_{y,q-1} \arrow[d, "\zeta_{n,y}"] \\
 & (W_ni_{y,x})_*K^{-q+1}_{n,\overline{\{y\}}} \arrow[d, "-\mathrm{Tr}_{W_ni_{y,x}}"] \\
K^{-q}_{n,\overline{\{x\}}} \arrow[r, "d_{\overline{\{x\}}}"] & K^{-q+1}_{n,\overline{\{x\}}}
\end{tikzcd}$$

commutes. Here $i_{y,x} : \overline{\{y\}} \hookrightarrow \overline{\{x\}}$ denotes the canonical closed immersion.

Since the definition of the differential maps in C^M_X involves normalization, consider the normalization $\rho : X' \to \overline{\{x\}}$ of $\overline{\{x\}}$, and form the cartesian square

$$\begin{tikzcd}
\overline{\{y\}} \times_{\overline{\{x\}}} X' \arrow[r, hook] \arrow[d] & X' = \overline{\{x'\}} \arrow[d, "\rho"] \\
\overline{\{y\}} \arrow[r, hook, "i_{y,x}"] & \overline{\{x\}}.
\end{tikzcd}$$

Denote the generic point of X' by x'. Suppose y' is one of the generic points of the irreducible components of $\overline{\{y\}} \times_{\overline{\{x\}}} X'$, and denote by Y' the irreducible component corresponding to y'. In particular, y' is a codimension 1 point in the normal scheme X', thus is regular. Because the base field k is perfect, y' is also a smooth point in X'. According to Remark 1.4, the degree $[-q,-q+1]$ terms of $K_{n,X'}$ are of the form

$$(W_n\iota_{x'})_*H^0_{x'}(W_n\Omega^q_{X'}) \xrightarrow{\delta} \bigoplus_{y'\in X'_{(q-1)}} (W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^q_{X'}) \to \dots,$$

where δ denotes the differential map of the residual complex $K_{n,X}$. After localizing at a single $y'\in X'^{(1)}$ in the Zariski sense, one gets

$$(W_n\iota_{x'})_*H^0_{x'}(W_n\Omega^q_{X'}) \xrightarrow{\delta_{y'}} (W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^q_{X'}) \to \dots.$$

Consider the following diagrams. Write $\iota_{x'}:\operatorname{Spec} k(x')\hookrightarrow X'$, $\iota_{y'}:\operatorname{Spec} k(y')\hookrightarrow X'$ the inclusions, $i_{y',x'}: Y'=\overline{\{y'\}}\hookrightarrow X'$ the closed immersion, we have a diagram

(5.1.3)
$$\begin{tikzcd}
(W_n\iota_{x'})_*\mathcal{K}^M_{x',q} \arrow[r, "\partial^{x'}_{y'}"] \arrow[d, "d\log"] & (W_n\iota_{y'})_*\mathcal{K}^M_{y',q-1} \arrow[d, "d\log"] \\
(W_n\iota_{x'})_*W_n\Omega^q_{k(x')} \arrow[dr, "\delta_{y'}"] & (W_n\iota_{y'})_*W_n\Omega^{q-1}_{k(y')} \arrow[d, "-\operatorname{Tr}_{W_n(i_{y',x'})}"] \\
 & (W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^q_{X'}).
\end{tikzcd}$$

For any $y'\in\rho^{-1}(y)\subset X'^{(1)}$, we have a diagram

(5.1.4)
$$\begin{tikzcd}
(W_n\rho)_*\mathcal{K}^M_{y',q-1} \arrow[r, "\mathrm{Nm}_{y'/y}"] \arrow[d, "d\log"] & \mathcal{K}^M_{y,q-1} \arrow[d, "d\log"] \\
(W_n\rho)_*W_n\Omega^{q-1}_{k(y')} \arrow[r, "\operatorname{Tr}_{W_n\rho}"] & W_n\Omega^{q-1}_{k(y)}.
\end{tikzcd}$$

Write $i_{y',x'}: Y'=\overline{\{y'\}}\hookrightarrow X'$, $i_{y,x}:\overline{\{y\}}\hookrightarrow\overline{\{x\}}$, we have a diagram

(5.1.5)
$$\begin{tikzcd}
(W_n\rho)_*(W_n\iota_{y'})_*W_n\Omega^{q-1}_{k(y')} \arrow[r, "\operatorname{Tr}_{W_n\rho}"] \arrow[d, "\operatorname{Tr}_{W_n(i_{y',x'})}"'] & (W_n\iota_y)_*W_n\Omega^{q-1}_{k(y)} \arrow[d, "\operatorname{Tr}_{W_n(i_{y,x})}"] \\
(W_n\rho)_*(W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^q_{X'}) \arrow[r, "\operatorname{Tr}_{W_n\rho}"] & K^{-(q-1)}_{n,\overline{\{x\}}},
\end{tikzcd}$$

and a diagram

(5.1.6)
$$\begin{tikzcd}
(W_n\iota_{x'})_*W_n\Omega^q_{k(x')} \arrow[r, "d_{X'}=\sum\delta_{y'}"] \arrow[d, "\operatorname{Tr}_{W_n\rho}"', "\simeq"] & \bigoplus_{y'\in\rho^{-1}(y)}(W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^{q-1}_{X'}) \arrow[d, "\operatorname{Tr}_{W_n\rho}"] \\
K^{-q}_{n,\overline{\{x\}}} \arrow[r, "d_{\overline{\{x\}}}"] & K^{-(q-1)}_{n,\overline{\{x\}}}.
\end{tikzcd}$$

All the trace maps above are trace maps of residual complexes at a certain degree. (5.1.5) is the degree $q-1$ part of the diagram

$$\begin{tikzcd}
(W_n\rho)_*(W_n i_{y',x'})_* K_{n,Y'} \arrow[r, "\mathrm{Tr}_{W_n\rho}"] \arrow[d, "\mathrm{Tr}_{W_n(i_{y',x'})}"'] & (W_n i_{y,x})_* K_{n,\overline{\{y\}}} \arrow[d, "\mathrm{Tr}_{W_n(i_{y,x})}"] \\
(W_n\rho)_* K_{n,X'} \arrow[r, "\mathrm{Tr}_{W_n\rho}"] & K_{n,\overline{\{x\}}}
\end{tikzcd}$$

(the trace map on top is the trace map of the restriction of $W_n\rho$ to W_nY'), and thus is commutative by the functoriality of the Grothendieck trace map with respect to composition of morphisms (Proposition 1.2(4)). (5.1.6) is simply the degree $-q$ to $-q+1$ terms of the trace map $\mathrm{Tr}_{W_n\rho} : (W_n\rho)_*K_{n,X'} \to K_{n,\overline{\{x\}}}$, thus is also commutative. It remains to check the commutativity of (5.1.3) and (5.1.4). And these are Lemma 5.2 and Lemma 5.3.

One notices that diagram (5.1.2) decomposes into the four diagrams (5.1.3)-(5.1.6):

$$\begin{tikzcd}
(W_n\rho)_*(W_n\iota_{x'})_*\mathcal{K}^M_{x',q} \arrow[r, "\oplus_{y'|y}\partial^{x'}_{y'}"] \arrow[d, "d\log"] \arrow[dr, phantom, "(5.1.3)"] & \bigoplus_{y'|y}(W_n\rho)_*(W_n\iota_{y'})_*\mathcal{K}^M_{y',q-1} \arrow[r, "\sum_{y'/y}\mathrm{Nm}_{y'/y}"] \arrow[d, "d\log"] \arrow[dr, phantom, "(5.1.4)"] & (W_n\iota_y)_*\mathcal{K}^M_{y,q-1} \arrow[d, "d\log"] \\
(W_n\rho)_*(W_n\iota_{x'})_*W_n\Omega^q_{k(x')} \arrow[dd, "\mathrm{Tr}_{W_n\rho}"] \arrow[dr, "\oplus_{y'|y}\delta_{y'}"] & \bigoplus_{y'|y}(W_n\rho)_*(W_n\iota_{y'})_*W_n\Omega^{q-1}_{k(y')} \arrow[r, "\mathrm{Tr}_{W_n\rho}"] \arrow[d, "\oplus_{y'|y} -\mathrm{Tr}_{W_n(i_{y',x'})}"] \arrow[dr, phantom, "(5.1.5)"] & (W_n\iota_y)_*W_n\Omega^{q-1}_{k(y)} \arrow[d, "-\mathrm{Tr}_{W_n(i_{y,x})}"] \\
 & \bigoplus_{y'|y}(W_n\rho)_*(W_n\iota_{y'})_*H^1_{y'}(W_n\Omega^q_{X'}) \arrow[r, "\mathrm{Tr}_{W_n\rho}"] \arrow[d, phantom, "(5.1.6)"] & K^{-(q-1)}_{n,\overline{\{x\}}} \arrow[d, equal] \\
K^{-q}_{n,\overline{\{x\}}} \arrow[rr, "d_{\overline{\{x\}}}"] & & K^{-(q-1)}_{n,\overline{\{x\}}}
\end{tikzcd}$$

Here by symbol $y'|y$ we mean that $y' \in \rho^{-1}(y)$. Notice that we have added a minus sign to both vertical arrows of (5.1.5) in the corresponding square above, but this does not affect its commutativity. Since one can canonically identify

$$(W_n\rho)_*(W_n\iota_{x'})_*\mathcal{K}^M_{x',q} \quad \text{with} \quad (W_n\iota_x)_*\mathcal{K}^M_{x,q},$$

to show the commutativity of the diagram (5.1.2), it only remains to show Lemma 5.2 and Lemma 5.3. $\square$

Lemma 5.2. *For an integral normal scheme X', with $x' \in X'$ being the generic point and $y' \in X'^{(1)}$ being a codimension 1 point, the diagram* (5.1.3) *is commutative.*

Proof. Given a $y' \in X'^{(1)}$ lying over y, the abelian group $K^M_q(x')$ is generated by

$$\{\pi', u_1, \ldots, u_{q-1}\} \text{ and } \{v_1, \ldots, v_{q-1}, v_q\},$$

where $u_1, \ldots, u_{q-1}, v_1, \ldots, v_{q-1}, v_q \in \mathcal{O}^*_{X',y'}$, and π' is a chosen uniformizer of the discrete valuation ring $\mathcal{O}_{X',y'}$. It suffices to check the commutativity for these generators.

In the first case, the left-bottom composition gives

$$\begin{aligned}(\delta_{y'} \circ d\log)(\{\pi', u_1, \dots, u_{q-1}\}) &= \delta_{y'}(d\log[\pi']_n d\log[u_1]_n \dots d\log[u_{q-1}]_n) \\ &= \begin{bmatrix} d[\pi']_n d\log[u_1]_n \dots d\log[u_{q-1}]_n \\ [\pi']_n \end{bmatrix}.\end{aligned}$$

The last equality above is given by [6, A.1.2]. Here we have used the fact that $[\pi']$ is a regular element in W_nX', since π' is regular in X'. The top-right composition gives

$$\begin{aligned}&(-\operatorname{Tr}_{W_n(i_{y',x'})} \circ d\log \circ \partial_{y'}^{x'})(\{\pi', u_1, \dots, u_{q-1}\}) \\ &\qquad = (-\operatorname{Tr}_{W_n(i_{y',x'})} \circ d\log)\{\overline{u}_1, \dots, \overline{u}_{q-1}\} \\ &\qquad = -\operatorname{Tr}_{W_n(i_{y',x'})}(d\log[\overline{u}_1]_n \dots d\log[\overline{u}_{q-1}]_n) \\ &\qquad = \begin{bmatrix} d[\pi']_n d\log[\overline{u}_1]_n \dots d\log[\overline{u}_{q-1}]_n \\ [\pi']_n \end{bmatrix}.\end{aligned}$$

The last equality is given by [6, A.2.12]. So the diagram (5.1.3) is commutative in this case.

In the second case, since $\partial_{y'}^{x'}(\{v_1, \dots, v_q\}) = 0$, we need to check the left-bottom composite also gives zero. In fact,

$$\begin{aligned}(\delta_{y'} \circ d\log)(\{v_1, \dots, v_q\}) &= \delta_{y'}(d\log[v_1]_n \dots d\log[v_q]_n) \\ &= \begin{bmatrix} [\pi']_n \cdot d\log[v_1]_n \dots d\log[v_q]_n \\ [\pi']_n \end{bmatrix} \\ &= 0.\end{aligned}$$

The second equality is due to [6, A.1.2]. The last equality is because, in a small neighborhood V of y', the element $[\pi']_n \cdot d\log[v_1]_n \dots d\log[v_q]_n \in W_n\Omega_V^q$ lies in the $W_n\mathcal{O}_V$-submodule $[\pi']_n \cdot W_n\Omega_V^q$. $\square$

Lemma 5.3 (Compatibility of Milnor norm and Grothendieck trace)**.** *Let F/E be a finite field extension with both fields E and F being of transcendence degree $q-1$ over k. Suppose there exists a finite morphism g between integral separated finite type k-schemes, such that F is the function field of the source of g and E is the function field of the target of g, and the field extension F/E is induced via the map g. Then the following diagram commutes*

$$\begin{array}{ccc} K_{q-1}^M(F) & \xrightarrow{\operatorname{Nm}_{F/E}} & K_{q-1}^M(E) \\ \downarrow{\scriptstyle d\log} & & \downarrow{\scriptstyle d\log} \\ W_n\Omega_F^{q-1} & \xrightarrow{\operatorname{Tr}_{W_n g}} & W_n\Omega_E^{q-1}. \end{array}$$

Here the norm map $\operatorname{Nm}_{F/E}$ denotes the norm map from Milnor K-theory, and $\operatorname{Tr}_{W_n g}$ denotes the Grothendieck trace map associated to the finite morphism g.

