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THE HETEROTIC G_2 SYSTEM ON CONTACT CALABI–YAU 7-MANIFOLDS

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Abstract. We obtain non-trivial approximate solutions to the heterotic G_2 system on the total spaces of non-trivial circle bundles over Calabi–Yau 3-orbifolds, which satisfy the equations up to an arbitrarily small error, by adjusting the size of the S^1 fibres in proportion to a power of the string constant α'. Each approximate solution provides a cocalibrated G_2-structure, the torsion of which realises a non-trivial scalar field, a constant (trivial) dilaton field and an H-flux with non-trivial Chern–Simons defect. The approximate solutions also include a connection on the tangent bundle which, together with a non-flat G_2-instanton induced from the horizontal Calabi–Yau metric, satisfy the anomaly-free condition, also known as the heterotic Bianchi identity. The approximate solutions fail to be genuine solutions solely because the connections on the tangent bundle are only G_2-instantons up to higher order corrections in α'.

Contents

1. Introduction

The heterotic G_2 system intertwines geometric and gauge-theoretic degrees of freedom over a 7-manifold with G_2-structure, subject to instanton-type equations and a prescribed Chern–Simons defect. The latter constraint is required by what physicists refer to as the Green–Schwartz anomaly cancellation mechanism. This setup fits in the broader context of so-called Hull–Strominger systems on manifolds with special geometry, particularly in real dimensions 6, 7 and 8, which arise as low-energy effective theories of the heterotic string. Its motivation aside, we should

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stress from the outset that the language and arguments in this paper are primarily aimed at a mathematical audience.

To the best of our knowledge, the present problem was first formulated in the mathematics literature by Fernandez et al. [FIUV11], who found 'the first explicit compact valid supersymmetric heterotic solutions with non-zero flux, non-flat instanton and constant dilaton' on some carefully chosen generalised Heisenberg nilmanifolds. Moreover, they somewhat inspired our approach, by invoking the methods of Kobayashi [Kob56] to guarantee, albeit non-constructively, the existence of circle fibrations which *partially* satisfy the heterotic G_2 system [FIUV11, Theorem 6.4]. For a comprehensive survey of the problem's origins in the string theory literature, we refer the reader to that paper's Introduction and references therein.

Over recent years, such Hull–Strominger systems have attracted substantial interest. For instance, García-Fernández et al. have addressed description of infinitesimal moduli of solutions to these systems over a Calabi–Yau [GFRT17] or G_2-manifold [CGFT16] base, as well as an interpretation of the problem from the perspective of generalised Ricci flow on a Courant algebroid [GF19]. More recently still, Fino et al. [FGV21] have found solutions to the Hull–Strominger system in 6 dimensions using 2-torus bundles over K3 orbifolds, extending the fundamental work of Fu–Yau [FY08], which also has some relation to our study.

Our approach to the heterotic G_2 system will follow most closely the thorough investigation by de la Ossa et al. in [dlOLS16, dlOLS18a, dlOLS18b], who propose, among various contributions, a physically viable formulation of the problem for G_2-structures with torsion. Indeed, we study the system over so-called contact Calabi–Yau (cCY) 7-manifolds, which carry cocalibrated G_2-structures; cCY manifolds were introduced by [HV15], and gauge theory on 7-dimensional cCY was proposed in [CARSE20] and further studied in [PSE19]. Our base 7-manifolds include the total spaces of S^1-(orbi)bundles over every weighted Calabi–Yau 3-fold famously listed by Candelas-Lynker-Schimmrigk [CLS90], seen as links of isolated hypersurface singularities on $S^9 \subset \mathbb{C}^5$. In particular, we obtain *constructive* approximate solutions to the heterotic G_2-system over compact *simply-connected* (actually, 2-connected) 7-manifolds as in Example 2.3, which can be made arbitrarily close to genuine solutions by shrinking the circle fibres.

1.1. Heterotic G_2 system or G_2-Hull–Strominger system.

Definition 1.1. On a 7-manifold with G_2-structure (K^7, φ), we let $\psi = *\varphi \in \Omega^4(K)$ and recall the following characterisations of some components of $\Omega^\bullet(K)$ corresponding to irreducible G_2-representations:

$$\Omega^2_{14}(K) = \{\beta \in \Omega^2(K) : \beta \wedge \varphi = - * \beta\} = \{\beta \in \Omega^2(K) : \beta \wedge \psi = 0\},$$
$$\Omega^3_{27}(K) = \{\gamma \in \Omega^3(K) : \gamma \wedge \varphi = 0, \gamma \wedge \psi = 0\}.$$

The *torsion* of φ is completely described by the quantities $\tau_0 \in C^\infty(K)$, $\tau_1 \in \Omega^1(K)$, $\tau_2 \in \Omega^2_{14}(K)$ and $\tau_3 \in \Omega^3_{27}(K)$, which satisfy

$$d\varphi = \tau_0 \psi + 3\tau_1 \wedge \varphi + *\tau_3 \quad \text{and} \quad d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi.$$

Given a smooth G-bundle $F \to K$, for some compact semi-simple Lie group G, let $\mathcal{A}(F)$ denote its space of smooth G-connections.

Definition 1.2. The *heterotic* G_2 *system* or G_2*-Hull–Strominger system* on a 7-manifold with G_2-structure (K, φ) is comprised of the following degrees of freedom:

- Geometric fields (tensors):

$\lambda \in \mathbb{R}$ (scalar field), $\quad \mu \in C^\infty(K)$ (dilaton), $\quad$ and $\quad H \in \Omega^3(K)$ (flux).

- Gauge fields (connections):

$$A \in \mathcal{A}(E), \quad \text{and} \quad \theta \in \mathcal{A}(TK),$$

where $E \to K$ is a vector bundle and both connections are respectively G$_2$-instantons:

$$F_A \wedge \psi = 0 \quad \text{and} \quad R_\theta \wedge \psi = 0.$$

The geometric fields satisfy the following relations with the torsion of the G$_2$-structure φ:

$$\begin{aligned} \tau_0 &= \frac{3}{7}\lambda & H^\perp &= -\frac{1}{2}d\mu^\# \lrcorner \psi - \tau_3 \\ \tau_1 &= \frac{1}{2}d\mu & H &= \frac{\lambda}{14}\varphi \oplus H^\perp \\ \tau_2 &= 0 & \tau_3 &= -H^\perp - \frac{1}{2}d\mu^\# \lrcorner \psi. \end{aligned} \tag{1}$$

Given a (small) real constant $\alpha' \neq 0$, related to the string scale, the flux compensates exactly the Chern–Simons defect between the gauge fields via the *anomaly-free condition*, also referred to as the *heterotic Bianchi identity*:

$$dH = \frac{\alpha'}{4}\left(\operatorname{tr} F_A \wedge F_A - \operatorname{tr} R_\theta \wedge R_\theta\right), \tag{2}$$

where F_A is the curvature of A, R_θ is the Riemann curvature tensor of θ.

Remark 1.3. In the physics literature one obtains the heterotic G$_2$ system by truncating a system of equations involving (formal) power series in α'. Consequently, one finds statements such as that θ need only be a G$_2$-instanton on TK 'up to $O(\alpha')$-corrections', cf. [dlOS14, Appendix B]. One natural way to formulate the G$_2$-instanton condition to order $O(\alpha')^k$ is

$$|R_\theta \wedge \psi|_g = O(\alpha')^k, \quad \text{as } \alpha' \to 0,$$

where $|.|_g$ is the pointwise C^0-norm with respect to the G$_2$-metric g defined by $\varphi = *\psi$. (Note that θ, ψ and g can all depend on α'.) We will state our results in Theorem 1 in those terms. However, we should stress that we still adopt Definition 1.2 for a genuine solution to the heterotic G$_2$ system, in accordance with the mathematics literature on Hull–Strominger systems in 6 and 7 dimensions; see e.g. [FIUV11, Iva10, CGFT16, GF16, GFRT17, GFRT20].

Remark 1.4. For physical reasons one typically assumes $\alpha' > 0$ in (2), so we are not interested in the case $dH = 0$. Hence, (2) only has any hope of occurring under the so-called *omalous* condition:

$$p_1(E) = p_1(K) \in H^4_{dR}(K). \tag{3}$$

Omalous bundles can be systematically constructed for instance via monad techniques, as in the following example, which is derived trivially by combining results from [HJ13, CARSE20]. In this paper, though, we will follow a different approach, cf. Theorem 1.

Example 1.5. When K is the link in S^9 associated to the Fermat quintic $V \subset \mathbb{P}^4$, the cohomology of the monad

$$0 \longrightarrow \mathcal{O}_V(-1)^{\oplus 10} \longrightarrow \mathcal{O}_V^{\oplus 22} \longrightarrow \mathcal{O}_V(1)^{\oplus 10} \longrightarrow 0$$

is a rank 2 omalous bundle E, i.e. satisfying (3), with $c_1 = 0$ and $c_2 = 10$.

Remark 1.6. Fernandez et al. [FIUV11] argue that one can replace the G_2-instanton condition on R_θ by a more general second order condition, and still satisfy the equations of motion which motivate the heterotic G_2 system. However, Ivanov concluded separately that in this context both conditions are equivalent [Iva10, §2.3.1].

1.2. **Gauge theory on contact Calabi–Yau (cCY) manifolds.** Let (M^{2n+1}, η, ξ) denote a contact manifold, with contact form η and Reeb vector field ξ [BG08]. When M is endowed in addition with a Sasakian structure, namely an integrable transverse complex structure J and a compatible metric g, Biswas-Schumacher [BS10] propose a natural notion of Sasakian holomorphic structure for complex vector bundles $E \to M$.

We recall that a connection A on a complex vector bundle over a Kähler manifold is said to be *Hermitian Yang-Mills (HYM)* if

$$\hat{F}_A := (F_A, \omega) = 0 \quad \text{and} \quad F_A^{0,2} = 0. \tag{4}$$

This notion extends to Sasakian bundles, by taking $\omega := d\eta \in \Omega^{1,1}(M)$ as a 'transverse Kähler form', and defining HYM connections to be the solutions of (4) in that sense. The well-known concept of *Chern connection* also extends, namely as a connection mutually compatible with the holomorphic structure (*integrable*) and a given Hermitian bundle metric (*unitary*), see [BS10, § 3].

An important class of Sasakian manifolds are those endowed with a *contact Calabi–Yau (cCY)* structure [Definition 2.1], the Riemannian metrics of which have transverse holonomy $\mathrm{SU}(2n+1)$, in the sense of foliations, corresponding to the existence of a global transverse holomorphic volume form $\Omega \in \Omega^{n,0}(M)$ [HV15]. When $n = 3$, cCY 7-manifolds are naturally endowed with a G_2-structure defined by the 3-form

$$\varphi := \eta \wedge d\eta + \operatorname{Re}\Omega, \tag{5}$$

which is *cocalibrated*, in the sense that its Hodge dual $\psi := *_g\varphi$ is closed under the de Rham differential. When a 3-form φ on a 7-manifold defines a G_2-structure, the condition

$$F_A \wedge \psi = 0 \tag{6}$$

is referred to as the G_2*-instanton equation.* On holomorphic Sasakian bundles over closed cCY 7-manifolds, it has the distinctive feature that integrable solutions (i.e. compatible with the holomorphic Sasakian structure) are indeed Yang-Mills critical points, even though the G_2-structure has torsion [CARSE20].