Remark 5.4. The compatibility of the trace map with the norm and the pushforward of cycles in various settings has been known by the experts, and many definitions and properties of the trace map in the literature reflect this viewpoint. But since we have not found a proof of the compatibility of the Milnor norm with the trace map defined via the Grothendieck duality theory, we include a proof here.

Proof. We start the proof by some reductions. Since both $\mathrm{Nm}_{F/E}$ and $\mathrm{Tr}_{F/E}$ are independent of the choice of towers of simple field extensions, without loss of generality, one can suppose F is a finite simple field extension over E. Now $F = E(a) = \frac{E[T]}{f(T)}$ for some monic irreducible polynomial $f(T) \in E[T]$ with $a \in F$ being one of its roots. This realizes $\operatorname{Spec} F$ as an F-valued point P of $\mathbf{P}^1_E$, namely,

$$\begin{tikzcd}
\operatorname{Spec} F = P \arrow[r, hook, "i_P"] \arrow[dr, "g"'] & \mathbf{P}^1_E \arrow[d, "\pi"] \\
& \operatorname{Spec} E.
\end{tikzcd}$$

All the three morphisms on above are morphisms of finite type (although not between schemes of finite type over k), so it makes sense to talk about the associated trace maps for residual complexes. But for the particular residual complexes we are interested in, we need to enlarge the schemes involved to schemes of finite type over k, while preserving the morphism classes (e.g., closed immersion, smooth morphism, etc.) of the morphisms between them.

To this end, take Y to be any separated smooth connected scheme of finite type over k with E being the function field. Since $\mathbf{P}^1_E$ is the generic fiber of $Y \times_k \mathbf{P}^1_k$, by possibly shrinking Y to an affine neighborhood $\operatorname{Spec} B$ of $pr_1(P)$ (here $pr_1 : Y \times_k \mathbf{P}^1_k \to Y$ is the first projection map) one can extend the above diagram to the following:

$$\begin{tikzcd}
\operatorname{Spec} F \in W \arrow[r, hook, "i_W"] \arrow[dr, "g"'] & \mathbf{P}^1_Y \arrow[d, "\pi"] \\
& Y = \operatorname{Spec} B \ni \operatorname{Spec} E.
\end{tikzcd}$$

Here $W := \overline{\{P\}}^{\mathbf{P}^1_Y}$ is the closure of the point P in $\mathbf{P}^1_Y$. This is a commutative diagram of finite type k-schemes. In particular, it makes sense to talk about the residual complexes $K_{n,Y}, K_{n,W}$ and $K_{n,\mathbf{P}^1_Y}$.

Now it remains to show the commutativity of the following diagram

$$\begin{tikzcd}
K^M_{q-1}(E(a)) \arrow[r, "\mathrm{Nm}_{E(a)/E}"] \arrow[d, "d\log"] & K^M_{q-1}(E) \arrow[d, "d\log"] \\
W_n\Omega^{q-1}_{E(a)} \arrow[r, "\mathrm{Tr}_{W_n g}"] & W_n\Omega^{q-1}_E,
\end{tikzcd} \tag{5.1.7}$$

where $\mathrm{Tr}_{W_n g}$ denotes the trace map for residual complexes $\mathrm{Tr}_{W_n g} : (W_n g)_* K_{n,W} \to K_{n,Y}$ at degree $-(q-1)$.

We do induction on $[E(a) : E]$. If $[E(a) : E] = 1$, then both the Grothendieck trace $\mathrm{Tr}_{W_n g} : W_n\Omega^{q-1}_{E(a)/k} \to W_n\Omega^{q-1}_{E/k}$ and the norm map $\mathrm{Nm}_{E(a)/E} : K^M_{q-1}(E(a)) \to K^M_{q-1}(E)$ are the identity, therefore the claim holds. Now the induction step. Suppose the diagram (5.1.7) commutes for $[E(a) : E] \leq r - 1$. We will need to prove the commutativity for $[E(a) : E] = r$.

First note that $\mathrm{Tr}_{W_n g} : (W_n g)_* K_{n,W} \to K_{n,Y}$ naturally decomposes into

$$(W_n g)_* K_{n,W} \xrightarrow{(W_n\pi)_* \mathrm{Tr}_{W_n i_P W}} (W_n\pi)_* K_{n,\mathbf{P}^1_Y} \xrightarrow{\mathrm{Tr}_{W_n\pi}} K_{n,Y}, \tag{5.1.8}$$

by Proposition 1.2(4). $H^1_P(W_n\Omega^q_{\mathbf{P}^1_Y})$ is a direct summand of the degree $-(q-1)$ part of $K_{n,\mathbf{P}^1_Y}$. One can canonically identify

$$H^1_P(W_n\Omega^q_{\mathbf{P}^1_Y}) = H^1_P(W_n\Omega^q_{\mathbf{P}^1_E}), \tag{5.1.9}$$

via pulling back along the natural map $\mathbf{P}^1_E \hookrightarrow \mathbf{P}^1_Y$. Thus on degree $-(q-1)$ and at the point P, the map (5.1.8) is canonically identified with

$$W_n\Omega^{q-1}_{E(a)} \xrightarrow{\mathrm{Tr}_{W_n i_P W}} H^1_P(W_n\Omega^q_{\mathbf{P}^1_E}) \xrightarrow{\mathrm{Tr}_{W_n\pi}} W_n\Omega^{q-1}_E.$$

Consider the diagram

$$\begin{tikzcd}
K^M_q(E(T)) \arrow[r, "\partial_P"] \arrow[dd, "d\log"] \arrow[dr, phantom, "(-1)"] & K^M_{q-1}(E(a)) \arrow[r, "\mathrm{Nm}_{E(a)/E}"] \arrow[d, "d\log"] & K^M_{q-1}(E) \arrow[dd, "d\log"] \\
 & W_n\Omega^{q-1}_{E(a)} \arrow[d, "\mathrm{Tr}_{W_n i_P W}"] \arrow[dr, "\mathrm{Tr}_{W_n g}"] & \\
W_n\Omega^q_{E(T)} \arrow[r, "\delta_P"] & H^1_P(W_n\Omega^q_{\mathbf{P}^1_E}) \arrow[r, "\mathrm{Tr}_{W_n\pi}"] & W_n\Omega^{q-1}_E.
\end{tikzcd}$$

We have used the identification (5.1.9) in this diagram. We have seen that the left square is commutative up to sign -1, as a special case of Lemma 5.2 (i.e. take normal scheme $X' = \mathbf{P}^1_E$ and $y' := P = \operatorname{Spec} F$). Since ∂_P is surjective, to show the commutativity of the trapezoid on the right, it suffices to show that the composite square is commutative up to -1. For any element

$$s := \{s_1, \ldots, s_{q-1}\} \in K^M_{q-1}(E(a)),$$

one can always find a lift

$$\widetilde{s} := \{f, \widetilde{s}_1, \ldots, \widetilde{s}_{q-1}\} \in K^M_q(E(T)),$$

such that each of the $s_i = s_i(T)$ is a polynomial of degree $\leq r-1$ (e.g. decompose $E(a)$ as an r-dimensional E-vector space $E(a) = \bigoplus_{j=0}^{r-1} Ea^j$ and suppose $s_i = \sum_{j=0}^{r-1} b_{i,j}a^j$ with $b_{i,j} \in E$, then $\widetilde{s}_i = \widetilde{s}_i(T) = \sum_{j=0}^{r-1} b_{i,j}T^j$ satisfies the condition), and $\partial_P(\widetilde{s}) = s$. Denote by

$$y_{i,1}, \ldots, y_{i,a_i} \quad (1 \leq i \leq q-1)$$

the closed points of $\mathbf{P}^1_E$ corresponding to the irreducible factors of the polynomials $\widetilde{s}_1, \ldots, \widetilde{s}_{q-1}$. Note that the local section $\widetilde{s}_{i,l}$ cutting out $y_{i,l}$ is by definition an irreducible factor of $\widetilde{s}_i$, and therefore $\deg \widetilde{s}_{i,l} < r$ for all i and all l.

We claim that

$$\sum_{y \in (\mathbf{P}^1_E)_{(0)}} (\mathrm{Tr}_{W_n\pi})_y \circ \delta_y = 0 : W_n\Omega^q_{E(T)/k} \to W_n\Omega^{q-1}_{E/k}. \tag{5.1.10}$$

In fact,

$$0 \to W_n\Omega^q_{\mathbf{P}^1_E} \to W_n\Omega^q_{E(T)} \to \bigoplus_{y \in (\mathbf{P}^1_E)_{(0)}} (W_n\iota_y)_* H^1_y(W_n\Omega^q_{\mathbf{P}^1_E}) \to 0 \tag{5.1.11}$$

is an exact sequence [7, 1.5.9], where $\iota_y : y \hookrightarrow \mathbf{P}^1_E$ is the natural inclusion of the point y. Taking the long exact sequence with respect to the global section functor, one arrives at the following diagram with the row being a complex

$$\begin{tikzcd} W_n\Omega^q_{E(T)} \arrow[r, "\delta"] & \bigoplus_{y\in(\mathbf{P}^1_E)_{(0)}} H^1_y(W_n\Omega^q_{\mathbf{P}^1_E}) \arrow[r] \arrow[dr, "\sum_y(\mathrm{Tr}_{W_n\pi})_y"'] & H^1(\mathbf{P}^1_E, W_n\Omega^q_{\mathbf{P}^1_E}) \arrow[d, "\mathrm{Tr}_{W_n\pi}"] \\ & & W_n\Omega^{q-1}_{E/k}. \end{tikzcd}$$

The trace maps on left of the above are induced from the degree 0 part of $\mathrm{Tr}_{W_n\pi} : (W_n\pi)_*K_{n,\mathbf{P}^1_Y} \to K_{n,Y}$. The trace map on the right of the above is induced also by $\mathrm{Tr}_{W_n\pi} : (W_n\pi)_*K_{n,\mathbf{P}^1_Y} \to K_{n,Y}$, while the global cohomology group is calculated via (5.1.11), i.e., one uses the last two terms of (5.1.11) as an injective resolution of the sheaf $W_n\Omega^q_{\mathbf{P}^1_E}$, and then $\mathrm{Tr}_{W_n\pi} : (W_n\pi)_*K_{n,\mathbf{P}^1_Y} \to K_{n,Y}$ induces the map of complexes (sitting in degrees $[-1,0]$) on global sections, and then the map of cohomologies on degree 0 gives our trace map $H^1(\mathbf{P}^1_E, W_n\Omega^q_E) \to W_n\Omega^{q-1}_E$ on the right. From the construction of these trace maps, the above diagram is by definition commutative. Therefore (5.1.10) holds.

One notices that $\delta_y \circ d\log(\widetilde{s}) = 0$ unless $y \in \{p, y_{1,1}, \ldots, y_{q-1,a_{q-1}}, \infty\}$. Now we calculate

$$\begin{aligned}
&(\mathrm{Tr}_{W_ng} \circ d\log)(s) \\
&\quad = (\mathrm{Tr}_{W_ng} \circ d\log \circ \partial_P)(\widetilde{s}) \\
&\quad = -((\mathrm{Tr}_{W_n\pi})_P \circ \delta_P \circ d\log)(\widetilde{s}) \quad \text{(Lemma 5.2)} \\
&\quad = \sum_{y\in\{y_{1,1},\ldots,y_{q-1,a_{q-1}},\infty\}} ((\mathrm{Tr}_{W_n\pi})_y \circ \delta_y \circ d\log)(\widetilde{s}) \quad (5.1.10) \\
&\quad = -\sum_{y\in\{y_{1,1},\ldots,y_{q-1,a_{q-1}},\infty\}} (d\log \circ \mathrm{Nm}_{E(k(y))/E} \circ \partial_y)(\widetilde{s}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(induction hypothesis)} \\
&\quad = (d\log \circ \mathrm{Nm}_{E(a)/E} \circ \partial_P)(\widetilde{s}) \quad [37, 2.2 \text{ (RC)}] \\
&\quad = (d\log \circ \mathrm{Nm}_{E(a)/E})(s).
\end{aligned}$$

This finishes the induction. □

5.2. **Functoriality of $\zeta_{n,X,t} : C^M_{X,t} \to K_{n,X,t}$.** Let k denote a perfect field of positive characteristic p.

Proposition 5.5 (Proper pushforward)**.** *ζ is compatible with proper pushforward. I.e., for $f : X \to Y$ a proper map, the following diagram is commutative*

$$\begin{tikzcd} (W_nf)_*C^M_{X,t} \arrow[r, "\zeta_{n,X,t}"] \arrow[d, "f_*"] & (W_nf)_*K_{n,X,t} \arrow[d, "f_*"] \\ C^M_{Y,t} \arrow[r, "\zeta_{n,Y,t}"] & K_{n,Y,t}. \end{tikzcd}$$

Here f_* on the left denotes the pushforward map for Kato's complex of Milnor K-theory (cf. Section 3), and f_* on the right denotes the Grothendieck trace map $\mathrm{Tr}_{W_nf,t}$ *for residual complexes.*

Proof. We only need to prove the proposition for $t = \mathrm{Zar}$ and for degree $i \in [-d, 0]$. Then by the very definition of the ζ map and the compatibility of the trace map with morphism compositions Proposition 1.2(4), it suffices to check the commutativity at points $x \in X_{(q)}$, $y \in Y_{(q)}$, where $q = -i$:

$$\begin{tikzcd} K_q^M(x) \arrow[r, "d\log"] \arrow[d, "f_*"'] & W_n\Omega^q_{k(x)} \arrow[d, "f_*"] \\ K_q^M(y) \arrow[r, "d\log"] & W_n\Omega^q_{k(y)}. \end{tikzcd}$$

(1) If $y \neq f(x)$, both pushforward maps are zero maps, therefore we have the desired commutativity.
(2) If $y = f(x)$, by definition of ζ and the pushforward maps, we need to show the commutativity of the following diagram for the finite field extension $k(y) \subset k(x)$

$$\begin{tikzcd} K_q^M(x) \arrow[r, "d\log"] \arrow[d, "\mathrm{Nm}_{k(x)/k(y)}"'] & W_n\Omega^q_{k(x)} \arrow[d, "\mathrm{Tr}_{W_nf}"] \\ K_q^M(y) \arrow[r, "d\log"] & W_n\Omega^q_{k(y)}. \end{tikzcd}$$

This is precisely Lemma 5.3. $\square$

Proposition 5.6 (Étale pullback)**.** *ζ is compatible with étale pullbacks. I.e., for $f : X \to Y$ an étale morphism, the following diagram is commutative*

$$\begin{tikzcd} C^M_{Y,t} \arrow[r, "\zeta_{n,Y,t}"] \arrow[d, "f^*"] & K_{n,Y,t} \arrow[d, "f^*"] \\ (W_nf)_*C^M_{X,t} \arrow[r, "\zeta_{n,X,t}"] & (W_nf)_*K_{n,X,t}. \end{tikzcd}$$

Here f^* on the left denotes the pullback map for Kato's complex of Milnor K-theory (cf. Section 3), and f^* on the right denotes the pullback map for residual complexes (1.5.1).

Proof. It suffices to prove the proposition for $t = \mathrm{Zar}$. Take $y \in Y_{(q)}$. Consider the cartesian diagram

$$\begin{tikzcd} X \times_Y \overline{\{y\}} =: W \arrow[r, "f|_W"] \arrow[d, hook, "i_W"] & \overline{\{y\}} \arrow[d, hook, "i_y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Then the desired diagram at point y decomposes in the following way at degree $-q$:

$$\begin{tikzcd} K^M_q(y) \arrow[r, "d\log"] \arrow[d, "f^*"'] & W_n\Omega^q_{k(y)} = K^{-q}_{n,\overline{\{y\}}} \arrow[r, "\mathrm{Tr}_{W_n i_y}"] \arrow[d, "(f|_W)^*"] & K^{-q}_{n,Y} \arrow[d, "f^*"] \\ \bigoplus_{x\in W_{(q)}} K^M_q(x) \arrow[r, "d\log"] & \bigoplus_{x\in W_{(q)}} W_n\Omega^q_{k(x)} = K^{-q}_{n,W} \arrow[r, "\mathrm{Tr}_{W_n i_W}"] & K^{-q}_{n,X}. \end{tikzcd}$$

The left square commutes because both f^* and $(f\mid_W)^*$ are induced by the natural map $f^*:\mathcal{O}_Y\to f_*\mathcal{O}_X$. The right square commutes due to Lemma 1.24. □

5.3. **Extend to $K_{n,X,log,t}$.** Recall the complex $K_{n,X,log,t} := \mathrm{Cone}(K_{n,X,t} \xrightarrow{C'_t-1} K_{n,X,t})[-1]$, i.e.,

$$K^i_{n,X,log,t} = K^i_{n,X,t} \oplus K^{i-1}_{n,X,t}.$$

Notice that

$$K_{n,X,t} \to K_{n,X,log,t}, \quad a \mapsto (a,0) \tag{5.3.1}$$

is not a chain map. Nevertheless,

Proposition 5.7. *We keep the same assumptions as in Proposition* 5.1. *The chain map $\zeta_{n,X,t}: C^M_{X,t} \to K_{n,X,t}$ composed with* (5.3.1) *gives a chain map*

$$\zeta_{n,X,log,t} := (5.3.1)\circ\zeta_{n,X,t} : C^M_{X,t} \to K_{n,X,log,t}$$

of complexes of abelian sheaves on $(W_nX)_t$.

We will also use the shortened notation $\zeta_{log,t}$ for $\zeta_{n,X,log,t}$. If $t=\mathrm{Zar}$, the subscript Zar will also be omitted.

Proof. Given $x\in X_{(q)}$, we prove commutativity of the following diagram

$$\begin{tikzcd} \iota_{x*}\mathcal{K}^M_{x,q,t} \arrow[r, "d\log"] \arrow[d, equal] & \iota_{x*}W_n\Omega^q_{k(x),log,t} \arrow[r, "\mathrm{Tr}_{W_n i_x,t}"] \arrow[d, hook] & K^{-q}_{n,X,t} \arrow[r, "C'_{X,t}-1"] & K^{-q}_{n,X,t} \arrow[d, equal] \\ \iota_{x*}\mathcal{K}^M_{x,q,t} \arrow[r, "d\log"] & \iota_{x*}W_n\Omega^q_{k(x),t} \arrow[r, "C'_{\overline{\{x\}},t}-1"] & (i_{x,*}K_{n,\overline{\{x\}},t})^{-q} \arrow[r, "\mathrm{Tr}_{W_n i_x,t}"] & K^{-q}_{n,X,t}. \end{tikzcd}$$

The left square naturally commutes. The right square also commutes, because C' is compatible with the Grothendieck trace map $\mathrm{Tr}_{W_n i_x}$ (the proofs of Proposition 1.20 and Proposition 1.29 give the case for $t=\mathrm{Zar}$ and $t=\text{ét}$, respectively). Now because $C'_{\overline{\{x\}},t}-1: W_n\Omega^q_{k(x),t}\to W_n\Omega^q_{k(x),t}$, which is identified with $C_{\overline{\{x\}},t}-1$ as a result of Theorem 1.9 and Proposition 1.26, annihilates $W_n\Omega^q_{k(x),log,t}$, the composite of the second row is zero. Thus the composite of the first row is zero. This yields a unique chain map

$$\zeta_{n,X,log,t}: C^M_{X,t}\to K_{n,X,log,t},$$

i.e., on degree $i=-q$, we have $\zeta_{n,X,log,t}=\sum_{x\in X_{(q)}}\zeta_{n,x,log,t}$ with

$$\begin{aligned} \zeta^i_{n,x,log,t}: \mathcal{K}^M_{x,q,t} &\to K^i_{n,X,log,t} = K^i_{n,X,t}\oplus K^{i-1}_{n,x,t}, \\ s=\{s_1\dots,s_q\} &\mapsto (\zeta^i_{n,X,t}(s),0). \end{aligned}$$

□

As a direct corollary of Proposition 5.5 and Proposition 5.6, one has Proposition 5.8.

Proposition 5.8 (Functoriality)**.**

(1) $\zeta_{log,t}$ *is compatible with proper pushforward. I.e., for* $f : X \to Y$ *a proper map, the following diagram of complexes is commutative*

$$\begin{tikzcd}
(W_nf)_*C^M_{X,t} \arrow[r, "\zeta_{n,X,log,t}"] \arrow[d, "f_*"] & (W_nf)_*K_{n,X,log,t} \arrow[d, "f_*"] \\
C^M_{Y,t} \arrow[r, "\zeta_{n,Y,log,t}"] & K_{n,Y,log,t}.
\end{tikzcd}$$

Here f_* *on the left denotes the pushforward map for Kato's complex of Milnor* K*-theory (cf. Section* 3*), and* f_* *on the right denotes* $\mathrm{Tr}_{W_nf,log,t}$ *as defined in Proposition* 1.20 *and Proposition* 1.29*.*

(2) $\zeta_{log,t}$ *is compatible with étale pullbacks. I.e., for* $f : X \to Y$ *an étale morphism, the following diagram of complexes is commutative*

$$\begin{tikzcd}
C^M_{Y,t} \arrow[r, "\zeta_{n,Y,log,t}"] \arrow[d, "f^*"] & K_{n,Y,log,t} \arrow[d, "f^*"] \\
(W_nf)_*C^M_{X,t} \arrow[r, "\zeta_{n,X,log,t}"] & (W_nf)_*K_{n,X,log,t}.
\end{tikzcd}$$

Here f^* *on the left denotes the pullback map for Kato's complex of Milnor* K*-theory (cf. Section* 3*), and* f^* *on the right denotes the pullback map defined in Proposition* 1.23*.*

5.4. **The map** $\overline{\zeta}_{n,X,log,t} : C^M_{X,t}/p^n \simeq \widetilde{\nu}_{n,X,t} \to K_{n,X,log,t}$ **is a quasi-isomorphism.** Since $\zeta_{n,X,t}$ is termwise defined via the $d\log$ map, it annihilates $p^nC^M_{X,t}$. Therefore $\zeta_{n,X,log,t}$ annihilates $p^nC^M_{X,t}$ as well, and it induces a chain map

$$\overline{\zeta}_{n,X,log,t} : C^M_{X,t}/p^n \to K_{n,X,log,t}.$$

Since the $d\log$ map induces an isomorphism of complexes $C^M_{X,t}/p^n \simeq \widetilde{\nu}_{n,X,t}$, to show $\overline{\zeta}_{n,X,log,t}$ is a quasi-isomorphism, it is equivalent to showing

$$\overline{\zeta}_{n,X,log,t} : \widetilde{\nu}_{n,X,t} \to K_{n,X,log,t}$$

is a quasi-isomorphism.

Lemma 5.9. *Suppose* X *is separated smooth over the perfect field* k *of characteristic* $p > 0$*. Then for any level* n*,*

$$\overline{\zeta}_{n,X,log,\text{ét}} : \widetilde{\nu}_{n,X,\text{ét}} \to K_{n,X,log,\text{ét}}$$

is a quasi-isomorphism. If we moreover have $k = \overline{k}$*, then*

$$\overline{\zeta}_{n,X,log,\mathrm{Zar}} : \widetilde{\nu}_{n,X,\mathrm{Zar}} \to K_{n,X,log,\mathrm{Zar}}$$

is also a quasi-isomorphism.

Proof. This is a local problem, thus it suffices to prove the statement for each connected component of X. Therefore we assume X is of pure dimension d over k. Then for any level n, we have a quasi-isomorphism [17, Cor 1.6]

$$W_n\Omega^d_{X,log,t}[d] \xrightarrow{\simeq} \widetilde{\nu}_{n,X,t}.$$

We also have

$$W_n\Omega^d_{X,log,\text{ét}}[d] \xrightarrow{\simeq} K_{n,X,log,\text{ét}} \quad \text{(by Proposition 1.28), and}$$
$$W_n\Omega^d_{X,log,\text{Zar}}[d] \xrightarrow{\simeq} K_{n,X,log,\text{Zar}} \quad \text{if } k=\overline{k} \text{ (by Proposition 1.19).}$$

On degree $-d$, we have a diagram

$$\begin{tikzcd}
\widetilde{\nu}^{-d}_{n,X,t} = \bigoplus_{x\in X^{(0)}} (W_n\iota_x)_*W_n\Omega^d_{k(x),log,t} \arrow[r, "\overline{\zeta}^{-d}_{n,x,log,t}"] & K^{-d}_{n,X,log,t} = \bigoplus_{x\in X^{(0)}} (W_n\iota_x)_*H^0_x(W_n\Omega^d_{X,t}) \\
W_n\Omega^d_{X,log,t} \arrow[u] \arrow[r, "(-1)^d"] & W_n\Omega^d_{X,log,t} \arrow[u]
\end{tikzcd}$$

which is naturally commutative, due to the definition of $\overline{\zeta}_{n,X,log,t}$. It induces quasi-isomorphisms as stated in the lemma. □

Theorem 5.10. *Let X be a separated scheme of finite type over a perfect field k of characteristic $p>0$. Then the chain map*

$$\overline{\zeta}_{n,X,log,\text{ét}} : \widetilde{\nu}_{n,X,\text{ét}} \to K_{n,X,log,\text{ét}}$$

is a quasi-isomorphism. Moreover if $k=\overline{k}$,

$$\overline{\zeta}_{n,X,log,\text{Zar}} : \widetilde{\nu}_{n,X,\text{Zar}} \to K_{n,X,log,\text{Zar}}$$

is also a quasi-isomorphism.