1.3. **Statement of main result.**

Definition 1.7. Let V be a Calabi–Yau 3-orbifold with metric g_V, volume form vol_V, Kähler form ω and holomorphic volume form Ω satisfying

$$\mathrm{vol}_V = \frac{\omega^3}{3!} = \frac{\operatorname{Re}\Omega \wedge \operatorname{Im}\Omega}{4}. \tag{7}$$

Suppose that the total space of $\pi : K \to V$ is a contact Calabi–Yau 7-manifold, i.e. K is a S^1-(orbi)bundle, with connection 1-form η, such that[1] $d\eta = \omega$. For every $\varepsilon > 0$, we define a S^1-invariant G_2-structure on K by

$$\varphi_\varepsilon = \varepsilon\eta \wedge \omega + \operatorname{Re}\Omega, \tag{8}$$

$$\psi_\varepsilon = \frac{1}{2}\omega^2 - \varepsilon\eta \wedge \operatorname{Im}\Omega. \tag{9}$$

The metric induced from this G_2-structure and its corresponding volume form are

$$g_\varepsilon = \varepsilon^2\eta \otimes \eta + g_V \quad \text{and} \quad \mathrm{vol}_\varepsilon = \varepsilon\eta \wedge \mathrm{vol}_V. \tag{10}$$

NB.: The choice of $\varepsilon > 0$ will a posteriori depend on the string parameter α' in (2).

We will see that producing geometric fields satisfying the prescribed relations (1) with the torsion of the G_2-structure (8) is rather straightforward. The actual problem consists in obtaining gauge fields that satisfy the heterotic Bianchi identity (2) on the contact Calabi–Yau K^7. We introduce therefore the following data:

- Let $A := \pi^*\Gamma_V$ be the pullback of the Levi-Civita connection of g_V to $E := \pi^*TV \to K$. Then A is a G_2-instanton on E, since it is the pullback of a HYM connection on TV [CARSE20, §4.3]. Moreover, A is a Yang-Mills connection and it minimises the Yang-Mills energy among Chern connections, with respect to the natural Sasakian holomorphic structure of E [ibid., Theorem 1.4].
- For each fixed $\varepsilon > 0$, let θ_ε denote the Levi-Civita connection of the metric g_ε on K of Definition 1.7. It was shown in [FI03] that there is a unique metric connection which makes φ_ε parallel and has totally skew-symmetric torsion (which may be identified with H_ε), often called the *Bismut connection* (and also sometimes called the *canonical connection*). Following work in [Hul86], another natural metric connection which appears in the physics literature is the *Hull connection*, whose torsion has the opposite sign to the Bismut connection. The Bismut and Hull connections fit in a 1-parameter family $\{\theta_\varepsilon^\delta\}$, which are modifications of θ_ε by a prescribed torsion component governed by the parameter $\delta \in \mathbb{R}$ and the flux H_ε. We further extend it to a 2-parameter family $\{\theta_\varepsilon^{\delta,k}\}$, with[2] $k \in \mathbb{R} \setminus \{0\}$, corresponding to 'squashings' of the connections $\theta_\varepsilon^\delta$. Finally, we define a 'twist' by an additional parameter $m \in \mathbb{R}$, to obtain our overall family of connections $\{\theta_{\varepsilon,m}^{\delta,k}\}$ on TK [Proposition 3.21]. Whilst typically $\theta_{\varepsilon,m}^{\delta,k}$ will *not* be a G_2-instanton on TK, it does satisfy the G_2-instanton condition up to $O(\alpha')$-corrections (in the sense of Remark 1.3) for various parameter choices.

Theorem 1. *Let $(K^7, \eta, \xi, J, \Omega)$ be a contact Calabi–Yau 7-manifold, fibering by $\pi : K^7 \to V$ over the Calabi–Yau 3-fold $(V, g_V, \omega, J, \Omega)$, and let $E := \pi^*TV \to K$.*

Given any $\alpha' > 0$, there exist $k(\alpha'), \varepsilon(\alpha') > 0$ and $m, \delta \in \mathbb{R}$ such that the following assertions hold:

(i) *The G_2-structure (8) is coclosed and satisfies the torsion conditions (1), with scalar field $\lambda = \frac{\varepsilon}{2}$, constant dilaton $\mu \in \mathbb{R}$, and flux $H_\varepsilon = -\varepsilon^2\eta \wedge \omega + \varepsilon \operatorname{Re}\Omega$.*

[1] For ease of notation, we omit the pullback π^* for forms and tensors defined on K which are pulled back from V.

[2] Choosing $k = 0$ would in fact require the S^1-fibration $K \to V$ to be trivial, see Remark 3.6.

(ii) *The connection* $A := \pi^*\Gamma_V$ *is a* G_2*-instanton on* E*, with respect to the dual* 4*-form* (9)*.*

(iii) *There exists a connection* $\theta := \theta^{\delta,k}_{\varepsilon,m}$ *on* TK*, with torsion*

$$H^{\delta,k}_{\varepsilon,m} = \big(1 - k - \frac{km}{2}\big)\varepsilon^2\omega\otimes\eta + \frac{km\varepsilon^2}{2}\eta\wedge\omega + k\delta H_\varepsilon,$$

which satisfies the G_2*-instanton condition* (6) *to order* $O(\alpha')^2$ *with respect to the dual* 4*-form* (9)*; i.e.*

$$|R_\theta\wedge\psi_\varepsilon|_{g_\varepsilon} = O(\alpha')^2 \quad \text{as } \alpha'\to 0. \tag{11}$$

(iv) *The data* $(H_\varepsilon, A, \theta)$ *satisfy the heterotic Bianchi identity* (2)*:*

$$dH_\varepsilon = \frac{\alpha'}{4}(\operatorname{tr} F_A^2 - \operatorname{tr} R_\theta^2). \tag{12}$$

(v) $\lim_{\alpha'\to 0}\varepsilon(\alpha') = 0$ *and* $\lim_{\alpha'\to 0} k(\alpha') = \infty$.

The various components of the proof are developed throughout the paper, and aggregated in §4.4.

2. Contact Calabi–Yau geometry: scalar field, dilaton, and flux

One may interpret special structure group reductions on compact odd-dimensional Riemannian manifolds as 'transverse even-dimensional' structures with respect to a S^1-action. So for instance contact geometry may be seen as transverse symplectic geometry, almost-contact geometry as tranverse almost-complex geometry, and in the same way Sasakian geometry as transverse Kähler geometry. In particular, one may consider reduction of the transverse holonomy group; indeed Sasakian manifolds with transverse holonomy $\mathrm{SU}(n)$ are studied by Habib and Vezzoni [HV15, § 6.2.1]:

Definition 2.1. A Sasakian manifold $(K^{2n+1}, \eta, \xi, J, \Omega)$ is said to be a *contact Calabi–Yau manifold* (cCY) if Ω is a nowhere-vanishing transverse form of horizontal type $(n, 0)$, such that

$$\Omega\wedge\bar\Omega = (-1)^{\frac{n(n+1)}{2}}\mathbf{i}^n\omega^n \quad \text{and} \quad d\Omega = 0, \quad \text{with} \quad \omega = d\eta.$$

Let us specialise to real dimension 7. It is well-known that, for a Calabi–Yau 3-fold (V, ω, Ω), the product $V\times \mathrm{S}^1$ has a natural torsion-free G_2-structure defined by: $\varphi := dt\wedge\omega + \operatorname{Re}\Omega$, where t is the coordinate on S^1. The Hodge dual of φ is

$$\psi := *\varphi = \frac{1}{2}\omega\wedge\omega + dt\wedge\operatorname{Im}\Omega \tag{13}$$

and the induced metric $g_\varphi = g_V + dt\otimes dt$ is the Riemannian product metric on $V\times S^1$ with holonomy $\mathrm{Hol}(g_\varphi) = \mathrm{SU}(3)\subset \mathrm{G}_2$. A contact Calabi–Yau structure essentially emulates all of these features, albeit its G_2-structure has some symmetric torsion.

Proposition 2.2 ([HV15, §6.2.1])**.** *Every cCY manifold* $(K^7, \eta, \xi, J, \Omega)$ *is an* S^1*-bundle* $\pi : K\to V$ *over a Calabi–Yau 3-orbifold* (V, ω, Ω)*, with connection 1-form* η *and curvature*

$$d\eta = \omega, \tag{14}$$

and it carries a cocalibrated G_2*-structure*

$$\varphi := \eta \wedge \omega + \operatorname{Re}\Omega, \tag{15}$$

with torsion $d\varphi = \omega \wedge \omega$ *and Hodge dual* 4*-form* $\psi = *\varphi = \frac{1}{2}\omega \wedge \omega + \eta \wedge \operatorname{Im}\Omega$*.*

Example 2.3 (Calabi–Yau links for $k = 1$)**.** Given a rational weight vector $w = (w_0, \dots, w_4) \in \mathbb{Q}^5$, a w-weighted homogeneous polynomial $f \in \mathbb{C}[z_0, \dots, z_4]$ of degree $d = \sum_{i=0}^{4} w_i$ cuts out an affine hypersurface $\mathcal{V} = (f)$ with an isolated singularity at $0 \in \mathbb{C}^5$.

Its link $K_f := \mathcal{V} \cap S^9 \subset \mathbb{C}^5$ on a local 9-sphere is a compact and 2-connected smooth cCY 7-manifold, fibering by circles over a Calabi–Yau 3-orbifold $V \subset \mathbb{P}^4(w)$ by the weighted Hopf fibration [CARSE20, Theorem 1.1]:

$$\begin{tikzcd} K_f^7 \arrow[r, hook] \arrow[d] & S^9 \arrow[d] \\ V \arrow[r, hook] & \mathbb{P}^4(w) \end{tikzcd}$$

In particular, V can be assumed to be any of the weighted Calabi–Yau 3-folds listed by Candelas-Lynker-Schimmrigk [CLS90]. For a detailed survey on Calabi–Yau links, see [CARSE20, §2]. The $\mathbb{C}$-family of Fermat quintics yields but the simplest of instances, and indeed the only one for which the base V is smooth.

2.1. **Torsion forms and flux of the** G_2**-structure** φ_ε**.** We begin by addressing the heterotic G_2 system conditions (1) on the G_2-structure, as prescribed by [dlOLS16]. In particular, we identify the components of the torsion corresponding to the scalar field, the dilaton and the flux, as asserted in Theorem 1(i).

We see from (8), (9), (14), and the fact that V is Calabi–Yau, that

$$d\varphi_\varepsilon = \varepsilon\omega^2 \quad \text{and} \quad d\psi_\varepsilon = 0, \tag{16}$$

so that the G_2-structures of Definition 1.7 are coclosed. We can now compute their torsion forms.