Proof. One can assume that X is reduced. In fact, the complex $\widetilde{\nu}_{n,X,t}$ is defined to be the same complex as $\widetilde{\nu}_{n,X_{\text{red}},t}$ (see (4.0.1)), and we have a quasi-isomorphism $K_{n,X_{\text{red}},log,t} \xrightarrow{\simeq} K_{n,X,log,t}$ given by the trace map, according to Proposition 1.21 and Proposition 1.30. One notices that $\overline{\zeta}_{n,X_{\text{red}},log,t}$ is compatible with $\overline{\zeta}_{n,X,log,t}$ because of the functoriality of the map $\zeta_{log,t}$ with respect to proper maps Proposition 5.8(1). As long as we have a quasi-isomorphism

$$\overline{\zeta}_{n,X_{\text{red}},log,t} : \widetilde{\nu}_{n,X_{\text{red}},t} \to K_{n,X_{\text{red}},log},$$

we will get automatically that

$$\overline{\zeta}_{n,X,log,t} : \widetilde{\nu}_{n,X_{\text{red}},t} = \widetilde{\nu}_{n,X,t} \xrightarrow{\overline{\zeta}_{n,X_{\text{red}},log,t}} K_{n,X_{\text{red}},log} \xrightarrow{\simeq} K_{n,X,log,t}$$

is a quasi-isomorphism.

Now we do induction on the dimension of the reduced scheme X. Suppose X is of dimension d, and suppose $\overline{\zeta}_{n,Y,log,t}$ is a quasi-isomorphism for schemes of dimension $\leq d-1$. Now decompose X into the singular part Z and the smooth part U

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z.$$

Then Z has dimension $\leq d-1$. Consider the following diagram in the derived category of complexes of $\mathbb{Z}/p^n$-modules
(5.4.1)

$$\begin{tikzcd}
i_*\widetilde{\nu}_{n,Z,t} \arrow[r] \arrow[d, "i_*\overline{\zeta}_{n,Z,log,t}"] & \widetilde{\nu}_{n,X,t} \arrow[r] \arrow[d, "\overline{\zeta}_{n,X,log,t}"] & Rj_*\widetilde{\nu}_{n,U,t} \arrow[r, "+1"] \arrow[d, "Rj_*\overline{\zeta}_{n,U,log,t}"] & i_*\widetilde{\nu}_{n,Z,t}[1] \arrow[d, "i_*\overline{\zeta}_{n,Z,log,t}[1]"] \\
i_*K_{n,Z,log,t} \arrow[r, "\mathrm{Tr}_{W_n i,log}"] & K_{n,X,log,t} \arrow[r] & Rj_*K_{n,U,log,t} \arrow[r, "+1"] & i_*K_{n,Z,log,t}[1],
\end{tikzcd}$$

where the two rows are distinguished triangles coming from Proposition 1.22, Proposition 1.31 and Proposition 4.1. We show that the three squares in (5.4.1) are commutative in the derived category. The left square is commutative because of Proposition 5.8(1). The middle square is induced from the diagram

(5.4.2)
$$\begin{tikzcd}
\widetilde{\nu}_{n,X,t} \arrow[r] \arrow[d, "\overline{\zeta}_{n,X,log,t}"] & j_*\widetilde{\nu}_{n,U,t} \arrow[d, "j_*\overline{\zeta}_{n,U,log,t}"] \\
K_{n,X,log,t} \arrow[r] & j_*K_{n,U,log,t}
\end{tikzcd}$$

of chain complexes. Let $x \in X_{(q)}$. If $x \in X_{(q)} \cap U$, both $\widetilde{\nu}_{n,X,t} \to j_*\widetilde{\nu}_{n,U,t}$ and $K_{n,X,log,t} \to j_*K_{n,U,log,t}$ give identity maps at x, therefore the square (5.4.2) commutes in this case. If $x \in X_{(q)} \cap Z$, both of these give the zero map at x, therefore the square (5.4.2) is also commutative. The right square of (5.4.1) can be decomposed in the following way (cf. (1.4.4) and (1.6.3)):

$$\begin{tikzcd}
Rj_*\widetilde{\nu}_{n,U,t} \arrow[r, "+1"] \arrow[d, "Rj_*\overline{\zeta}_{n,U,log,t}"] & R\underline{\Gamma}_Z(\widetilde{\nu}_{n,X,t})[1] \arrow[d, "R\underline{\Gamma}_Z(\overline{\zeta}_{n,X,log,t})[1]"] & i_*\widetilde{\nu}_{n,Z,t}[1] \arrow[l, "i_*", "\simeq"'] \arrow[d, "i_*\overline{\zeta}_{n,Z,log,t}[1]"] \\
Rj_*K_{n,U,log,t} \arrow[r, "+1"] & R\underline{\Gamma}_Z(K_{n,X,log,t})[1] & i_*K_{n,Z,log,t}[1]. \arrow[l, "i_*", "\simeq"']
\end{tikzcd}$$

The map i_* on the first row is induced by the norm map of Milnor K-theory. It is clearly an isomorphism of complexes if $t = \mathrm{Zar}$. It is a quasi-isomorphism if $t = \text{ét}$ due to the purity theorem [35, p.130 Cor.]. The map i_* on the second row is induced from $\mathrm{Tr}_{W_n i,log,t}$ as defined in Proposition 1.20 and Proposition 1.29, and it is an isomorphism due to Proposition 1.22(1) if $t = \mathrm{Zar}$, and Proposition 1.31 if $t = \text{ét}$. The first square commutes by the naturality of the $+1$ map. The second commutes because of the compatibility of $\zeta_{log,t}$ with the proper pushforward Proposition 5.8(1). We thus deduce that the right square of (5.4.1) commutes.

Now consider over any perfect field k for either of the two cases:

(1) $t = \text{ét}$ and k is a perfect field, or
(2) $t = \mathrm{Zar}$ and $k = \overline{k}$.

The left vertical arrow of (5.4.1) is a quasi-isomorphism because of the induction hypothesis. The third one counting from the left is also a quasi-isomorphism because of Lemma 5.9. Thus so is the second one. □

6. Combine $\psi_{X,t} : \mathbb{Z}^c_{X,t} \to C^M_{X,t}$ with $\zeta_{n,X,log,t} : C^M_{X,t} \to K_{n,X,log,t}$

6.1. **The map $\psi_{X,t} : \mathbb{Z}^c_{X,t} \to C^M_{X,t}$.** In [43, 2.14], Zhong constructed a map of abelian groups $\psi_{X,t}(X) : \mathbb{Z}^c_X(X) \to C^M_{X,\mathrm{Zar}}(X)$ based on the Nesterenko-Suslin-Totaro isomorphism [36, Thm. 4.9], [41]. It is straightforward to check that Zhong's construction induces a well-defined map of complexes $\psi := \psi_{X,t} : \mathbb{Z}^c_{X,t} \to C^M_{X,t}$ of sheaves for both $t = \mathrm{Zar}$ and $t = \text{ét}$. Zhong in [43, 2.15] proved that ψ is covariant with respect to proper morphisms, and contravariant with respect to quasi-finite flat morphisms.

6.2. **The map $\overline{\zeta}_{n,X,log,t} \circ \overline{\psi}_{X,t} : \mathbb{Z}^c_{X,t}/p^n \xrightarrow{\simeq} K_{n,X,log,t}$ is a quasi-isomorphism.** In [43, 2.16] Zhong proved that $\psi_{X,\text{ét}}$ combined with the Bloch-Gabber-Kato isomorphism [4, 2.8] gives a quasi-isomorphism

$$\overline{\psi}_{X,\text{ét}} : \mathbb{Z}^c_{X,\text{ét}}/p^n \xrightarrow{\simeq} \widetilde{\nu}_{n,X,\text{ét}}.$$

In the proof, Zhong actually showed that these two complexes of sheaves on each section of the big Zariski site over X are quasi-isomorphic. Therefore by restriction to the small Zariski site, we have

$$\overline{\psi}_{X,\mathrm{Zar}} : \mathbb{Z}^c_{X,\mathrm{Zar}}/p^n \xrightarrow{\simeq} \widetilde{\nu}_{n,X,\mathrm{Zar}}.$$

Combining Zhong's quasi-isomorphism with Theorem 5.10:

Theorem 6.1. *Let X be a separated scheme of finite type over k with k being a perfect field of positive characteristic p. Then the following composition*

$$\overline{\zeta}_{n,X,log,\text{ét}} \circ \overline{\psi}_{X,\text{ét}} : \mathbb{Z}^c_{X,\text{ét}}/p^n \xrightarrow{\simeq} K_{n,X,log,\text{ét}}$$

is a quasi-isomorphism. If moreover $k = \overline{k}$, then the following composition

$$\overline{\zeta}_{n,X,log,\mathrm{Zar}} \circ \overline{\psi}_{X,\mathrm{Zar}} : \mathbb{Z}^c_{X,\mathrm{Zar}}/p^n \xrightarrow{\simeq} K_{n,X,log,\mathrm{Zar}}$$

is also a quasi-isomorphism.

Remark 6.2. From the construction of the maps $\overline{\zeta}_{n,X,log,t}$ and $\overline{\psi}_{X,t}$, we can describe explicitly their composite map. We write here only the Zariski case, and the étale case is just given by the Zariski version on the small étale site and then doing the étale sheafification.

Let U be a Zariski open subset of X. Let $Z \in (\mathbb{Z}^c_{X,\mathrm{Zar}})^i(U) = z_0(U, -i)$ be a prime cycle.

- If $i \in [-d, 0]$ and $\dim p_U(Z) = -i$, set $q = -i$. Then Z as a cycle of dimension q in $U \times \Delta^q$ is dominant over some $u = u(Z) \in U_{(q)}$ under the projection $p_U : U \times \Delta^q \to U$. By slight abuse of notation, we denote by $T_0, \dots, T_q \in k(Z)$ the pullbacks of the corresponding coordinates via $Z \hookrightarrow U \times \Delta^q$. Since Z intersects all faces properly, $T_0, \dots, T_q \in k(Z)^*$. Thus $\{\frac{-T_0}{T_q}, \dots, \frac{-T_{q-1}}{T_q}\} \in K^M_q(k(Z))$ is well-defined. Take the Zariski closure of $\operatorname{Spec} k(Z)$ in $U \times \Delta^q$, and denote it by Z'. Then p_U maps Z' to $\overline{\{u\}}^U = \overline{\{u\}}^X \cap U$. Denote by $i_u : \overline{\{u\}}^X \hookrightarrow X$ the closed immersion, and denote the composition

$$Z' \xrightarrow{p_U} \overline{\{u\}}^U \hookrightarrow \overline{\{u\}}^X \xhookrightarrow{i_u} X$$

by h. The map h is clearly generically finite, then there exists an open neighborhood V of u in X such that the restriction $h : h^{-1}(V) \to V$ is finite. Then $W_n h : W_n(h^{-1}(V)) \to W_n V$ is also finite. Therefore it makes sense to consider the trace map $\mathrm{Tr}_{W_n h}$ near the generic point of Z'.

Similarly, it makes sense to consider the trace map $\mathrm{Tr}_{W_n p_U}$ near the generic point of Z'. Then we calculate

$$\begin{aligned}\zeta_{log}(\psi(Z)) &= (-1)^i \mathrm{Tr}_{W_n i_u}(d\log(\mathrm{Nm}_{k(Z)/k(u(Z))}\{\frac{-T_0}{T_q},\dots,\frac{-T_{q-1}}{T_q}\}))\\ &= (-1)^i \mathrm{Tr}_{W_n i_u}(\mathrm{Tr}_{W_n p_U}\, d\log\{\frac{-T_0}{T_q},\dots,\frac{-T_{q-1}}{T_q}\}) \quad \text{(Lemma 5.3)}\\ &= (-1)^i \mathrm{Tr}_{W_n h}(\frac{T_q dT_0 - T_0 dT_q}{T_0 T_q}\cdots\frac{T_q dT_{q-1} - T_{q-1} dT_q}{T_{q-1}T_q}).\end{aligned}$$

Here in the last step we have used the functoriality of the trace map with respect to composition of morphisms (Proposition 1.2(4)).

- If $i \notin [-d, 0]$ or $\dim p_U(Z) \neq -i$, we have $\zeta_{log}(\psi(Z)) = 0$.

Combining the functoriality of Zhong's map ψ with Proposition 5.8, one arrives at Proposition 6.3.

Proposition 6.3 (Functoriality). *The composition $\overline{\zeta}_{n,X,log,t} \circ \overline{\psi}_{X,t} : \mathbb{Z}^c_{X,t}/p^n \xrightarrow{\simeq} K_{n,X,log,t}$ is covariant with respect to proper morphisms, and contravariant with respect to étale morphisms for both $t = \mathrm{Zar}$ and $t = \text{ét}$.*

Part 3. Applications

7. De Rham-Witt analysis of $\tilde{\nu}_{n,X,t}$ and $K_{n,X,log,t}$

Let X be a separated scheme of finite type over k of dimension d. In this section we will use terminologies as defined in [7, §1], such as Witt residual complexes, etc.

Recall that Ekedahl defined a map of complexes of $W_n\mathcal{O}_X$-modules (cf. [7, Def. 1.8.3])

$$\underline{p} := \underline{p}_{\{K_{n,X}\}_n} : R_* K_{n-1,X,t} \to K_{n,X,t}.$$

By abuse of notations, we denote by $R : W_{n-1}X \hookrightarrow W_nX$ the closed immersion induced by the restriction map on the structure sheaves $R : W_n\mathcal{O}_X \to W_{n-1}\mathcal{O}_X$.