Lemma 2.4. *For each* $\varepsilon > 0$*, the* G_2*-structure on* K^7 *defined by* (8)*–*(9) *has torsion forms*

$$\begin{aligned} \tau_0 &= \frac{6}{7}\varepsilon, & \tau_1 &= 0, \\ \tau_2 &= 0, & \tau_3 &= \frac{8}{7}\varepsilon^2\eta \wedge \omega - \frac{6}{7}\varepsilon \operatorname{Re}\Omega. \end{aligned}$$

Proof. The fact that τ_1 and τ_2 vanish is an immediate consequence of (16). Again by (16) and definition of the torsion forms, we have:

$$d\varphi_\varepsilon = \varepsilon\omega^2 = \tau_0\psi_\varepsilon + *_\varepsilon\tau_3. \tag{17}$$

Thus, using $\omega \wedge \Omega = 0$, we find

$$7\tau_0 \mathrm{vol}_\varepsilon = d\varphi_\varepsilon \wedge \varphi_\varepsilon = \varepsilon\omega^2 \wedge (\varepsilon\eta \wedge \omega) = 6\varepsilon(\varepsilon\eta \wedge \frac{\omega^3}{3!}). \tag{18}$$

We further deduce from (18) and the expression of volume form (10) that

$$\tau_0 = \frac{6}{7}\varepsilon. \tag{19}$$

Moreover, substituting (19) into (17), we see that

$$*_\varepsilon\tau_3 = d\varphi_\varepsilon - \tau_0\psi_\varepsilon = \varepsilon\omega^2 - \frac{6}{7}\varepsilon(\frac{1}{2}\omega^2 - \varepsilon\eta\wedge\operatorname{Im}\Omega) = \frac{4}{7}\varepsilon\omega^2 + \frac{6}{7}\varepsilon^2\eta\wedge\operatorname{Im}\Omega. \tag{20}$$

Therefore, using (10) and (20) we obtain

$$\tau_3 = \frac{8}{7}\varepsilon *_\varepsilon (\frac{1}{2}\omega^2) + \frac{6}{7}\varepsilon *_\varepsilon (\varepsilon\eta\wedge\operatorname{Im}\Omega) = \frac{8}{7}\varepsilon^2\eta\wedge\omega - \frac{6}{7}\varepsilon\operatorname{Re}\Omega. \qquad \square$$

We may compute the flux of the G_2 structure φ_ε as follows.

Lemma 2.5. *In the situation of Lemma* 2.4*, the flux of the* G_2 *structure* φ_ε *is*

$$H_\varepsilon = -\varepsilon^2\eta\wedge\omega + \varepsilon\operatorname{Re}\Omega. \tag{21}$$

Hence,

$$dH_\varepsilon = -\varepsilon^2\omega^2. \tag{22}$$

Proof. From Definition 1.2 and the Lemma, we compute directly:

$$\begin{aligned} H_\varepsilon &= \frac{\lambda}{14}\varphi_\varepsilon + (H_\varepsilon)^\perp = \frac{\tau_0}{6}\varphi_\varepsilon - \tau_3 \\ &= \frac{1}{7}\varepsilon(\varepsilon\eta\wedge\omega + \operatorname{Re}\Omega) - (\frac{8}{7}\varepsilon^2\eta\wedge\omega - \frac{6}{7}\varepsilon\operatorname{Re}\Omega) \\ &= -\varepsilon^2\eta\wedge\omega + \varepsilon\operatorname{Re}\Omega. \end{aligned} \qquad \square$$

2.2. **Local orthonormal coframe.** One key strategy in our construction consists in varying the length of the S^1-fibres on K as a function of the string parameter α'. With that in mind, we adopt a useful local orthonormal coframe as follows.

Definition 2.6. Given $\varepsilon > 0$, let $(K^7, \varphi_\varepsilon)$ be as in Definition 1.7. We choose the local Sasakian real orthonormal coframe on K:

$$e_0 = \varepsilon\eta, \quad e_1, \quad e_2, \quad e_3, \quad Je_1, \quad Je_2, \quad Je_3, \tag{23}$$

where J is the transverse complex structure (from the Calabi–Yau 3-fold V) acting on 1-forms, and we have a basic SU(3)-coframe $\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}$, the pullback of an SU(3)-coframe on V, such that

$$\omega = e_1\wedge Je_1 + e_2\wedge Je_2 + e_3\wedge Je_3, \tag{24}$$

$$\Omega = (e_1 + iJe_1)\wedge(e_2 + iJe_2)\wedge(e_3 + iJe_3). \tag{25}$$

Remark 2.7. It is worth noting from (25) that

$$\operatorname{Re}\Omega = e_1\wedge e_2\wedge e_3 - e_1\wedge Je_2\wedge Je_3 - e_2\wedge Je_3\wedge Je_1 - e_3\wedge Je_1\wedge Je_2, \tag{26}$$

$$\operatorname{Im}\Omega = Je_1\wedge e_2\wedge e_3 + Je_2\wedge e_3\wedge e_1 + Je_3\wedge e_1\wedge e_2 - Je_1\wedge Je_2\wedge Je_3. \tag{27}$$

Using (24) and (27), we easily derive the precise expression of ψ_ε in this frame:

$$\begin{aligned} \psi_\varepsilon &= \frac{1}{2}\omega^2 - \varepsilon\eta\wedge\operatorname{Im}\Omega \\ &= e_2\wedge Je_2\wedge e_3\wedge Je_3 + e_3\wedge Je_3\wedge e_1\wedge Je_1 + e_1\wedge Je_1\wedge e_2\wedge Je_2 \\ &\quad - e_0\wedge(Je_1\wedge e_2\wedge e_3 + Je_2\wedge e_3\wedge e_1 + Je_3\wedge e_1\wedge e_2 - Je_1\wedge Je_2\wedge Je_3). \end{aligned} \tag{28}$$

Lemma 2.8. *In terms of the local coframe* (23) *and the natural matrix operations described in Appendix* A*, the basic* 3*-covectors* $e = (e_1, e_2, e_3)^{\mathrm{T}}$ *and* $Je = (Je_1, Je_2, Je_3)^{\mathrm{T}}$ *in* $\Omega^1(K)^{\oplus 3}$ *have the following properties.*

(a) *The vectors*

$$e \times Je \quad \textit{and} \quad e \times e - Je \times Je$$

consist of basic forms of type $(2,0)+(0,2)$.

(b) *The vector*

$$e \times e + Je \times Je$$

and the off-diagonal part of

$$[e] \wedge [Je] - [Je] \wedge [e] \tag{29}$$

consist of basic forms of type $(1,1)$ *which are also primitive (i.e. wedge with* ω^2 *to give zero). The diagonal part of* (29) *consists of basic forms of type* $(1,1)$.

Proof. For (a), we notice that

$$e_2 \wedge Je_3 - e_3 \wedge Je_2 = \mathrm{Im}((e_2 + iJe_2) \wedge (e_3 + iJe_3)),$$
$$e_2 \wedge e_3 - Je_2 \wedge Je_3 = \mathrm{Re}((e_2 + iJe_2) \wedge (e_3 + iJe_3)).$$

We deduce that $e \times Je$ and $e \times e - Je \times Je$ consist of basic forms of type $(2,0)+(0,2)$ as claimed.

For (b), we observe that

$$e_2 \wedge e_3 + Je_2 \wedge Je_3 = \mathrm{Re}((e_2 + iJe_2) \wedge (e_3 - iJe_3)),$$

and hence $e \times e + Je \times Je$ consists of primitive forms of basic type $(1,1)$. We now note that

$$[e] \wedge [Je] - [Je] \wedge [e] = e \wedge Je^{\mathrm{T}} - Je \wedge e^{\mathrm{T}} - 2\omega I \tag{30}$$

by Lemma A.3. Since

$$e_2 \wedge Je_3 + e_3 \wedge Je_2 = \mathrm{Im}((e_2 - iJe_2) \wedge (e_3 + iJe_3)),$$

we deduce that the off-diagonal part of $[e] \wedge [Je] - [Je] \wedge [e]$ consists of forms of basic type $(1,1)$ which are primitive also. Finally, we now see from (30) that the diagonal entries in $[e] \wedge [Je] - [Je] \wedge [e]$ define the diagonal matrix

$$-2\,\mathrm{diag}(e_2 \wedge Je_2 + e_3 \wedge Je_3, e_3 \wedge Je_3 + e_1 \wedge Je_1, e_1 \wedge Je_1 + e_2 \wedge Je_2), \tag{31}$$

which clearly consists of basic forms of type $(1,1)$. □

3. Gauge fields: G_2-instanton, Bismut, Hull and twisted connections

It is well-known that the pullback of a basic HYM connection to the total space of a contact Calabi–Yau (cCY) 7-manifold is a G_2-instanton, with respect to the standard G_2-structure [CARSE20, §4.3]. Since the Levi-Civita connection of the Calabi–Yau (V, g_V) on TV is HYM, the following result establishes Theorem 1(ii).

Lemma 3.1. *Let* $E = \pi^* TV$ *be the pullback of* TV *to* K *via the projection* $\pi : K \to V$. *Let* A *be the connection on* E *given by the pullback of the Levi-Civita connection of* g_V. *Then* A *is a* G_2*-instanton on* E *with holonomy contained in* $\mathrm{SU}(3)$.

In this section we give formulae for the connections $\theta^{\delta,k}_{\varepsilon,m}$ and A and their curvatures with respect to the local coframe in Definition 2.6. Using the freedom given by all three parameters, we will show that $\theta^{\delta,k}_{\varepsilon,m}$ can be chosen to satisfy the G_2-instanton condition, at least to higher orders of the string scale α'.

3.1. **The $\mathbf{G_2}$-instanton A and the 'squashings' θ_ε^k of the Levi-Civita connection.**

3.1.1. *Local connection matrices.* Since the choice of a local Sasakian coframe on K naturally trivialises $E = \pi^*TV \hookrightarrow TK$, we now want to relate the local matrix of the Levi-Civita connection θ_ε on TK to (the pullback of) the gauge field A. To that end, we compute the first structure equations of our natural coframe:

Proposition 3.2. *The coframe* (23) *on K satisfies the following structure equations:*

$$de_0 = \varepsilon\omega = \varepsilon(e_1 \wedge Je_1 + e_2 \wedge Je_2 + e_3 \wedge Je_3), \tag{32}$$

$$de_i = -a_{ij} \wedge e_j - b_{ij} \wedge Je_j, \tag{33}$$

$$d(Je_i) = b_{ij} \wedge e_j - a_{ij} \wedge Je_j, \tag{34}$$

for some local 1-forms a_{ij}, b_{ij}, using the summation convention, with $1 \le i, j \le 3$. Moreover,

$$a_{ji} = -a_{ij}, \quad b_{ji} = b_{ij}, \quad \sum_{i=1}^{3} b_{ii} = 0, \tag{35}$$

so the matrix $a := (a_{ij})$ is skew-symmetric, and the matrix $b := (b_{ij})$ is symmetric traceless. Letting $I := (\delta_{ij})$ and $e := (e_1\ e_2\ e_3)^{\mathrm{T}}$, the structure equations (32)–(34) *can be written in terms of 7×7 matrices:*

$$d\begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} = -\begin{pmatrix} 0 & \frac{\varepsilon}{2}Je^{\mathrm{T}} & -\frac{\varepsilon}{2}e^{\mathrm{T}} \\ -\frac{\varepsilon}{2}Je & a & b - \frac{\varepsilon}{2}e_0 I \\ \frac{\varepsilon}{2}e & -b + \frac{\varepsilon}{2}e_0 I & a \end{pmatrix} \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix}. \tag{36}$$

Proof. The first equation (32) is a direct consequence of (23) and (24). The relationship between the derivatives of e_i and Je_i and the properties of the a_{ij} and b_{ij} are a consequence of J being covariantly constant (on V) and A having holonomy contained in SU(3), since A arises from a torsion-free SU(3)-structure. □

It will be useful later to have the following corollary of the structure equations, which is an elementary computation using (36).