Lemma 7.1. *The map $\underline{p} : R_*K_{n-1,X,t} \to K_{n,X,t}$ induces a map of complexes of abelian sheaves*

$$\underline{p} : K_{n-1,X,log,t} \to K_{n,X,log,t} \tag{7.0.1}$$

by applying $\underline{p}$ on each summand.

Proof. It suffices to show that $C'_t : K_{n,X,t} \to K_{n,X,t}$ commutes with $\underline{p}$ for both $t = \text{ét}$ and $t = \mathrm{Zar}$. For $t = \text{ét}$, $C'_{\text{ét}}$ is the composition of $\tau^{-1} : K_{n,X,\text{ét}} \to (W_nF_X)_*K_{n,X,\text{ét}}$ and $\epsilon^*(C'_{\mathrm{Zar}}) : (W_nF_X)_*K_{n,X,\text{ét}} \to K_{n,X,\text{ét}}$. Since τ^{-1} is functorial with respect to any map of abelian sheaves, we know that

$$\begin{tikzcd} R_*K_{n-1,X,\text{ét}} \arrow[r, "\tau^{-1}"] \arrow[d, "\underline{p}"] & (W_nF_X)_*R_*K_{n-1,X,\text{ét}} \arrow[d, "\underline{p}"] \\ K_{n,X,\text{ét}} \arrow[r, "\tau^{-1}"] & (W_nF_X)_*K_{n,X,\text{ét}} \end{tikzcd}$$

is commutative, thus it suffices to prove the proposition for $t = \mathrm{Zar}$. That is, it suffices to show the diagrams (7.0.2) and (7.0.3) commute:

$$\begin{tikzcd} R_*K_{n-1,X} \arrow[r, "R_*(1.2.2)", "\simeq"'] \arrow[d, "\underline{p}"] & R_*(W_{n-1}F_X)^{\triangle}K_{n-1,X} \arrow[d, "\underline{p}_{\{(W_nF_X)^{\triangle}K_{n,X}\}_n}"] \\ K_{n,X} \arrow[r, "(1.2.2)", "\simeq"'] & (W_nF_X)^{\triangle}K_{n,X}, \end{tikzcd} \tag{7.0.2}$$

$$\begin{tikzcd} (W_nF_X)_*R_*(W_{n-1}F_X)^{\triangle}K_{n-1,X} \arrow[r, "\simeq"] \arrow[d, "(W_nF_X)_*\underline{p}_{\{(W_nF_X)^{\triangle}K_{n,X}\}_n}"] & R_*(W_{n-1}F_X)_*(W_{n-1}F_X)^{\triangle}K_{n-1,X} \arrow[r, "R_*\mathrm{Tr}_{W_{n-1}F_X}"] & R_*K_{n-1,X} \arrow[d, "\underline{p}"] \\ (W_nF_X)_*(W_nF_X)^{\triangle}K_{n,X} \arrow[rr, "\mathrm{Tr}_{W_nF_X}"] & & K_{n,X}. \end{tikzcd} \tag{7.0.3}$$

Here $\underline{p} := \underline{p}_{\{K_{n,X}\}_n}$ is the lift-and-multiplication-by-p map associated to the Witt residual complex $\{K_{n,X}\}_n$, while $\underline{p}_{\{(W_nF_X)^{\triangle}K_{n,X}\}_n}$ denotes the one associated to Witt residual system $\{(W_nF_X)^{\triangle}K_{n,X}\}_n$ (cf. [7, 1.8.7]). By definition, the map

$$\underline{p}_{\{(W_nF_X)^{\triangle}K_{n,X}\}_n} : R_*(W_{n-1}F_X)^{\triangle}K_{n-1,X} \to (W_nF_X)^{\triangle}K_{n,X}$$

is given by the adjunction map of

$$(W_{n-1}F_X)^{\triangle}K_{n-1,X} \xrightarrow[\simeq]{(W_{n-1}F_X)^{\triangle}({}^a\underline{p})} (W_{n-1}F_X)^{\triangle}R^{\triangle}K_{n,X} \simeq R^{\triangle}(W_nF_X)^{\triangle}K_{n,X},$$

where ${}^a\underline{p}$ is the adjunction of $\underline{p}$ for residual complexes (cf. [7, Def. 1.8.3]). The second diagram (7.0.3) commutes because the trace map $\mathrm{Tr}_{W_nF_X}$ induces a well-defined map between Witt residual complexes [7, Lemma 1.8.9].

It remains to show the commutativity of (7.0.2). According to the definition of $\underline{p}_{(W_nF_X)^{\triangle}K_{n,X}}$ in [7, 1.8.7], we are reduced to showing the adjunction square commutes:

$$\begin{tikzcd} R^{\triangle}K_{n,X} \arrow[r, "R^{\triangle}(1.2.2)"] & R^{\triangle}(W_nF_X)^{\triangle}K_{n,X} \arrow[r, "\simeq"] & (W_{n-1}F_X)^{\triangle}R^{\triangle}K_{n,X} \\ K_{n-1,X} \arrow[u, "{}^a\underline{p}"] \arrow[rr, "(1.2.2)"] & & (W_{n-1}F_X)^{\triangle}K_{n-1,X}. \arrow[u, "(W_{n-1}F_X)^{\triangle}({}^a\underline{p})"'] \end{tikzcd}$$

And this is $(W_{n-1}\pi)^{\triangle}$ applied to the following diagram

$$\begin{tikzcd} R^{\triangle}W_nk \arrow[r, "R^{\triangle}(1.2.1)"] & R^{\triangle}(W_nF_k)^{\triangle}W_nk \arrow[r, "\simeq"] & (W_{n-1}F_k)^{\triangle}R^{\triangle}W_nk \\ W_{n-1}k \arrow[u, "{}^a\underline{p}"] \arrow[rr, "(1.2.1)"] & & (W_{n-1}F_k)^{\triangle}W_{n-1}k. \arrow[u, "(W_{n-1}F_k)^{\triangle}({}^a\underline{p})"'] \end{tikzcd}$$

We are reduced to showing its commutativity. Notice that this diagram is over $\operatorname{Spec} W_{n-1}k$, where the only possible filtration is the one-element set consisting of the unique point of $\operatorname{Spec} W_{n-1}k$. This means that the Cousin functor associated to this filtration sends any dualizing complex to itself, and the map ${}^a\underline{p}$ in the sense of a map either between residual complexes [7, Def. 1.8.3] or between dualizing complexes [7, Def. 1.6.3] actually agrees.

Now we start the computation. Formulas for (1.2.1) and for ${}^a\underline{p}$ (in the sense of a map between dualizing complexes) are explicitly given in Section 1.2 and [7, 1.6.4(1)], respectively. Label the source and the target of W_nF_k by $\operatorname{Spec} W_nk_1$ and $\operatorname{Spec} k_2$ respectively. Take $a \in W_{n-1}k_1$. Denote by $\overline{W_nF_k} : (\operatorname{Spec} W_nk_1, W_nk_1) \to (\operatorname{Spec} W_nk_2, (W_nF_k)_*(W_nk_1))$, and

$$\overline{R} : (\operatorname{Spec} W_{n-1}k_i, W_{n-1}k_i) \to (\operatorname{Spec} W_nk_i, R_*W_{n-1}k_i)(i = 1, 2)$$

the natural maps of ringed spaces. Now the down-right composition

$$((W_{n-1}F_k)^{\triangle}({}^a\underline{p})) \circ (1.2.1)$$

equals the Cousin functor $E_{(W_{n-1}F_k)^{\triangle}R^{\triangle}Z^{\bullet}(W_nk)}$ applied to the following composition

$$\begin{aligned} W_{n-1}k_1 &\xrightarrow[\simeq]{(1.2.1)} \overline{W_{n-1}F_k}^* \operatorname{Hom}_{W_{n-1}k_2}((W_{n-1}F_k)_*(W_{n-1}k_1), W_{n-1}k_2) \\ &\xrightarrow[\simeq]{{}^a\underline{p}} \overline{W_{n-1}F_k}^* \operatorname{Hom}_{W_{n-1}k_2}((W_{n-1}F_k)_*(W_{n-1}k_1), \\ &\qquad \overline{R}^* \operatorname{Hom}_{W_nk_2}(R_*W_{n-1}k_2, W_nk_2)), \\ a &\mapsto [(W_{n-1}F_k)_*1 \mapsto (W_{n-1}F_k)^{-1}(a)] \\ &\mapsto [(W_{n-1}F_k)_*1 \mapsto [R_*1 \mapsto \underline{p}(W_{n-1}F_k)^{-1}(a)]]. \end{aligned}$$

And $R^{\triangle}(1.2.1) \circ ({}^a\underline{p})$ equals the Cousin functor $E_{(W_{n-1}F_k)^{\triangle}R^{\triangle}Z^{\bullet}(W_nk)}$ applied the following composition

$$\begin{aligned} W_{n-1}k_1 &\xrightarrow[\simeq]{{}^a\underline{p}} \overline{R}^* \operatorname{Hom}_{W_nk_1}(R_*W_{n-1}k_1, W_nk_1) \\ &\xrightarrow{(1.2.1)} \overline{R}^* \operatorname{Hom}_{W_nk_1}(R_*W_{n-1}k_1, \overline{W_nF_k}^* \operatorname{Hom}_{W_nk_2}((W_nF_k)_*(W_nk_1), W_nk_2)), \\ a &\mapsto [R_*1 \mapsto \underline{p}(a)] \\ &\mapsto [R_*1 \mapsto [(W_nF_k)_*1 \mapsto (W_nF_k)^{-1}\underline{p}(a)]]. \end{aligned}$$

It remains to identify $\underline{p}((W_{n-1}F_k)^{-1}a)$ and $(W_nF_k)^{-1}\underline{p}(a)$. And this is straightforward: write $a = \sum_{i=0}^{n-2} V^i[a_i] \in W_{n-1}k_1$,

$$\begin{aligned} (W_nF_k)^{-1}\underline{p}(a) &= \sum_{i=0}^{n-2}(W_nF_k)^{-1}\underline{p}(V^i[a_i]) = \sum_{i=0}^{n-2}(W_nF_k)^{-1}(V^{i+1}[a_i^p]) \\ &= \sum_{i=0}^{n-2}(V^{i+1}[a_i]) = \underline{p}\sum_{i=0}^{n-2}(V^i[a_i^{1/p}]) = \underline{p}((W_{n-1}F_k)^{-1}a). \end{aligned} \tag{7.0.4}$$

Hence we finish the proof. □

However we don't naturally have a restriction map R between residual complexes. Nevertheless, we can use the quasi-isomorphism $\overline{\zeta}_{n,X,log,t} : \widetilde{\nu}_{n,X,t} \xrightarrow{\simeq} K_{n,X,log,t}$ to build up a map

$$R : K_{n,X,log,t} \to K_{n-1,X,log,t} \tag{7.0.5}$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$. For this we will need to show that $\underline{p}$ and R induce chain maps for $\widetilde{\nu}_{n,X,t}$. This should be well-known to experts, we add here again due to a lack of reference.

Lemma 7.2.

$$\underline{p} : \widetilde{\nu}_{n,X,t} \to \widetilde{\nu}_{n+1,X,t}, \quad R : \widetilde{\nu}_{n+1,X,t} \to \widetilde{\nu}_{n,X,t}$$

given by $\underline{p}$ *and* R *termwise, are well-defined maps of complexes for both* $t = \mathrm{Zar}$ *and* $t = \text{ét}$.

Proof. It suffices to prove for $t = \mathrm{Zar}$. Let $x \in X_{(q)}$ be a point of dimension q. Let $\rho : X' \to \overline{\{x\}}$ be the normalization of $\overline{\{x\}}$. Let x' be the generic point of X' and $y' \in X'^{(1)}$ be a codimension 1 point. Denote $y := \rho(y')$. It suffices to check the commutativity of the following diagrams in (1) and (2).