Proposition 3.3. *Using the notation of Definition* A.1, *the coframe in Definition* 2.6 *satisfies*

$$\begin{aligned} d([e]) &= -a \wedge [e] - [e] \wedge a + b \wedge [Je] - [Je] \wedge b, \\ d([Je]) &= -a \wedge [Je] - [Je] \wedge a - b \wedge [e] + [e] \wedge b. \end{aligned} \tag{37}$$

The matrix in (36) represents the Levi-Civita connection θ_ε in the given local coframe, and setting $\varepsilon = 0$ in that matrix gives the matrix of A. Hence, we have the following.

Corollary 3.4. *If we let*

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix} \tag{38}$$

and

$$B = \begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & 0 & -e_0 I \\ e & e_0 I & 0 \end{pmatrix}, \tag{39}$$

then the Levi-Civita connection θ_ε of the metric g_ε in (10) *is given locally by*

$$\theta_\varepsilon = \begin{pmatrix} 0 & \frac{\varepsilon}{2}Je^{\mathrm{T}} & -\frac{\varepsilon}{2}e^{\mathrm{T}} \\ -\frac{\varepsilon}{2}Je & a & b-\frac{\varepsilon}{2}e_0 I \\ \frac{\varepsilon}{2}e & -b+\frac{\varepsilon}{2}e_0 I & a \end{pmatrix} = A + \frac{\varepsilon}{2}B.$$

Corollary 3.4 allows us to define a family of connections θ_ε^k on TK as follows.

Definition 3.5. For each $0 \neq k \in \mathbb{R}$, let θ_ε^k be the connection on TK given, in the local coframe of Definition 2.6, by

$$\theta_\varepsilon^k := A + \frac{k\varepsilon}{2}B,$$

with A and B as in Corollary 3.4.

Remark 3.6. The *trivial case* $k=0$ can only occur when $K = S^1 \times V$ is a trivial bundle over V, and then the connection on TK will be equal to the pullback of the Levi-Civita connection on V (trivial along S^1). Since we are assuming that $K \to V$ is a non-trivial S^1-bundle, we require $k \neq 0$.

Remark 3.7. Notice that

$$\begin{aligned} d\begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} &= -\left(A + \frac{k\varepsilon}{2}B\right) \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} + \frac{(k-1)\varepsilon}{2} B \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} \\ &= -\theta_\varepsilon^k \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} + (1-k)\varepsilon \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, we may view θ_ε^k as a metric connection on K, with torsion $(1-k)\varepsilon\omega \otimes e_0$. Since $k \neq 0$, we see from Corollary 3.4 and Definition 3.5 that we may view θ_ε^k as a 'squashing' of the Levi-Civita connection θ_ε of the metric g_ε on K.

3.1.2. *Local curvature matrices.* We begin by relating the curvature of the connections θ_ε^k in Definition 3.5 to the curvature F_A of A.

Proposition 3.8. *In the local coframe of Definition* 2.6*, the curvature $R_{\theta_\varepsilon^k}$ of the connection θ_ε^k from Definition* 3.5 *satisfies*

$$R_{\theta_\varepsilon^k} = F_A + \frac{k\varepsilon^2}{2}\omega\mathcal{I} + \frac{k^2\varepsilon^2}{4} B \wedge B,$$

where

$$\mathcal{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -I \\ 0 & I & 0 \end{pmatrix} \tag{40}$$

and

$$\begin{aligned}B\wedge B &= \begin{pmatrix} 0 & e_0\wedge e^{\mathrm{T}} & e_0\wedge Je^{\mathrm{T}}\\ -e_0\wedge e & -Je\wedge Je^{\mathrm{T}} & Je\wedge e^{\mathrm{T}}\\ -e_0\wedge Je & e\wedge Je^{\mathrm{T}} & -e\wedge e^{\mathrm{T}}\end{pmatrix}\\ &= e_0\wedge\begin{pmatrix} 0 & e^{\mathrm{T}} & Je^{\mathrm{T}}\\ -e & 0 & 0\\ -Je & 0 & 0\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0\\ 0 & -Je\wedge Je^{\mathrm{T}} & Je\wedge e^{\mathrm{T}}\\ 0 & e\wedge Je^{\mathrm{T}} & -e\wedge e^{\mathrm{T}}\end{pmatrix}.\end{aligned} \tag{41}$$

Proof. From the relation between θ^k_ε and A in Corollary 3.4, we see that

$$\begin{aligned}R_{\theta^k_\varepsilon} &= d\theta^k_\varepsilon + \theta^k_\varepsilon\wedge\theta^k_\varepsilon\\ &= dA + \frac{k\varepsilon}{2}dB + (A+\frac{k\varepsilon}{2}B)\wedge(A+\frac{k\varepsilon}{2}B)\\ &= F_A + \frac{k\varepsilon}{2}(dB + A\wedge B + B\wedge A) + \frac{k^2\varepsilon^2}{4}B\wedge B.\end{aligned} \tag{42}$$

The first term of interest in (42) is

$$\begin{aligned}&dB + A\wedge B + B\wedge A\\ &= \begin{pmatrix} 0 & d(Je^{\mathrm{T}})+Je^{\mathrm{T}}\wedge a+e^{\mathrm{T}}\wedge b & -d(e^{\mathrm{T}})+Je^{\mathrm{T}}\wedge b-e^{\mathrm{T}}\wedge a\\ -d(Je)-a\wedge Je+b\wedge e & b\wedge e_0 I+e_0 I\wedge b & -d(e_0)I-a\wedge e_0 I-e_0 I\wedge a\\ d(e)+b\wedge Je+a\wedge e & d(e_0)I+a\wedge e_0 I+e_0 I\wedge a & b\wedge e_0 I+e_0 I\wedge b\end{pmatrix}\\ &= \varepsilon\omega\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -I\\ 0 & I & 0\end{pmatrix} = \varepsilon\omega\mathcal{I}\end{aligned} \tag{43}$$

as a consequence of the structure equations for the coframe in Proposition 3.2. Equation (41) follows directly from (39). □

At this point, it is worth recalling that A is a G_2-instanton, in fact the lift of a connection with holonomy SU(3) on V, so F_A must take values in $\mathfrak{su}(3)\subseteq\mathfrak{g}_2$:

$$F_A = \begin{pmatrix} 0 & 0 & 0\\ 0 & \alpha & \beta\\ 0 & -\beta & \alpha\end{pmatrix}, \tag{44}$$

where α is a skew-symmetric 3×3 matrix of 2-forms, and β is a symmetric traceless 3×3 matrix of 2-forms.

Lemma 3.9. *The block-elements of the curvature matrix* (44) *of A, in the local coframe* (23), *satisfy:*

$$\begin{aligned}\alpha\wedge e + \beta\wedge Je &= 0,\\ \alpha\wedge Je - \beta\wedge e &= 0.\end{aligned} \tag{45}$$

Moreover, using the notation of Definition A.1, *we have*

$$\begin{aligned}\alpha\wedge[e] - [e]\wedge\alpha - \beta\wedge[Je] - [Je]\wedge\beta &= 0,\\ \alpha\wedge[Je] + [Je]\wedge\alpha + \beta\wedge[e] - [e]\wedge\beta.&\end{aligned} \tag{46}$$

Proof. Differentiating the defining relation

$$d\begin{pmatrix}0\\ e\\ Je\end{pmatrix} = -A\wedge\begin{pmatrix}0\\ e\\ Je\end{pmatrix},$$

we obtain

$$\begin{aligned}
0 &= -dA \wedge \begin{pmatrix} 0 \\ e \\ Je \end{pmatrix} + A \wedge d \begin{pmatrix} 0 \\ e \\ Je \end{pmatrix} = -(dA + A \wedge A) \wedge \begin{pmatrix} 0 \\ e \\ Je \end{pmatrix} \\
&= -F_A \wedge \begin{pmatrix} 0 \\ e \\ Je \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix} \wedge \begin{pmatrix} 0 \\ e \\ Je \end{pmatrix} \\
&= - \begin{pmatrix} 0 \\ \alpha \wedge e + \beta \wedge Je \\ \alpha \wedge Je - \beta \wedge e \end{pmatrix}.
\end{aligned}$$

Equation (46) follows similarly from the structure equations (37). □

3.2. **The 'squashed' Bismut and Hull connections on TK.** We now introduce an additional parameter to our connections which introduces a multiple of the flux H_ε as torsion. This, in particular, leads us to the Bismut and Hull connections.

3.2.1. *Local connection matrices and torsion.* We begin by identifying the flux H_ε with a locally defined matrix of 1-forms and a vector-valued 2-form as follows, so that we can define connections with torsion given by the flux.

Proposition 3.10. *In the local coframe of Definition* 2.6*, and using the notation from Definition* A.1*, let*

$$C := \begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & -[e] & e_0 I + [Je] \\ e & -e_0 I + [Je] & [e] \end{pmatrix} = \begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & -[e] & [Je] \\ e & [Je] & [e] \end{pmatrix} - e_0 \mathcal{I}. \tag{47}$$

Then we may raise an index on the 3*-form H_ε and view it as a vector-valued* 2*-form, as follows:*

$$H_\varepsilon = \frac{\varepsilon}{2} \begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & -[e] & e_0 I + [Je] \\ e & -e_0 I + [Je] & [e] \end{pmatrix} \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} = \frac{\varepsilon}{2} C \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix}. \tag{48}$$

Proof. By Lemma 2.5, (24) and (26), we have that

(49)

$$\begin{aligned}
H_\varepsilon &= -\varepsilon^2 \eta \wedge \omega + \varepsilon \operatorname{Re} \Omega \\
&= -\varepsilon e_0 \wedge (e_1 \wedge Je_1 + e_2 \wedge Je_2 + e_3 \wedge Je_3) \\
&\qquad + \varepsilon (e_1 \wedge e_2 \wedge e_3 - e_1 \wedge Je_2 \wedge Je_3 - e_2 \wedge Je_3 \wedge Je_1 - e_3 \wedge Je_1 \wedge Je_2).
\end{aligned}$$

We raise an index, so that H_ε is a vector-valued 2-form, and use Lemma A.3 to deduce the claim:

$$H_\varepsilon = \varepsilon \begin{pmatrix} -e_1\wedge Je_1 - e_2\wedge Je_2 - e_3\wedge Je_3 \\ e_0\wedge Je_1 + e_2\wedge e_3 - Je_2\wedge Je_3 \\ e_0\wedge Je_2 + e_3\wedge e_1 - Je_3\wedge Je_1 \\ e_0\wedge Je_3 + e_1\wedge e_2 - Je_1\wedge Je_2 \\ -e_0\wedge e_1 - e_2\wedge Je_3 + e_3\wedge Je_2 \\ -e_0\wedge e_1 - e_3\wedge Je_1 + e_1\wedge Je_3 \\ -e_0\wedge e_1 - e_1\wedge Je_2 + e_2\wedge Je_1 \end{pmatrix}$$

$$= \frac{\varepsilon}{2}\begin{pmatrix} 0 & Je^{\mathrm{T}} & -e^{\mathrm{T}} \\ -Je & -[e] & e_0 I + [Je] \\ e & -e_0 I + [Je] & [e] \end{pmatrix} \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix}. \qquad \square$$

Corollary 3.11. *In the terms of Definition* 3.5 *and Proposition* 3.10*, let* $\tau_\varepsilon := \varepsilon C$*; then each local matrix*

$$\theta_\varepsilon^{\delta,k} = \theta_\varepsilon^k + \frac{k\delta}{2}\tau_\varepsilon = A + \frac{k\varepsilon}{2}B + \frac{k\varepsilon\delta}{2}C, \quad \textit{for} \quad k\neq 0 \textit{ and } \delta\in\mathbb{R}, \tag{50}$$

defines a connection on TK*, with torsion*

$$H_\varepsilon^{\delta,k} = (1-k)\varepsilon\omega\otimes e_0 + k\delta H_\varepsilon. \tag{51}$$

Explicitly,

$$\theta_\varepsilon^{\delta,k} = A + \frac{k\varepsilon}{2}B + \frac{k\varepsilon\delta}{2}C$$

$$= \begin{pmatrix} 0 & \frac{k\varepsilon(1+\delta)}{2}Je^{\mathrm{T}} & -\frac{k\varepsilon(1+\delta)}{2}e^{\mathrm{T}} \\ -\frac{k\varepsilon(1+\delta)}{2}Je & a - \frac{k\varepsilon\delta}{2}[e] & b - \frac{k\varepsilon(1-\delta)}{2}e_0 I + \frac{k\varepsilon\delta}{2}[Je] \\ \frac{k\varepsilon(1+\delta)}{2}e & -b + \frac{k\varepsilon(1-\delta)}{2}e_0 I + \frac{k\varepsilon\delta}{2}[Je] & a + \frac{k\varepsilon\delta}{2}[e] \end{pmatrix}.$$

Proof. We see from (36) and Proposition 3.10 that

$$d\begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} = -(A + \frac{k\varepsilon}{2}B + \frac{k\varepsilon\delta}{2}C)\wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} + (1-k)\varepsilon\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} + \frac{k\varepsilon\delta}{2}C\wedge\begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} r$$

$$= -\theta_\varepsilon^{\delta,k}\wedge\begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} + (1-k)\varepsilon\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} + k\delta H_\varepsilon. \qquad \square$$

Remark 3.12. It is possible to further deform the connection, and indeed the whole heterotic G_2 system, by allowing a non-trivial (non-constant) dilaton, which is equivalent to performing a conformal transformation on the G_2-structure. However, since there are in general no distinguished functions on K to define the dilaton, we will not pursue this possibility here.

Definition 3.13. Taking $\delta = +1$ in (50) gives

$$\theta_\varepsilon^{+,k} = A + \frac{k\varepsilon}{2}(B+C) = \begin{pmatrix} 0 & k\varepsilon Je^{\mathrm{T}} & -k\varepsilon e^{\mathrm{T}} \\ -k\varepsilon Je & a - \frac{k\varepsilon}{2}[e] & b + \frac{k\varepsilon}{2}[Je] \\ k\varepsilon e & -b + \frac{k\varepsilon}{2}[Je] & a + \frac{k\varepsilon}{2}[e] \end{pmatrix}. \tag{52}$$

We see from our choice of coframe that $\theta_\varepsilon^{+,k}$ takes values in $\mathfrak{g}_2 \subseteq \Lambda^2$, see e.g. [Lot11], and hence $\theta_\varepsilon^{+,k}$ has holonomy contained in G_2, as its curvature will necessarily take values in $\mathfrak{g}_2$.

Further, setting $k = 1$ in (52) gives what is often called the *Bismut connection* θ_ε^+ for φ_ε, the unique metric connection which makes φ_ε parallel and has totally skew-symmetric torsion (which is the flux H_ε).

Remark 3.14. The Bismut connection has been the subject of much study, and is a natural connection in this context. It is therefore tempting to use the Bismut connection (and more generally the connections $\theta_\varepsilon^{+,k}$ in Definition 3.13) when studying the heterotic G_2 system, particularly because of its holonomy property. However, inspired by the ideas in [Hul86, MS11], one could also consider a connection, known as the *Hull connection*, whose torsion has the *opposite sign* to the Bismut connection when trying to satisfy the heterotic Bianchi identity (2). This also motivates our discussion of the connections $\theta_\varepsilon^{\delta,k}$ for $\delta < 0$ as well as $\delta \geq 0$.

As a consequence of the previous remark, we will also be interested in the Hull connection, formally defined below.

Definition 3.15. Taking $\delta = -1$ in (50) gives

$$\begin{aligned}\theta_\varepsilon^{-,k} &= A + \frac{k\varepsilon}{2}(B - C) \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & a + \frac{k\varepsilon}{2}[e] & b - k\varepsilon e_0 I - \frac{k\varepsilon}{2}[Je] \\ 0 & -b + k\varepsilon e_0 I - \frac{k\varepsilon}{2}[Je] & a - \frac{k\varepsilon}{2}[e] \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & a + \frac{k\varepsilon}{2}[e] & b - \frac{k\varepsilon}{2}[Je] \\ 0 & -b - \frac{k\varepsilon}{2}[Je] & a - \frac{k\varepsilon}{2}[e] \end{pmatrix} + k\varepsilon e_0 \mathcal{I}.\end{aligned} \tag{53}$$

Setting $k = 1$ in (53) gives the *Hull connection* θ_ε^- associated to the G_2-structure φ_ε.

Remark 3.16. As in the case of θ_ε^k, we may view the connections $\theta_\varepsilon^{+,k}$ and $\theta_\varepsilon^{-,k}$, respectively, as 'squashed' versions of the Bismut and Hull connections θ_ε^+ and θ_ε^-.

3.2.2. *Local curvature matrices.* Now, we want to determine the curvature of $\theta_\varepsilon^{\delta,k}$ in Corollary 3.11, with a particular emphasis on the cases $\delta = \pm 1$. We begin with the result for all δ.

Proposition 3.17. *The curvature $R_\varepsilon^{\delta,k}$ of the connection $\theta_\varepsilon^{\delta,k}$ in* (50) *satisfies*

$$R_\varepsilon^{\delta,k} = F_A + \frac{k\varepsilon^2(1-\delta)}{2}\omega \mathcal{I} + \frac{k^2\varepsilon^2}{4} Q^\delta, \tag{54}$$

where $\mathcal{I}$ is given in (40)*,*

$$Q^\delta := (B + \delta C) \wedge (B + \delta C) = (1-\delta)Q_-^\delta + (1+\delta)Q_+^\delta + \delta^2 Q_0 \tag{55}$$

and

$$(56)\quad Q_-^\delta = e_0 \wedge \begin{pmatrix} 0 & (1+\delta)e^{\mathrm{T}} & (1+\delta)Je^{\mathrm{T}} \\ -(1+\delta)e & -2\delta[Je] & -2\delta[e] \\ -(1+\delta)Je & -2\delta[e] & 2\delta[Je] \end{pmatrix},$$

$$(57)\quad Q_+^\delta = \begin{pmatrix} 0 & 2\delta(e\times Je)^{\mathrm{T}} & \delta(e\times e - Je\times Je)^{\mathrm{T}} \\ -2\delta(e\times Je) & -(1+\delta)(Je\wedge Je^{\mathrm{T}}) & (1+\delta)(Je\wedge e^{\mathrm{T}}) \\ -\delta(e\times e - Je\times Je) & (1+\delta)(e\wedge Je^{\mathrm{T}}) & -(1+\delta)(e\wedge e^{\mathrm{T}}) \end{pmatrix},$$

$$(58)\quad Q_0 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -[e\times e + Je\times Je] & -2([e]\wedge[Je]-[Je]\wedge[e]) \\ 0 & 2([e]\wedge[Je]-[Je]\wedge[e]) & -[e\times e+Je\times Je] \end{pmatrix}.$$

Proof. We begin by observing that, by Corollary 3.11 and (43),

(59)

$$\begin{aligned} R_\varepsilon^{\delta,k} &= d\theta_\varepsilon^{\delta,k} + \theta_\varepsilon^{\delta,k}\wedge\theta_\varepsilon^{\delta,k} \\ &= dA + \frac{k\varepsilon}{2}dB + \frac{k\delta\varepsilon}{2}dC + \left(A + \frac{k\varepsilon}{2}B + \frac{k\varepsilon\delta}{2}C\right)\wedge\left(A + \frac{k\varepsilon}{2}B + \frac{k\varepsilon\delta}{2}C\right) \\ &= F_A + \frac{k\varepsilon}{2}(dB + A\wedge B + B\wedge A) + \frac{k\varepsilon\delta}{2}(dC + A\wedge C + C\wedge A) \\ &\qquad + \frac{k^2\varepsilon^2}{4}(B+\delta C)\wedge(B+\delta C) \\ &= F_A + \frac{k\varepsilon^2}{2}\omega\mathcal{I} + \frac{k\varepsilon\delta}{2}(dC + A\wedge C + C\wedge A) + \frac{k^2\varepsilon^2}{4}(B+\delta C)\wedge(B+\delta C). \end{aligned}$$

We may easily compute $dC + A\wedge C + C\wedge A$ appearing in (59). We first see that

$$(dC + A\wedge C + C\wedge A)_{1j} = (dB + A\wedge B + B\wedge A)_{1j} = 0.$$

Therefore,

$$(dC + A\wedge C + C\wedge A)_{j1} = 0$$

as well by skew-symmetry. We may therefore write $dC + A\wedge C + C\wedge A$ in the block form

$$dC + A\wedge C + C\wedge A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & d \\ 0 & -d^{\mathrm{T}} & -c \end{pmatrix}.$$

We then find that

$$c = -b\wedge e_0 I - e_0 I\wedge b - d([e]) - a\wedge[e] - [e]\wedge a + b\wedge[Je] - [Je]\wedge b = 0$$

using the structure equations (37) in Proposition 3.3. We also find that

$$\begin{aligned} d &= d(e_0)I + d([Je]) + a\wedge e_0 I + e_0 I\wedge a + a\wedge[Je] + [Je]\wedge a + b\wedge[e] - [e]\wedge b \\ &= \varepsilon\omega I, \end{aligned}$$

using (32) and (37). Overall, we deduce that

$$dC + A\wedge C + C\wedge A = \varepsilon\omega\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -I & 0 \end{pmatrix} = -\varepsilon\omega\mathcal{I}.$$

Hence, (54) follows.