(1) Firstly,

$$\begin{array}{ccc} W_n\Omega^q_{x',log} & \xrightarrow{\partial} & W_n\Omega^{q-1}_{y',log} \\ \downarrow{\scriptstyle \underline{p}} & & \downarrow{\scriptstyle \underline{p}} \\ W_{n+1}\Omega^q_{x',log} & \xrightarrow{\partial} & W_{n+1}\Omega^{q-1}_{y',log}, \end{array} \qquad \begin{array}{ccc} W_{n+1}\Omega^q_{x',log} & \xrightarrow{\partial} & W_{n+1}\Omega^{q-1}_{y',log} \\ \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R} \\ W_n\Omega^q_{x',log} & \xrightarrow{\partial} & W_n\Omega^{q-1}_{y',log}. \end{array}$$

Notice that $p = \underline{p} \circ R$. Suppose π' is a uniformizer of discrete valuation ring $\mathcal{O}_{X',y'}$ and $u_1, \ldots, u_q$ are invertible elements in $\mathcal{O}_{X',y'}$. Calculate

$$\begin{aligned} &\underline{p}(\partial(d\log[\pi']_n d\log[u_2]_n \ldots d\log[u_q]_n)) \\ &\quad = \underline{p}(d\log[u_2]_n \ldots d\log[u_q]_n) \\ &\quad = p(d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1}) \\ &\quad = p(\partial(d\log[\pi']_{n+1} d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1})) \\ &\quad = \partial(p(d\log[\pi']_{n+1} d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1})) \\ &\quad = \partial(\underline{p}(d\log[\pi']_n d\log[u_2]_n \ldots d\log[u_q]_n)), \end{aligned}$$

and

$$\begin{aligned} &\underline{p}(\partial(d\log[u_1]_n d\log[u_2]_n \ldots d\log[u_q]_n)) \\ &\quad = 0 \\ &\quad = p(\partial(d\log[u_1]_{n+1} d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1})) \\ &\quad = \partial(p(d\log[u_1]_{n+1} d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1})) \\ &\quad = \partial(\underline{p}(d\log[u_1]_n d\log[u_2]_n \ldots d\log[u_q]_n)). \end{aligned}$$

This proves the first diagram. Now the second.

$$\begin{aligned} &R(\partial(d\log[\pi']_{n+1} d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1})) \\ &\quad = R(d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1}) \\ &\quad = d\log[u_2]_n \ldots d\log[u_q]_n \\ &\quad = \partial(d\log[\pi']_n d\log V[u_2]_n \ldots d\log V[u_q]_n) \\ &\quad = \partial(R(d\log[\pi']_{n+1} d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1})), \end{aligned}$$

and

$$\begin{aligned} &R(\partial(d\log[u_1]_{n+1} d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1})) \\ &\quad = 0 \\ &\quad = \partial(d\log[u_1]_n d\log[u_2]_n \ldots d\log[u_q]_n) \\ &\quad = \partial(R(d\log[u_1]_{n+1} d\log[u_2]_{n+1} \ldots d\log[u_q]_{n+1})). \end{aligned}$$

(2) Secondly,

$$\begin{tikzcd} W_n\Omega^{q-1}_{y',log} \arrow[r, "\mathrm{tr}"] \arrow[d, "\underline{p}"] & W_n\Omega^{q-1}_{y,log} \arrow[d, "\underline{p}"] \\ W_{n+1}\Omega^{q-1}_{y',log} \arrow[r, "\mathrm{tr}"] & W_{n+1}\Omega^{q-1}_{y,log}, \end{tikzcd} \qquad \begin{tikzcd} W_{n+1}\Omega^{q-1}_{y',log} \arrow[r, "\mathrm{tr}"] \arrow[d, "R"] & W_{n+1}\Omega^{q-1}_{y,log} \arrow[d, "R"] \\ W_n\Omega^{q-1}_{y',log} \arrow[r, "\mathrm{tr}"] & W_n\Omega^{q-1}_{y,log}. \end{tikzcd}$$

Notice that $\rho : X' \to \overline{\{x\}}^X$ can be restricted to a map from $\overline{\{y'\}}^{X'}$ to $\overline{\{y\}}^X$ ($\overline{\{x\}}^X$ denotes the closure of x in X, and similarly for $\overline{\{y'\}}^{X'}$, $\overline{\{y\}}^X$). Furthermore, y' (resp. y) belongs to the smooth locus of $\overline{\{y'\}}^{X'}$ (resp. $\overline{\{y\}}^X$), and there $\underline{p}$ and R come from the restriction of the $\underline{p}$ and R on the respective smooth locus. The map tr, induced by Milnor's norm map, agrees with the Grothendieck trace map $\mathrm{Tr}_{W_n\rho}$ due to Lemma 5.3. And according to compatibility of the Grothendieck trace map with the Witt system structure (i.e. de Rham-Witt system structure with zero differential) on canonical sheaves [7, 4.1.4(6)], we arrive at the desired commutativity. □

Lemma 7.3. *Assume either*

- $t = \mathrm{Zar}$ *and* $k = \overline{k}$, *or*
- $t = \text{ét}$ *and* k *being a perfect field of characteristic* $p > 0$.

Then we have the following short exact sequence

$$0 \to \widetilde{\nu}_{i,X,t} \xrightarrow{\underline{p}^j} \widetilde{\nu}_{i+j,X,t} \xrightarrow{R^i} \widetilde{\nu}_{j,X,t} \to 0, \tag{7.0.6}$$

in the category of complexes of sheaves over X_t, *and a distinguished triangle*

$$K_{i,X,log,t} \xrightarrow{\underline{p}^j} K_{i+j,X,log,t} \xrightarrow{R^i} K_{j,X,log,t} \xrightarrow{+1} \tag{7.0.7}$$

in the derived category $D^b(X_t, \mathbb{Z}/p^n)$.

Proof. (1) Because of Lemma 7.2, it suffices to show

$$0 \to W_i\Omega^q_{x,log,t} \xrightarrow{\underline{p}^j} W_{i+j}\Omega^q_{x,log,t} \xrightarrow{R^i} W_j\Omega^q_{x,log,t} \to 0$$

is short exact for any given point $x \in X_{(q)}$. And this is true for $t = \text{ét}$ because of [8, Lemme 3]. And for $t = \mathrm{Zar}$, one further needs $R^1\epsilon_* W_n\Omega^q_{x,log,\text{ét}} = 0$ for any $x \in X_{(q)}$ if $k = \overline{k}$, which is proved in [39, Cor. 2.3].

(2) Now it suffices to show that $\underline{p}$ and R for the system $\{K_{n,X,log,t}\}_n$ are compatible with $\underline{p}$ and R of the system $\{\widetilde{\nu}_{n,X,t}\}_n$, via the quasi-isomorphism $\overline{\zeta}_{n,X,log,t}$. The compatibility for R is clear by definition. It remains to check the compatibility for $\underline{p}$. Because $\overline{\zeta}_{n,X,log,t} = (5.3.1) \circ \overline{\zeta}_{n,X,t}$, it suffices to check compatibility of $\underline{p} : \widetilde{\nu}_{n-1,X,t} \to \widetilde{\nu}_{n,X,t}$ with $\underline{p} : K_{n-1,X,t} \to K_{n,X,t}$ via $\overline{\zeta}_{n,X,t}$. At a given degree $-q$ and given point $x \in X_{(q)}$, the map $\overline{\zeta}_{n,X,t} : \widetilde{\nu}_{n,X,t} \to K_{n,X,t}$ factors as

$$(W_n\iota_x)_* W_n\Omega^q_{x,log,t} \to (W_n i_x)_* W_n\Omega^q_{x,t} = (W_n i_x)_* K^q_{n,\overline{\{x\}},t} \xrightarrow{(-1)^q \mathrm{Tr}_{W_n i_x,t}} K^q_{n,X,t}.$$

The first arrow is the inclusion map and is naturally compatible with $\underline{p}$. The compatibility of $\underline{p}$ via the trace map is given in [7, Lemma 1.8.9]. □

8. Higher Chow groups of zero cycles

Let k be a perfect field of characteristic $p > 0$. In this whole section, X denotes a separated scheme of finite type over k of dimension d with structure map $\pi : X \to k$.

8.1. **First properties.**

Proposition 8.1. *There is a distinguished triangle*

$$\mathbb{Z}^c_{X,\text{ét}}/p^n \to K_{n,X,\text{ét}} \xrightarrow{C'_{\text{ét}}-1} K_{n,X,\text{ét}} \xrightarrow{+1}$$

in the derived category $D^b(X_{\text{ét}}, \mathbb{Z}/p^n)$. If $k = \overline{k}$, one also has the Zariski counterpart. Namely, we have a distinguished triangle

$$\mathbb{Z}^c_X/p^n \to K_{n,X} \xrightarrow{C'-1} K_{n,X} \xrightarrow{+1} \tag{8.1.1}$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$.

In particular, if $k = \overline{k}$ and X is Cohen-Macaulay of pure dimension d, then $\mathbb{Z}^c_X/p^n$ is concentrated at degree $-d$, and the triangle (8.1.1) *becomes*

$$\mathbb{Z}^c_X/p^n \to W_n\omega_X[d] \xrightarrow{C'-1} W_n\omega_X[d] \xrightarrow{+1}$$

in this case. Here $W_n\omega_X$ is the only non-vanishing cohomology sheaf of $K_{n,X}$ (if $n = 1$, $W_1\omega_X = \omega_X$ is the usual dualizing sheaf on X).

Proof. This is direct from the main result Theorem 6.1 and Remark 1.17. □

Proposition 8.2. *Assume $k = \overline{k}$. Then higher Chow groups of zero cycles equal the C'-invariant part of the cohomology groups of Grothendieck's coherent dualizing complex, i.e.,*

$$\mathrm{CH}_0(X, q; \mathbb{Z}/p^n) = H^{-q}(W_nX, K_{n,X})^{C'-1}.$$

Proof. This follows directly from Proposition 1.15 and the main result Theorem 6.1. □

Corollary 8.3 (Relation with p-torsion Poincaré duality)**.** *There is an isomorphism*

$$K_{n,X,log,\text{ét}} \simeq R\pi^!(\mathbb{Z}/p^n)$$

in $D^b(X_{\text{ét}}, \mathbb{Z}/p^n)$, where $R\pi^!$ is the extraordinary inverse image functor defined in [40, Exposé XVIII, Thm 3.1.4].

Proof. This follows directly from the main Theorem 5.10 and [26, Thm. 4.6.2]. □

Corollary 8.4 (Affine vanishing)**.** *Suppose X is affine and Cohen-Macaulay of pure dimension d. Then*

(1) *If $t = \mathrm{Zar}$ and $k = \overline{k}$,*

$$\mathrm{CH}_0(X, q, \mathbb{Z}/p^n) = 0$$

for $q \neq d$.

(2) *If $t = \text{ét}$,*

$$R^{-q}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_X/p^n) = 0$$

for $q \neq d, d-1$. If one further assumes $k = \overline{k}$ or smoothness, the possible non-vanishing occurs only in degree $q = d$.

Proof. If X is Cohen-Macaulay of pure dimension d, W_nX is also Cohen-Macaulay of pure dimension d by Serre's S_k-criterion, and $K_{n,X,t}$ is concentrated at degree $-d$ for all n [9, 3.5.1]. Now Serre's affine vanishing theorem implies $H^{-q}(W_nX, K_{n,X,t}) = 0$ for $q \neq d$. This implies that $R^{-q}\Gamma(W_nX, K_{n,X,log,t}) = 0$ unless $q = d, d-1$. With the given assumptions, Theorem 6.1 implies that $\mathrm{CH}_0(X, q, \mathbb{Z}/p^n) = R^{-q}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_X/p^n) = 0$ unless $q = d, d-1$. If one also assumes $k = \overline{k}$, Proposition 8.2 gives the vanishing result for $q = d-1$.

If X is smooth, $C_{\text{ét}} - 1 : W_n\Omega^d_{X,\text{ét}} \to W_n\Omega^d_{X,\text{ét}}$ is surjective by [16, 1.6(ii)] (see (1.3.30)). By the compatibility of $C_{\text{ét}}$ and $C'_{\text{ét}}$ Proposition 1.26, one deduces that $C' - 1 : \mathcal{H}^{-d}(K_{n,X,\text{ét}}) \to \mathcal{H}^{-d}(K_{n,X,\text{ét}})$ is surjective. □

Generalizing Bass's finiteness conjecture for K-groups (cf. [42, IV.6.8]), the finiteness of higher Chow groups in various arithmetic settings appears in the literature. The following result was first proved by Geisser [13, §5, eq. (12)] using the finiteness result from the étale cohomology theory, and here we deduce it as a corollary of our main theorem, which essentially relies on the finiteness of coherent cohomologies on a proper scheme. We remark that Geisser's result is more general than ours in that he allows arbitrary torsion coefficients.

Corollary 8.5 (Finiteness, Geisser). *Assume $k = \overline{k}$. Let X be proper over k. Then for any q,*

$$\mathrm{CH}_0(X, q; \mathbb{Z}/p^n)$$

is a finite $\mathbb{Z}/p^n$-module.

Proof. According to Theorem 6.1, $R^{-q}\Gamma(X, \mathbb{Z}^c_X/p^n) = R^{-q}\Gamma(X, K_{n,X,log})$. Thus it suffices to show that for every q, $R^{-q}\Gamma(X, K_{n,X,log})$ is a finite $\mathbb{Z}/p^n$-module. First of all, since the cohomology group $R^{-q}\Gamma(X, K_{n,X,log})$ is the C'-invariant part of $R^{-q}\Gamma(X, K_{n,X})$ by Proposition 1.15 and Proposition 1.27, $R^{-q}\Gamma(X, K_{n,X,log})$ is a module over the invariant ring $(W_nk)^{1-W_nF_X^{-1}} = \mathbb{Z}/p^n$. Because X is proper, $R^{-q}\Gamma(X, K_{n,X})$ is a finite W_nk-module by the local-to-global spectral sequence. Then Proposition A.7 gives us the result.

Alternatively, we can also do induction on n. In the $n = 1$ case, because $R^{-q}\Gamma(X, K_{X,log})$ is the C'-invariant part of the finite dimensional k-vector space $H^{-q}(X, K_X)$ again by Proposition 1.15 and Proposition 1.27, it is a finite $\mathbf{F}_p$-module by p^{-1}-linear algebra Proposition A.3. The desired result then follows from the long exact sequence associated to (7.0.7) by induction on n. □

We refer to Definition A.4 and Remark A.5(2) for the definition of the semisimplicity and the notation $(-)_{\mathrm{ss}}$ in this context.

Corollary 8.6 (Semisimplicity). *Assume $k = \overline{k}$. Let X be proper over k. Then for any q,*

$$H^{-q}(W_nX, K_{n,X})_{\mathrm{ss}} = \mathrm{CH}_0(X, q; \mathbb{Z}/p^n) \otimes_{\mathbb{Z}/p^n} W_nk.$$

Proof. Since X is proper, $H^{-q}(W_nX, K_{n,X})$ is a finite W_nk-module for any q. Then according to Proposition A.8,

$$H^{-q}(W_nX, K_{n,X})_{\mathrm{ss}} = H^{-q}(W_nX, K_{n,X})^{C'-1} \otimes_{\mathbb{Z}/p^n} W_nk.$$

The claim now follows from Proposition 8.2. □

8.2. **Étale descent.** The results Proposition 8.7, Proposition 8.8 in this subsection are well-known to experts.

Proposition 8.7 (Gros-Suwa). *Assume $k = \overline{k}$. Then one has a canonical isomorphism*

$$\widetilde{\nu}_{n,X,\mathrm{Zar}} = \epsilon_* \widetilde{\nu}_{n,X,\mathrm{\acute{e}t}} \xrightarrow{\simeq} R\epsilon_* \widetilde{\nu}_{n,X,\mathrm{\acute{e}t}}$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$.

Proof. If $k = \overline{k}$, terms of the étale complex $\widetilde{\nu}_{n,X,\mathrm{\acute{e}t}}$ are ϵ_*-acyclic according to [16, 3.16]. □

The étale descent of Bloch's cycle complex with $\mathbb{Z}$-coefficients is shown in [13, Thm 3.1], assuming the Beilinson-Lichtenbaum conjecture which is now proved by Rost and Voevodsky. Hence the étale descent holds conjecture-free. Note that one can also deduce the mod p^n version as a corollary of Proposition 8.7 via Zhong's quasi-isomorphism in Section 6.2 (which is dependent on the main result of Geisser-Levine [14, 1.1]).

Proposition 8.8 (Geisser-Levine). *Assume $k = \overline{k}$. Then one has a canonical isomorphism*

$$\mathbb{Z}^c_{X,\mathrm{Zar}}/p^n = \epsilon_* \mathbb{Z}^c_{X,\mathrm{\acute{e}t}}/p^n \xrightarrow{\simeq} R\epsilon_* \mathbb{Z}^c_{X,\mathrm{\acute{e}t}}/p^n$$

in the derived category $D^b(X, \mathbb{Z}/p^n)$. As a result,

$$\mathrm{CH}_0(X, q; \mathbb{Z}/p^n) \simeq R^{-q}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Z}^c_{X,\mathrm{\acute{e}t}}/p^n).$$

Proof. Clearly, we have the compatibility

$$\begin{tikzcd} \mathbb{Z}^c_{X,\mathrm{Zar}}/p^n \arrow[r, "\simeq"] \arrow[d, "\psi_{X,\mathrm{Zar}}"] & R\epsilon_* \mathbb{Z}^c_{X,\mathrm{\acute{e}t}}/p^n \arrow[d, "R\epsilon_*\psi_{X,\mathrm{\acute{e}t}}"] \\ \widetilde{\nu}_{n,X,\mathrm{Zar}} \arrow[r, "\simeq"] & R\epsilon_* \widetilde{\nu}_{n,X,\mathrm{\acute{e}t}}. \end{tikzcd}$$

Thus $\mathbb{Z}^c_{X,\mathrm{Zar}}/p^n \xrightarrow[\simeq]{\overline{\psi}_{X,\mathrm{Zar}}} \widetilde{\nu}_{n,X,\mathrm{Zar}} = \epsilon_* \widetilde{\nu}_{n,X,\mathrm{\acute{e}t}} \xrightarrow[\simeq]{\text{Proposition 8.7}} R\epsilon_* \widetilde{\nu}_{n,X,\mathrm{\acute{e}t}} \xrightarrow[\simeq]{R\epsilon_*\overline{\psi}_{X,\mathrm{\acute{e}t}}} R\epsilon_* \mathbb{Z}^c_{X,\mathrm{\acute{e}t}}/p^n$. □

Corollary 8.9. *Assume $k = \overline{k}$. Suppose X is affine and Cohen-Macaulay of pure dimension d. Then*

$$R^i \epsilon_* (\mathbb{Z}^c_{X,\mathrm{\acute{e}t}}/p^n) = R^i \epsilon_* \widetilde{\nu}_{n,X,\mathrm{\acute{e}t}} = 0, \quad i \neq -d.$$

Proof. This is a direct consequence of Proposition 8.8, Proposition 8.7 and Corollary 8.4. □

8.3. **Birational geometry and rational singularities.** Recall Definition 8.10 of *resolution-rational singularities*, which are more often called *rational singularities* before in the literature, but here we follow the terminology from [32] (see also Remark 8.11(1) and [32, (9.12.1)]).

Definition 8.10 (cf. [32, 9.1]). An integral k-scheme X is said to have *resolution-rational singularities* if

(1) there exists a birational proper morphism $f : \widetilde{X} \to X$ with $\widetilde{X}$ smooth (such an f is called a *resolution of singularities* or simply a *resolution* of X), and
(2) $R^i f_* \mathcal{O}_{\widetilde{X}} = R^i f_* \omega_{\widetilde{X}} = 0$ for $i \geq 1$. And $f_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$.

Such a map $f : \widetilde{X} \to X$ is called a *rational resolution* of X.

Note that the cohomological condition (2) is equivalent to the following condition

(2') $\mathcal{O}_X \simeq Rf_*\mathcal{O}_{\widetilde{X}}$, $f_*\omega_{\widetilde{X}} \simeq Rf_*\omega_{\widetilde{X}}$ in the derived category of abelian Zariski sheaves.

Remark 8.11.

(1) According to [32, 8.2], on integral k-schemes of pure dimension, our definitions for resolution-rational singularities and for rational resolutions are the same as the ones in [32, 9.1].
(2) Necessary conditions for an integral k-scheme to have resolution-rational singularities are that the scheme is normal and Cohen-Macaulay. The normality statement follows from the equality $f_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$, and the Cohen-Macaulay statement is a standard result, see e.g. [32, 8.3].
(3) According to [32, 9.6], resolution-rational singularities are pseudo-rational. By definition [32, 1.2], a k-scheme X is said to have *pseudo-rational singularities*, if it is normal Cohen-Macaulay, and for every normal scheme X', every projective birational morphism $f : X' \to X$, the composition $f_*\omega_{X'} \to Rf_*\omega_{X'} \xrightarrow{\mathrm{Tr}_f} \omega_X$ is an isomorphism.

Corollary 8.12. *Let X and Y be integral k-schemes of pure dimensions which have pseudo-rational singularities. Suppose there are proper birational k-morphisms $f : Z \to X$ and $g : Z \to Y$ where Z is a normal Cohen-Macaulay scheme. Then we have*

$$R^{-q}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_{\widetilde{X},\text{ét}}/p^n) = R^{-q}\Gamma(Y_{\text{ét}}, \mathbb{Z}^c_{Y,\text{ét}}/p^n)$$

for all q and all $n \geq 1$. If we assume furthermore $k = \overline{k}$, we also have

$$\mathrm{CH}_0(X, q, \mathbb{Z}/p^n) = \mathrm{CH}_0(Y, q, \mathbb{Z}/p^n)$$

for all q and all $n \geq 1$.

Remark 8.13.

(1) In particular, since for any rational resolution of singularities $f : \widetilde{X} \to X$, $\widetilde{X}$ and X are properly birational as k-schemes (i.e., take Z to be $\widetilde{X}$), one can compute the higher Chow groups of zero cycles of X via those of $\widetilde{X}$.
(2) Deleting the pseudo-rational singularities assumption (in particular, we relax the Cohen-Macaulay assumptions on X and Y), the proof still passes through with the following assumption: X and Y are linked by a chain of proper birational maps and each of these maps has its trace map being a quasi-isomorphism between the residual complexes. Such a proper birational map is called a *cohomological equivalence* in [32, 8.4].
(3) If normal Macaulayfications for integral varieties exist (e.g., conjecture [32, 1.13] is true), the assumption of Corollary 8.12 can be weakened to the following: *Let X and Y be integral k-schemes of pure dimensions which have pseudo-rational singularities and are properly birational, i.e., there are proper birational k-morphisms $f : Z \to X$ and $g : Z \to Y$ with Z being some integral scheme.* In fact, we can replace Z by a normal Cohen-Macaulay scheme by the following process. Using Chow's Lemma [32, 4.1], we know that there exist projective birational morphisms $f' : Z_1 \to Z$ and $g' : Z_2 \to Z$ such that the compositions $Z_1 \xrightarrow{f'} Z \xrightarrow{f} X$ and $Z_2 \xrightarrow{g'} Z \xrightarrow{g} Y$

are also birational and projective. Let $U \subset Z$ be an open dense subset such that f' and g' restricted to the preimage of U are isomorphisms. Take Z' to be the Zariski closure of the image of the diagonal of U in $Z_1 \times_Z Z_2$ with the reduced scheme structure. Then the two projections $Z' \to Z_1$ and $Z' \to Z_2$ are also projective and birational. This means that by replacing Z' with Z, f with $Z' \to X$ and g with $Z' \to Y$, we can assume our $f : Z \to X$, $g : Z \to Y$ to be projective birational and our Z to be integral. Using normal Macaulayfication [32, 1.13] we can additionally assume that Z is normal Cohen-Macaulay.

In particular, since the conjecture [32, 1.13] is known to be true for varieties of dimension at most 4 over algebraically closed fields (cf. [32, 1.14(iii)]), one can state Corollary 8.12 with this weakened assumption in this case.

Proof. Note that f and g are pseudo-rational modifications by [32, 9.7]. Suppose that X is of pure dimension d. Then so is Z. Now [32, 8.6] implies that the trace map of f induces an isomorphism

$$\mathrm{Tr}_f : Rf_* K_{Z,t} \xrightarrow{\simeq} K_{X,t}$$

in $D^b(X_t, \mathbb{Z}/p)$. Thus

$$\mathrm{Tr}_{f,log} : Rf_* K_{Z,log,t} \xrightarrow{\simeq} K_{X,log,t}$$

is also an isomorphism in $D^b(X_t, \mathbb{Z}/p)$. Consider the diagram

$$\begin{tikzcd}
f_* K_{Z,log,t} \arrow[r, "\underline{p}^{n-1}"] \arrow[d, "\mathrm{Tr}_{f,log}"] & f_* K_{n,Z,log,t} \arrow[r, "R"] \arrow[d, "\mathrm{Tr}_{W_n f,log}"] & f_* K_{n-1,Z,log,t} \arrow[r, "+1"] \arrow[d, "\mathrm{Tr}_{W_{n-1} f,log}"] & f_* K_{Z,log,t}[1] \arrow[d, "\mathrm{Tr}_{f,log}[1]"] \\
K_{X,log,t} \arrow[r, "\underline{p}^{n-1}"] & K_{n,X,log,t} \arrow[r, "R"] & K_{n-1,X,log,t} \arrow[r, "+1"] & K_{X,log,t}[1]
\end{tikzcd} \tag{8.3.1}$$

in $D^b(X_t, \mathbb{Z}/p)$. The first row is Rf_* applied to the triangle (7.0.7) on Z. The second row is the triangle (7.0.7) on X. The left square commutes on the level of complexes by the compatibility of the trace map with $\underline{p}$ [7, 1.8.9]. To prove commutativity of the middle square in the derived category, it suffices to show the square

$$\begin{tikzcd}
f_* \widetilde{\nu}_{n,Z,t} \arrow[r, "R"] \arrow[d, "f_*"] & f_* \widetilde{\nu}_{n-1,Z,t} \arrow[d, "f_*"] \\
\widetilde{\nu}_{n,X,t} \arrow[r, "R"] & \widetilde{\nu}_{n-1,X,t}
\end{tikzcd}$$

commutes on the level of complexes. Since the vertical maps f_* for Kato-Moser complexes are tr (cf. Section 4), which are by definition the reduction of the norm maps for Milnor K-theory, they agree with the Grothendieck trace maps $\mathrm{Tr}_{W_n f}$, $\mathrm{Tr}_{W_{n-1} f}$ by Lemma 5.3. And according to the compatibility of R with the Grothendieck trace maps [7, 4.1.4(6)], we arrive at the desired commutativity. The right square in (8.3.1) commutes by the naturality of the "+1" map. With all these commutativities we conclude that the vertical maps in (8.3.1) define a map of triangles. By induction on n we deduce that

$$\mathrm{Tr}_{W_n f,log} : Rf_* K_{n,Z,log,t} \xrightarrow{\simeq} K_{n,X,log,t}$$

is an isomorphism in $D^b(X_t, \mathbb{Z}/p^n)$ for every n. The main result Theorem 6.1 thus implies

$$R^{-q}\Gamma(Z_{\text{ét}}, \mathbb{Z}^c_{\widetilde{X},\text{ét}}/p^n) = R^{-q}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_{X,\text{ét}}/p^n)$$

for all q and n. If $k = \overline{k}$, the same theorem also implies that

$$\mathrm{CH}_0(Z, q, \mathbb{Z}/p^n) = \mathrm{CH}_0(X, q, \mathbb{Z}/p^n)$$

for all q and n.