We now need only verify (55). Recall from Corollary 3.11 that

$$B+\delta C=\begin{pmatrix} 0 & (1+\delta)Je^{\mathrm{T}} & -(1+\delta)e^{\mathrm{T}} \\ -(1+\delta)Je & -\delta[e] & -(1-\delta)e_0I+\delta[Je] \\ (1+\delta)e & (1-\delta)e_0I+\delta[Je] & \delta[e] \end{pmatrix}. \tag{60}$$

Using Lemma A.3, we start with the first row of $(B+\delta C)\wedge(B+\delta C)$ and find the non-zero entries

$$\begin{aligned}(1-\delta)(1+\delta)e_0\wedge e^{\mathrm{T}}-\delta(1+\delta)\big(Je^{\mathrm{T}}\wedge[e]+e^{\mathrm{T}}\wedge[Je]\big)&\\ =(1-\delta)(1+\delta)e_0\wedge e^{\mathrm{T}}+2\delta(1+\delta)(e\times Je)^{\mathrm{T}}&\end{aligned}$$

and

$$\begin{aligned}(1-\delta)(1+\delta)e_0\wedge Je^{\mathrm{T}}+\delta(1+\delta)\big(Je^{\mathrm{T}}\wedge[Je]-e^{\mathrm{T}}\wedge[e]\big)&\\ =(1-\delta)(1+\delta)e_0\wedge Je^{\mathrm{T}}+\delta(1+\delta)(e\times e-Je\times Je)^{\mathrm{T}}.&\end{aligned}$$

Moving to the middle block and again using Lemma A.3, we obtain

$$\begin{aligned}-(1+\delta)^2Je\wedge Je^{\mathrm{T}}+\delta^2\big([e]\wedge[e]+[Je]\wedge[Je]\big)-\delta(1-\delta)(e_0I\wedge[Je]-[Je]\wedge e_0I)&\\ =-(1+\delta)^2Je\wedge Je^{\mathrm{T}}-\frac{1}{2}\delta^2[e\times e+Je\times Je]-2\delta(1-\delta)e_0\wedge[Je].&\end{aligned}$$

Similarly, for the bottom right block, we obtain

$$-(1+\delta)^2e\wedge e^{\mathrm{T}}-\frac{1}{2}\delta^2[e\times e+Je\times Je]+2\delta(1-\delta)e_0\wedge[Je].$$

The remaining entries are defined by the middle right block, which is

$$(1+\delta)^2Je\wedge e^{\mathrm{T}}-\delta^2([e]\wedge[Je]-[Je]\wedge[e])-2\delta(1-\delta)e_0\wedge[e].$$

Equation (55) now follows. □

We now can specialize to the Bismut and Hull connections.

Corollary 3.18. *The curvature $R_{\theta_\varepsilon^+}$ of the Bismut connection θ_ε^+ satisfies*

$$R_{\theta_\varepsilon^+}=F_A+\frac{\varepsilon^2}{4}(B+C)\wedge(B+C), \tag{61}$$

where

$$\begin{aligned}&(B+C)\wedge(B+C)\\ &\quad=2\begin{pmatrix} 0 & 2(e\times Je)^{\mathrm{T}} & (e\times e-Je\times Je)^{\mathrm{T}} \\ -2(e\times Je) & -2(Je\wedge Je^{\mathrm{T}}) & 2(Je\wedge e^{\mathrm{T}}) \\ -(e\times e-Je\times Je) & 2(e\wedge Je^{\mathrm{T}}) & -2(e\wedge e^{\mathrm{T}}) \end{pmatrix}\\ &\qquad+\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -[e\times e+Je\times Je] & -2([e]\wedge[Je]-[Je]\wedge[e]) \\ 0 & 2([e]\wedge[Je]-[Je]\wedge[e]) & -[e\times e+Je\times Je] \end{pmatrix}.\end{aligned} \tag{62}$$

Corollary 3.19. *The curvature $R_{\theta_\varepsilon^-}$ of the Hull connection θ_ε^- satisfies*

$$R_{\theta_\varepsilon^-}=F_A+\varepsilon^2\omega\mathcal{I}+\frac{\varepsilon^2}{4}(B-C)\wedge(B-C), \tag{63}$$

where $\mathcal{I}$ is given in (40) *and*

$$
\begin{aligned}
&(B-C)\wedge(B-C)\\
&= 4e_0\wedge\begin{pmatrix}0&0&0\\0&[Je]&[e]\\0&[e]&-[Je]\end{pmatrix}\\
&\quad+\frac{1}{2}\begin{pmatrix}0&0&0\\0&-[e\times e+Je\times Je]&-2([e]\wedge[Je]-[Je]\wedge[e])\\0&2([e]\wedge[Je]-[Je]\wedge[e])&-[e\times e+Je\times Je]\end{pmatrix}.
\end{aligned}
\tag{64}
$$

3.3. **Connections: an extra twist.** It will be useful to 'twist' our connection by multiples of $e_0\mathcal{I}$. To discern the impact of this twist on the curvature of the connection, we have Lemma 3.20.

Lemma 3.20. *The local connection matrices A, B, C from Corollary* 3.4 *and Proposition* 3.10 *satisfy*

$$A\wedge e_0\mathcal{I}+e_0\mathcal{I}\wedge A=0, \tag{65}$$

$$B\wedge e_0\mathcal{I}+e_0\mathcal{I}\wedge B=e_0\wedge\begin{pmatrix}0&e^{\mathrm{T}}&Je^{\mathrm{T}}\\-e&0&0\\-Je&0&0\end{pmatrix}, \tag{66}$$

$$C\wedge e_0\mathcal{I}+e_0\mathcal{I}\wedge C=e_0\wedge\begin{pmatrix}0&e^{\mathrm{T}}&Je^{\mathrm{T}}\\-e&-2[Je]&-2[e]\\-Je&-2[e]&2[Je]\end{pmatrix}. \tag{67}$$

Proof. Given that A in (38) takes values in $\mathfrak{su}(3)\subseteq\mathfrak{u}(3)$ and $\mathcal{I}$ in (40) is central in $\mathfrak{u}(3)$, we immediately deduce (65). Moreover, we see from (39), (47) and (40) that

$$B\wedge e_0\mathcal{I}=\begin{pmatrix}0&-(e\wedge e_0)^{\mathrm{T}}&-(Je\wedge e_0)^{\mathrm{T}}\\0&0&0\\0&0&0\end{pmatrix},\qquad e_0\mathcal{I}\wedge B=\begin{pmatrix}0&0&0\\-e_0\wedge e&0&0\\-e_0\wedge Je&0&0\end{pmatrix}$$

and

$$C\wedge e_0\mathcal{I}=\begin{pmatrix}0&-(e\wedge e_0)^{\mathrm{T}}&-(Je\wedge e_0)^{\mathrm{T}}\\0&[Je]\wedge e_0&[e]\wedge e_0\\0&[e]\wedge e_0&-[Je]\wedge e_0\end{pmatrix},$$

$$e_0\mathcal{I}\wedge C=\begin{pmatrix}0&0&0\\-e_0\wedge e&-e_0\wedge[Je]&-e_0\wedge[e]\\-e_0\wedge Je&-e_0\wedge[e]&e_0\wedge[Je]\end{pmatrix}.$$

Equations (66) and (67) then follow. □

The previous lemma allows us to compute the curvature of a twisted connection, in particular establishing Theorem 1(iii), as follows.

Proposition 3.21. *In the local coframe from Definition* 2.6*, define a connection $\theta^{\delta,k}_{\varepsilon,m}$ on TK by*

$$\theta^{\delta,k}_{\varepsilon,m}=\theta^{\delta,k}_{\varepsilon}+\frac{km\varepsilon}{2}e_0\mathcal{I}. \tag{68}$$

Then its torsion is

$$H^{\delta,k}_{\varepsilon,m}=\big(1-k-\frac{km}{2}\big)\varepsilon\omega\otimes e_0+\frac{km\varepsilon}{2}e_0\wedge\omega+k\delta H_\varepsilon \tag{69}$$

and its curvature is given by

$$R^{\delta,k}_{\varepsilon,m} = F_A + \frac{k\varepsilon^2(1-\delta+m)}{2}\omega\mathcal{I} + \frac{k^2\varepsilon^2}{4}Q^\delta_m \tag{70}$$

where

$$Q^\delta_m = (1-\delta+m)\,Q^\delta_- + (1+\delta)Q^\delta_+ + \delta^2 Q_0 \tag{71}$$

for $Q^\delta_-, Q^\delta_+, Q_0$ *defined in* (56), (57) *and* (58), *respectively.*

Proof. Using (51) we see that

$$\begin{aligned} d\begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} &= -\theta^k_\delta \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} + (1-k)\varepsilon\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} + k\delta H_\varepsilon \\ &= -\theta^{\delta,k}_{\varepsilon,m} \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} + \frac{km\varepsilon}{2} e_0\mathcal{I} \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} + (1-k)\varepsilon\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} + k\delta H_\varepsilon. \end{aligned}$$

Since

$$\frac{km\varepsilon}{2} e_0\mathcal{I} \wedge \begin{pmatrix} e_0 \\ e \\ Je \end{pmatrix} = \frac{km\varepsilon}{2}\begin{pmatrix} 0 \\ -e_0 \wedge Je \\ e_0 \wedge e \end{pmatrix}$$

and raising an index on $e_0 \wedge \omega$ gives the vector-valued 2-form

$$\begin{pmatrix} \omega \\ -e_0 \wedge Je \\ e_0 \wedge e \end{pmatrix},$$

we quickly deduce (69).