Now replacing f by g everywhere in the above argument we get the result. □

8.4. **Galois descent.**

Corollary 8.14. *Let X and Y be separated schemes of finite type over k of dimension d. Let $f : Y \to X$ be a finite étale Galois map with Galois group G. Then*

$$R^{-d}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_X/p^n) = R^{-d}\Gamma(Y_{\text{ét}}, \mathbb{Z}^c_Y/p^n)^G.$$

If $k = \overline{k}$, we also have

$$\mathrm{CH}_0(X, d; \mathbb{Z}/p^n) = \mathrm{CH}_0(Y, d; \mathbb{Z}/p^n)^G.$$

Proof. The pullback f^* induces two canonical maps

$$f^* : \mathbb{Z}^c_{X,\text{ét}} \to (f_*\mathbb{Z}^c_{Y,\text{ét}})^G, \quad f^* : K_{n,X,log,\text{ét}} \to (f_*K_{n,Y,log,\text{ét}})^G.$$

Both of them are isomorphisms of complexes, because each term of these complexes is an étale sheaf. Because of the contravariant functoriality with respect to étale morphisms (Proposition 6.3 and [43, 2.15]), $\overline{\zeta}_{log} \circ \overline{\psi}$ is G-equivariant. That is, the diagram

$$\begin{tikzcd}
\mathbb{Z}^c_{X,\text{ét}}/p^n \arrow[r, "\overline{\zeta}_{log}\circ\overline{\psi}"] \arrow[d, "f^*"] & K_{n,X,log,\text{ét}} \arrow[d, "f^*"] \\
(f_*\mathbb{Z}^c_{Y,\text{ét}}/p^n)^G \arrow[r, "\overline{\zeta}_{log}\circ\overline{\psi}"] & (f_*K_{n,Y,log,\text{ét}})^G
\end{tikzcd}$$

commutes.

Applying $R^{-d}\Gamma(X_{\text{ét}}, -)$ to the isomorphism $f^* : K_{n,X,log,\text{ét}} \to (f_*K_{n,Y,log,\text{ét}})^G$, one gets

$$R^{-d}\Gamma(X_{\text{ét}}, K_{n,X,log,\text{ét}}) = R^{-d}\Gamma(X_{\text{ét}}, (f_*K_{n,Y,log,\text{ét}})^G).$$

Consider the local-to-global spectral sequence associated to the right hand side of this equality, there is only one non-zero term in the E_∞-page with total degree $-d$ (which is a term in the E_2-page), thus we have

$$R^{-d}\Gamma(X_{\text{ét}}, (f_*K_{n,Y,log,\text{ét}})^G) = H^0(X_{\text{ét}}, \mathcal{H}^{-d}((f_*K_{n,Y,log,\text{ét}})^G)).$$

Because $(-)^G$ commutes with taking kernels and with H^0, we have

$$H^0(X_{\text{ét}}, \mathcal{H}^{-d}((f_*K_{n,Y,log,\text{ét}})^G)) = H^0(X_{\text{ét}}, \mathcal{H}^{-d}(f_*K_{n,Y,log,\text{ét}}))^G.$$

Because f_* preserves kernels, we have

$$H^0(X_{\text{ét}}, \mathcal{H}^{-d}(f_*K_{n,Y,log,\text{ét}}))^G = H^0(Y_{\text{ét}}, \mathcal{H}^{-d}(K_{n,Y,log,\text{ét}}))^G.$$

Again by the observation from the spectral sequence, this means

$$H^0(Y_{\text{ét}}, \mathcal{H}^{-d}(K_{n,Y,log,\text{ét}}))^G = R^{-d}\Gamma(Y_{\text{ét}}, K_{n,Y,log,\text{ét}})^G.$$

Since $\overline{\zeta}_{log} \circ \overline{\psi}$ is G-equivariant, the main Theorem 6.1 implies

$$R^{-d}\Gamma(X_{\text{ét}}, \mathbb{Z}^c_X/p^n) = R^{-d}\Gamma(Y_{\text{ét}}, \mathbb{Z}^c_Y/p^n)^G.$$

If $k = \overline{k}$, Proposition 8.8 implies

$$\mathrm{CH}_0(X, d; \mathbb{Z}/p^n) = \mathrm{CH}_0(Y, d; \mathbb{Z}/p^n)^G.$$

□

Appendix A. Semilinear algebra

Definition A.1. Let k be a perfect field of positive characteristic p, and V be a finite dimensional k-vector space. A *p-linear map* (resp. p^{-1}-linear map) on V is a map $T : V \to V$, such that

$$T(v+w) = T(v) + T(w), \quad T(cv) = c^p T(v), \quad v, w \in V, c \in k$$
$$(\text{resp. } T(v+w) = T(v) + T(w), \quad T(cv) = c^{-p} T(v), \quad v, w \in V, c \in k).$$

We say a map $T : V \to V$ is *semilinear* if it is either p-linear or p^{-1}-linear. A semilinear map $T : V \to V$ is called *semisimple* if $\operatorname{Im} T = V$.

Remark A.2. Let T be a semilinear map.

(1) Note that

$$\{c \in k \mid c^p = c\} = \mathbf{F}_p = \{c \in k \mid c^{-p} = c\}.$$

The fixed point of T

$$V^{1-T} := \{v \in V \mid T(v) = v\}$$

is an $\mathbf{F}_p$-vector space.

(2) There is a descending chain of k-vector subspaces of V

$$\operatorname{Im} T \supset \operatorname{Im} T^2 \supset \cdots \supset \operatorname{Im} T^n \supset \ldots.$$

Since V is finite dimensional, it becomes stationary for some large $N \in \mathbb{N}$. Define

$$V_{\text{ss}} := \bigcap_{n \geq 1} \operatorname{Im}(T^n) = \operatorname{Im}(T^N) = \operatorname{Im}(T^{N+1}) = \ldots.$$

Obviously,

(a) V_{ss} is a k-vector subspace of V that is stable under T. T is semisimple on V_{ss}.

(b) $V^{1-T} \subset V_{\text{ss}}$.

The proof of the following result is given in [20] for p-linear maps, but an analogous proof also works for p^{-1}-linear maps.

Proposition A.3 ([20, Exposé XXII, Cor. 1.1.10, Prop. 1.2])**.** *Suppose k is a separably closed field of positive characteristic p. Then*

$$1 - T : V \to V$$

is surjective. And

$$V_{\text{ss}} \simeq V^{1-T} \otimes_{\mathbf{F}_p} k,$$

which in particular means V^{1-T} is a finite dimensional $\mathbf{F}_p$-vector space with $\dim_{\mathbf{F}_p} V^{1-T} = \dim_k V_{\text{ss}}$.

We generalize the definition of a semilinear map.

Definition A.4. Let k be a perfect field of positive characteristic p, and let W_nk be the ring of the n-th truncated Witt vectors of k. Let M be a finitely generated W_nk-module. A *p-linear map* (resp. *p^{-1}-linear map*) on M is a map $T : M \to M$, such that

$$T(m+m')=T(m)+T(m'),\quad T(cm)=W_nF_k(c)T(m),\quad m,m'\in M, c\in W_nk$$

$$(\text{resp. } T(m+m')=T(m)+T(m'),\quad T(cm)=W_nF_k^{-1}(c)T(m),\quad m,m'\in M, c\in W_nk).$$

Here F_k denotes the p-th power Frobenius on the field k. We say that T is *semilinear* if it is either p-linear or p^{-1}-linear in this sense. A semilinear map $T : V \to V$ is called *semisimple* if $\operatorname{Im} T = V$.

Remark A.5. Let T be a semilinear map in the sense of Definition A.4.

(1) Write $\sigma = W_nF_k$ (resp. $\sigma = W_nF_k^{-1}$). Then

$$(W_nk)^{1-\sigma} := \{c \in W_nk \mid \sigma(c) = c\} = \mathbb{Z}/p^n$$

for both cases. The fixed point of T

$$M^{1-T} := \{m \in M \mid T(m) = m\}$$

is a $\mathbb{Z}/p^n$-module.

(2) As in the case of vector spaces,

$$\operatorname{Im} T \supset \operatorname{Im} T^2 \supset \cdots \supset \operatorname{Im} T^n \supset \dots$$

is a descending chain of W_nk-submodules of M. It becomes stationary for some large $N \in \mathbb{N}$, because M as a finitely generated W_nk-module is artinian. Define the W_nk-submodule of M

$$M_{\mathrm{ss}} := \bigcap_{n\geq 1} \operatorname{Im}(T^n) = \operatorname{Im}(T^N) = \operatorname{Im}(T^{N+1}) = \dots.$$

Then

(a) M_{ss} is a W_nk-submodule of M that is stable under T. T is semisimple on M_{ss}.

(b) $M^{1-T} \subset M_{\mathrm{ss}}$.

(c) $(M/p)_{\mathrm{ss}} = M_{\mathrm{ss}}/p \subset M/p$.

Proposition A.6. *Let k be a separably closed field of positive characteristic p. Then*

$$1 - T : M \to M$$

is surjective.

Proof. Take $m \in M$. Because M is finitely generated as a W_nk-module, M/pM is a finite dimensional k-vector space. Then Proposition A.3 implies that there exists an $m' \in M$, such that $(1 - T)(m') - m \in pM$. That is, there exists an $m_1 \in M$ such that

$$(1 - T)(m') = m + pm_1.$$

Do the same process with m_1 instead of m, one gets an $m_1' \in M$ and an $m_2 \in M$ such that

$$(1 - T)(m_1') = m_1 + pm_2.$$

Thus

$$(1 - T)(m' - pm_1') = m - p^2m_2.$$

Repeat this process. After finitely many times, because $p^n = 0$ in $W_n k$,

$$(1-T)(m' - pm'_1 + \cdots + (-1)^{n-1} p^{n-1} m'_{n-1}) = m.$$

□

Proposition A.7. *Let k be a separably closed field of positive characteristic p. Then*

(1) $M^{1-T}/(pM)^{1-T} = (M/p)^{1-T}$.
(2) M^{1-T} *is a finite $\mathbb{Z}/p^n$-module.*

Proof. Since $W_n k$ is of p^n-torsion, we know that $p^m M = 0$ for some $m \leq n$. Do induction on the smallest number m such that $p^m M = 0$. If $m = 1$, the first claim is trivial, and $M = M/p$ is actually a finite dimensional k-vector space, thus the second claim follows from Proposition A.3.

Now we assume $m > 1$. Note that T induces a semilinear map on pM and pM is a finite $W_n k$-module, so by Proposition A.6 the map $1-T : pM \to pM$ is surjective. Now we have the two rows on the bottom of the following diagram being exact:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M^{1-T}/(pM)^{1-T} \arrow[r] \arrow[u] & M/p \arrow[r, "1-T"] \arrow[u] & M/p \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & M^{1-T} \arrow[r] \arrow[u] & M \arrow[r, "1-T"] \arrow[u] & M \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & (pM)^{1-T} \arrow[r] \arrow[u] & pM \arrow[r, "1-T"] \arrow[u] & pM \arrow[r] \arrow[u] & 0. \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The vertical maps between the last two rows are natural inclusions, and the first row is the cokernels of these inclusion maps. The snake lemma implies that the first row is exact, which means that

$$M^{1-T}/(pM)^{1-T} = (M/p)^{1-T}.$$

This is a finite $\mathbb{Z}/p^n$-module by the case $m = 1$. On the other hand, since $p^{m-1} \cdot pM = 0$, the induction hypothesis applied to the $W_n k$-module pM gives $(pM)^{1-T}$ which is a finite $\mathbb{Z}/p^n$-module. Now the vertical exact sequence on the left gives that M^{1-T} is a finite $\mathbb{Z}/p^n$-module. □

Proposition A.8. *Let k be a separably closed field of positive characteristic p. Then we have an identification of $W_n k$-modules*

$$M_{\mathrm{ss}} \simeq M^{1-T} \otimes_{\mathbb{Z}/p^n} W_n k.$$

Proof. For the finite dimensional k-vector space M/p, Proposition A.3 tells us that

$$(M/p)_{\mathrm{ss}} \simeq (M/p)^{1-T} \otimes_{\mathbf{F}_p} k.$$

In other words, there exist $m_1, \ldots, m_d \in M$ ($d = \dim_{\mathbf{F}_p} M/p$), such that $m_1 + pM, \ldots, m_d + pM \in (M/p)^{1-T}$ generate $(M/p)_{\mathrm{ss}}$ as a k-vector space. Because of Proposition A.7(1), one can choose $m_1, \ldots, m_d \in M^{1-T}$. Since M is a finite generated W_nk-module, M_{ss} as a submodule is also finitely generated over W_nk. Note moreover that $(M/p)_{\mathrm{ss}} = M_{\mathrm{ss}}/p$. Apply Nakayama's lemma, $m_1, \ldots, m_d \in M^{1-T}$ generate M_{ss} as a W_nk-module. $\square$

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