We know by definition that

$$\begin{aligned} R^{\delta,k}_{\varepsilon,m} &= d(\theta^{\delta,k}_\varepsilon + \frac{km\varepsilon}{2}e_0\mathcal{I}) + (\theta^{\delta,k}_\varepsilon + \frac{km\varepsilon}{2}e_0\mathcal{I}) \wedge (\theta^{\delta,k}_\varepsilon + \frac{km\varepsilon}{2}e_0\mathcal{I}) \\ &= R^{\delta,k}_\varepsilon + \frac{km\varepsilon^2}{2}\omega\mathcal{I} + \frac{km\varepsilon}{2}(\theta^{\delta,k}_\varepsilon \wedge e_0\mathcal{I} + e_0\mathcal{I} \wedge \theta^{\delta,k}_\varepsilon). \end{aligned}$$

Lemma 3.20 implies that

$$\begin{aligned} \theta^{\delta,k}_\varepsilon \wedge e_0\mathcal{I} + e_0\mathcal{I} \wedge \theta^{\delta,k}_\varepsilon &= \Big(A + \frac{k\varepsilon}{2}(B+\delta C)\Big) \wedge e_0\mathcal{I} + e_0\mathcal{I} \wedge \Big(A + \frac{k\varepsilon}{2}(B+\delta C)\Big) \\ &= \frac{k\varepsilon}{2} e_0 \wedge \begin{pmatrix} 0 & (1+\delta)e^{\mathrm{T}} & (1+\delta)Je^{\mathrm{T}} \\ -(1+\delta)e & -2\delta[Je] & -2\delta[e] \\ -(1+\delta)Je & -2\delta[e] & 2\delta[Je] \end{pmatrix} = \frac{k\varepsilon}{2}Q^\delta_- \end{aligned}$$

by (56). The result now follows from Proposition 3.17. $\square$

The following observation, which may have potential interest, is immediate from (69):

Corollary 3.22. *The connection* $\theta^{\delta,k}_{\varepsilon,m}$ *in* (68) *has totally skew-symmetric torsion if, and only if,*

$$1 - k\Big(1 + \frac{m}{2}\Big) = 0.$$

3.4. **The G_2-instanton condition.** One way to check the G_2-instanton condition is to verify the vanishing of the wedge product of the curvature with ψ_ε, cf. (9). Before doing this, we make some elementary observations.

Lemma 3.23. *In the local coframe* (23) *on a contact Calabi–Yau* 7*-manifold as in Definition* 1.7*, and using the notation from Definition* A.1*, the following identities hold:*

$$2(e\times Je)\wedge \operatorname{Im}\Omega = 4e\wedge\frac{\omega^2}{2},\qquad (e\times e - Je\times Je)\wedge\operatorname{Im}\Omega = 4Je\wedge\frac{\omega^2}{2}, \tag{72}$$

$$e\wedge e^{\mathrm{T}}\wedge\operatorname{Im}\Omega = [Je]\wedge\frac{\omega^2}{2},\qquad Je\wedge Je^{\mathrm{T}}\wedge\operatorname{Im}\Omega = -[Je]\wedge\frac{\omega^2}{2}, \tag{73}$$

$$[e\times e + Je\times Je]\wedge\operatorname{Im}\Omega = 0,\qquad ([e]\wedge[Je]-[Je]\wedge[e])\wedge\operatorname{Im}\Omega = 0, \tag{74}$$

$$[e\times e + Je\times Je]\wedge\frac{\omega^2}{2} = 0,\qquad ([e]\wedge[Je]-[Je]\wedge[e])\wedge\frac{\omega^2}{2} = -4\frac{\omega^3}{6}I, \tag{75}$$

$$e\wedge Je^{\mathrm{T}}\wedge\operatorname{Im}\Omega = [e]\wedge\frac{\omega^2}{2},\qquad e\wedge Je^{\mathrm{T}}\wedge\frac{\omega^2}{2} = \frac{\omega^3}{6}I. \tag{76}$$

$$Je\wedge e^{\mathrm{T}}\wedge\operatorname{Im}\Omega = [e]\wedge\frac{\omega^2}{2},\qquad Je\wedge e^{\mathrm{T}}\wedge\frac{\omega^2}{2} = -\frac{\omega^3}{6}I. \tag{77}$$

Proof. We observe from (27) that

$$\operatorname{Im}\Omega\wedge e_2\wedge Je_3 = Je_2\wedge e_3\wedge e_1\wedge e_2\wedge Je_3 = e_1\wedge(e_2\wedge Je_2\wedge e_3\wedge Je_3) = e_1\wedge\frac{\omega^2}{2}$$

and

$$\begin{aligned}\operatorname{Im}\Omega\wedge e_3\wedge Je_2 &= Je_3\wedge e_1\wedge e_2\wedge e_3\wedge Je_2\\ &= -e_1\wedge(e_2\wedge Je_2\wedge e_3\wedge Je_3) = -e_1\wedge\frac{\omega^2}{2}.\end{aligned}$$

Similarly, we may also compute

$$\begin{aligned}\operatorname{Im}\Omega\wedge e_2\wedge e_3 &= -Je_1\wedge Je_2\wedge Je_3\wedge e_2\wedge e_3\\ &= (e_2\wedge Je_2\wedge e_3\wedge Je_3)\wedge Je_1 = \frac{\omega^2}{2}\wedge Je_1,\end{aligned}$$

and

$$\begin{aligned}\operatorname{Im}\Omega\wedge Je_2\wedge Je_3 &= Je_1\wedge e_2\wedge e_3\wedge Je_2\wedge Je_3\\ &= -(e_2\wedge Je_2\wedge e_3\wedge Je_3)\wedge Je_1 = -\frac{\omega^2}{2}\wedge Je_1.\end{aligned}$$

Hence, (72), (73) and the first equations in (76) and (77) hold (noting that $e_j\wedge Je_j\wedge\operatorname{Im}\Omega = 0$).

We also notice that

$$e_1\wedge Je_1\wedge\frac{\omega^2}{2} = e_1\wedge Je_1\wedge e_2\wedge Je_2\wedge e_3\wedge Je_3 = \frac{\omega^3}{6},$$

from which the remaining identities in (76) and (77) follow (since clearly $e_j\wedge Je_k\wedge\omega^2 = 0$ for $j\neq k$).

The previous calculation, together with Lemma 2.8 and (31), shows that

$$([e] \wedge [Je] - [Je] \wedge [e]) \wedge \frac{\omega^2}{2} = -4(e_1 \wedge Je_1 \wedge e_2 \wedge Je_2 \wedge e_3 \wedge Je_3)I = -4\frac{\omega^3}{6}I$$

as claimed. The rest of (75) follows from Lemma 2.8. □

Proposition 3.24. *The curvature $R_{\theta_\varepsilon^{\delta,k}}$ of the connection $\theta_\varepsilon^{\delta,k}$ in* (50) *satisfies*

$$\begin{aligned} &R_{\theta_\varepsilon^{\delta,k}} \wedge \psi_\varepsilon \\ &= \frac{k\varepsilon^2(1-\delta)\big(6+k(1+3\delta)\big)}{4}\frac{\omega^3}{6}\mathcal{I} \\ &+ \frac{k^2\varepsilon^2}{4}e_0 \wedge \frac{\omega^2}{2} \wedge \begin{pmatrix} 0 & (1-5\delta)(1+\delta)e^{\mathrm{T}} & (1-5\delta)(1+\delta)Je^{\mathrm{T}} \\ (5\delta-1)(1+\delta)e & (\delta^2-4\delta-1)[Je] & (\delta^2-4\delta-1)[e] \\ (5\delta-1)(1+\delta)Je & (\delta^2-4\delta-1)[e] & -(\delta^2-4\delta-1)[Je] \end{pmatrix}. \end{aligned} \tag{78}$$

Therefore, $\theta_\varepsilon^{\delta,k}$ is never a G_2-instanton.

Remark 3.25. We see that $\theta_\varepsilon^{\delta,k}$ can be a G_2-instanton if and only if we are in the trivial case where $k = 0$, which we have excluded.

Proof. Since A is a G_2-instanton by Lemma 3.1, we deduce immediately from Proposition 3.17 that

$$\begin{aligned} R_{\theta_\varepsilon^{\delta,k}} \wedge \psi_\varepsilon &= F_A \wedge \psi_\varepsilon + \frac{k\varepsilon^2(1-\delta)}{2}(\omega \wedge \psi_\varepsilon)\mathcal{I} + \frac{k^2\varepsilon^2}{4}Q^\delta \wedge \psi_\varepsilon \\ &= \frac{k\varepsilon^2(1-\delta)}{4}\omega^3\mathcal{I} + \frac{k^2\varepsilon^2}{4}Q^\delta \wedge \psi_\varepsilon. \end{aligned} \tag{79}$$

We now study the term $Q^\delta \wedge \psi_\varepsilon$. We first note that

$$e_0 \wedge e \wedge \psi_\varepsilon = e_0 \wedge \frac{\omega^2}{2} \wedge e, \qquad e_0 \wedge Je \wedge \psi_\varepsilon = e_0 \wedge \frac{\omega^2}{2} \wedge Je.$$

Hence, from (56), we find that

$$Q_-^\delta \wedge \psi_\varepsilon = e_0 \wedge \frac{1}{2}\omega^2 \wedge \begin{pmatrix} 0 & (1+\delta)e^{\mathrm{T}} & (1+\delta)Je^{\mathrm{T}} \\ -(1+\delta)e & -2\delta[Je] & -2\delta[e] \\ -(1+\delta)Je & -2\delta[e] & 2\delta[Je] \end{pmatrix}. \tag{80}$$

By Lemmas 2.8 and 3.23 we find that

$$2(e \times Je) \wedge \psi_\varepsilon = -2e_0 \wedge \operatorname{Im}\Omega \wedge (e \times Je) = -4e_0 \wedge \frac{\omega^2}{2} \wedge e,$$

$$(e \times e - Je \times Je) \wedge \psi_\varepsilon = -e_0 \wedge \operatorname{Im}\Omega \wedge (e \times e - Je \times Je) = -4e_0 \wedge \frac{\omega^2}{2} \wedge Je.$$

We also see from Lemma 3.23 that

$$\begin{aligned} Je \wedge Je^{\mathrm{T}} \wedge \psi_\varepsilon &= -e_0 \wedge \operatorname{Im}\Omega \wedge Je \wedge Je^{\mathrm{T}} = e_0 \wedge \frac{\omega^2}{2} \wedge [Je], \\ e \wedge e^{\mathrm{T}} \wedge \psi_\varepsilon &= -e_0 \wedge \operatorname{Im}\Omega \wedge e \wedge e^{\mathrm{T}} = -e_0 \wedge \frac{\omega^2}{2} \wedge [Je], \\ Je \wedge e^{\mathrm{T}} \wedge \psi_\varepsilon &= -\frac{\omega^3}{6}I - e_0 \wedge \operatorname{Im}\Omega \wedge Je \wedge e^{\mathrm{T}} = -\frac{\omega^3}{6}I - e_0 \wedge \frac{\omega^2}{2} \wedge [e], \\ e \wedge Je^{\mathrm{T}} \wedge \psi_\varepsilon &= \frac{\omega^3}{6}I - e_0 \wedge \operatorname{Im}\Omega \wedge e \wedge Je^{\mathrm{T}} = \frac{\omega^3}{6}I - e_0 \wedge \frac{\omega^2}{2} \wedge [e]. \end{aligned}$$

We deduce that

$$Q_+^\delta \wedge \psi_\varepsilon = (1+\delta)\frac{\omega^3}{6}\mathcal{I} + e_0 \wedge \frac{\omega^2}{2} \wedge \begin{pmatrix} 0 & -4\delta e^{\mathrm{T}} & -4\delta Je^{\mathrm{T}} \\ 4\delta e & -(1+\delta)[Je] & -(1+\delta)[e] \\ 4\delta Je & -(1+\delta)[e] & (1+\delta)[Je] \end{pmatrix}.$$

Finally, it follows from Lemma 3.23 that

$$[e \times e + Je \times Je] \wedge \psi_\varepsilon = 0, \qquad ([e] \wedge [Je] - [Je] \wedge [e]) \wedge \psi_\varepsilon = -4\frac{\omega^3}{6}I.$$

Thus,

$$Q_0 \wedge \psi_\varepsilon = \frac{\omega^3}{6}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4I \\ 0 & -4I & 0 \end{pmatrix} = -4\frac{\omega^3}{6}\mathcal{I}.$$

Overall, we have

$$\begin{aligned} Q^\delta \wedge \psi_\varepsilon &= \big((1-\delta)Q_-^\delta + (1+\delta)Q_+^\delta + \delta^2 Q_0\big) \wedge \psi_\varepsilon \\ &= (1-\delta)e_0 \wedge \frac{1}{2}\omega^2 \wedge \begin{pmatrix} 0 & (1+\delta)e^{\mathrm{T}} & (1+\delta)Je^{\mathrm{T}} \\ -(1+\delta)e & -2\delta[Je] & -2\delta[e] \\ -(1+\delta)Je & -2\delta[e] & 2\delta[Je] \end{pmatrix} \\ &\quad + (1+\delta)^2\frac{\omega^3}{6}\mathcal{I} + (1+\delta)e_0^{(k)} \wedge \frac{\omega^2}{2} \wedge \begin{pmatrix} 0 & -4\delta e^{\mathrm{T}} & -4\delta Je^{\mathrm{T}} \\ 4\delta e & -(1+\delta)[Je] & -(1+\delta)[e] \\ 4\delta Je & -(1+\delta)[e] & (1+\delta)[Je] \end{pmatrix} \\ &\quad - 4\delta^2\frac{\omega^3}{6}\mathcal{I} \\ &= (1-\delta)(1+3\delta)\frac{\omega^3}{6}\mathcal{I} \\ &\quad + e_0 \wedge \frac{\omega^2}{2} \wedge \begin{pmatrix} 0 & (1+\delta)(1-5\delta)e^{\mathrm{T}} & (1+\delta)(1-5\delta)Je^{\mathrm{T}} \\ (1+\delta)(5\delta-1)e & (\delta^2-4\delta-1)[Je] & (\delta^2-4\delta-1)[e] \\ (1+\delta)(5\delta-1)Je & (\delta^2-4\delta-1)[e] & -(\delta^2-4\delta-1)[Je] \end{pmatrix}. \end{aligned}$$

We deduce from this equation and (79) that the coefficient of $\frac{\omega^3}{6}\mathcal{I}$ in $R_{\theta_\varepsilon^{\delta,k}} \wedge \psi_\varepsilon$ is

$$\frac{6k\varepsilon^2(1-\delta)}{4} + \frac{k^2\varepsilon^2(1-\delta)(1+3\delta)}{4} = \frac{k\varepsilon^2(1-\delta)\big(6+k(1+3\delta)\big)}{4}.$$

The claimed formula (78) now follows.

Since the quadratics $(1-5\delta)(1+\delta)$ and $\delta^2-4\delta-1$ in δ have no common roots, we see that if $\theta_\varepsilon^{\delta,k}$ were a G_2-instanton, then we must have $k=0$. □

Remark 3.26. In particular, we see that neither the Bismut nor the Hull connection are G_2-instantons.

A straightforward adaptation of the arguments leading to Proposition 3.24, using Proposition 3.21, gives the following result for $\theta_{\varepsilon,m}^{\delta,k}$.

Corollary 3.27. *The curvature $R^{\delta,k}_{\varepsilon,m}$ of the connection $\theta^{\delta,k}_{\varepsilon,m}$ in (68) satisfies*

$$
\begin{aligned}
(81)\quad & R^{\delta,k}_{\varepsilon,m}\wedge\psi_\varepsilon \\
&= \frac{k\varepsilon^2\big(6(1-\delta+m)+k(1-\delta)(1+3\delta)\big)}{4}\frac{\omega^3}{6}\mathcal{I}+\frac{k^2\varepsilon^2}{4}e_0\wedge\frac{\omega^2}{2}\\
&\wedge\begin{pmatrix} 0 & (1+m-5\delta)(1+\delta)e^{\mathrm{T}} & (1+m-5\delta)(1+\delta)Je^{\mathrm{T}}\\ (5\delta-1-m)(1+\delta)e & (\delta^2-2(2+m)\delta-1)[Je] & (\delta^2-2(2+m)\delta-1)[e]\\ (5\delta-1-m)(1+\delta)Je & (\delta^2-2(2+m)\delta-1)[e] & -(\delta^2-2(2+m)\delta-1)[Je]\end{pmatrix}.
\end{aligned}
$$

Therefore, $\theta^{\delta,k}_{\varepsilon,m}$ is never a G_2-instanton.

Proof. The key observation is (80) which shows, together with Proposition 3.21, that we must add

$$
\frac{km\varepsilon^2}{4}\omega^3\mathcal{I}+\frac{k^2\varepsilon^2}{4}me_0\wedge\frac{1}{2}\omega^2\wedge\begin{pmatrix} 0 & (1+\delta)e^{\mathrm{T}} & (1+\delta)Je^{\mathrm{T}}\\ -(1+\delta)e & -2\delta[Je] & -2\delta[e]\\ -(1+\delta)Je & -2\delta[e] & 2\delta[Je]\end{pmatrix}
$$

to the right-hand side of (78) to obtain $R_{\theta^{\delta,k}_{\varepsilon,m}}\wedge\psi_\varepsilon$. The claimed formula (81) then follows.

We deduce that, since $k\neq 0$, $\theta^{\delta,k}_{\varepsilon,m}$ is a G_2-instanton if and only if

$$
\begin{aligned}
(1-\delta)(6+k(1+3\delta))+6m&=0,\\
(5\delta-1-m)(1+\delta)&=0,\\
(\delta^2-1)-2(2+m)\delta&=0.
\end{aligned}
$$

One may see that the only real solutions have $\delta=-1$, meaning the second equation is satisfied for any m. The third equation forces $m=-2$ and the first equation gives $12-4k+6m=0$, which then forces $k=0$. □

Remark 3.28. Although $\theta^{\delta,k}_{\varepsilon,m}$ is never a G_2-instanton, we know by (11) and (81) that it is an 'approximate' G_2-instanton whenever

$$
\begin{gathered}
\frac{k\varepsilon^2\big(6(1-\delta+m)+k(1-\delta)(1+3\delta)\big)}{4},\\
\frac{k^2\varepsilon^2}{4}(1+m-5\delta)(1+\delta),\\
\frac{k^2\varepsilon^2}{4}(\delta^2-2(2+m)\delta-1)
\end{gathered}
$$

are all 'sufficiently small' in a suitable sense. This smallness will be related to the constant α' which we will determine in the next section on the anomaly-free condition (2).

4. The anomaly term

We wish to study the heterotic Bianchi identity for the connections $\theta=\theta^{\delta,k}_{\varepsilon,m}$ and G_2-structure φ_ε. By (2) and Lemma 2.5, this becomes

$$
dH_\varepsilon=-\varepsilon^2\omega^2=\frac{\alpha'}{4}(\operatorname{tr}F_A^2-\operatorname{tr}R_\theta^2). \tag{82}
$$

Proposition 3.21 allows us to study when this condition can be satisfied, since by (70), we have that

$$
\begin{aligned}
R_\theta^2 - F_A^2 = {} & \frac{k^2\varepsilon^4(1-\delta+m)^2}{4}\omega^2\mathcal{I}^2 + \frac{k\varepsilon^2(1-\delta+m)}{2}(F_A\wedge\omega\mathcal{I}+\omega\mathcal{I}\wedge F_A)\\
& + \frac{k^3\varepsilon^4(1-\delta+m)}{8}(\omega\mathcal{I}\wedge Q_m^\delta + Q_m^\delta\wedge\omega\mathcal{I})\\
& + \frac{k^2\varepsilon^2}{4}(F_A\wedge Q_m^\delta + Q_m^\delta\wedge F_A) + \frac{k^4\varepsilon^4}{16}(Q_m^\delta)^2.
\end{aligned}
\tag{83}
$$

4.1. **Terms involving the matrix $\mathcal{I}$.**

We begin by studying the trace of the first line on the right-hand side of (83).

Lemma 4.1. *For $\mathcal{I}$ as in* (40) *and F_A as in* (44) *we have that*

$$
\operatorname{tr}\mathcal{I}^2 = -6 \quad \text{and} \quad \operatorname{tr}(F_A\wedge\omega\mathcal{I}+\omega\mathcal{I}\wedge F_A) = 0. \tag{84}
$$

Proof. We first notice that

$$
\mathcal{I}^2 = -\begin{pmatrix} 0 & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{pmatrix}
$$

and hence the first equation in (84) holds.

We then deduce from the formula (44) for F_A that

$$
F_A\wedge\omega\mathcal{I}+\omega\mathcal{I}\wedge F_A = \begin{pmatrix} 0 & 0 & 0\\ 0 & 2\beta\wedge\omega & -2\alpha\wedge\omega\\ 0 & 2\alpha\wedge\omega & 2\beta\wedge\omega\end{pmatrix}.
$$

Since β is traceless, the second equation in (84) also holds. □

We deduce from (82) and Lemma 4.1 that

$$
\begin{aligned}
&\operatorname{tr}(R^2_{\theta^{\delta,k}_{\varepsilon,m}} - F_A^2)\\
&\quad = -\frac{3k^2\varepsilon^4(1-\delta+m)^2}{2}\omega^2 + \frac{k^3\varepsilon^4(1-\delta+m)}{8}\operatorname{tr}(\omega\mathcal{I}\wedge Q_m^\delta + Q_m^\delta\wedge\omega\mathcal{I})\\
&\qquad + \frac{k^2\varepsilon^2}{4}\operatorname{tr}(F_A\wedge Q_m^\delta + Q_m^\delta\wedge F_A) + \frac{k^4\varepsilon^4}{16}\operatorname{tr}(Q_m^\delta)^2.
\end{aligned}
\tag{85}
$$

We now wish to study the second term on the right-hand side of (85).

Lemma 4.2. *For $\mathcal{I}$ in* (40) *and Q_-^δ, Q_+^δ, Q_0 in* (56), (57) *and* (58), *we have*

$$
\operatorname{tr}(\omega\mathcal{I}\wedge Q_-^\delta + Q_-^\delta\wedge\omega\mathcal{I}) = 0, \tag{86}
$$

$$
\operatorname{tr}(\omega\mathcal{I}\wedge Q_+^\delta + Q_+^\delta\wedge\omega\mathcal{I}) = -4(1+\delta)\omega^2, \tag{87}
$$

$$
\operatorname{tr}(\omega\mathcal{I}\wedge Q_0 + Q_0\wedge\omega\mathcal{I}) = 16\omega^2. \tag{88}
$$

Hence, for Q_m^δ given in (71), *we have*

$$
\operatorname{tr}(\omega\mathcal{I}\wedge Q_m^\delta + Q_m^\delta\wedge\omega\mathcal{I}) = 4(4\delta^2-(1+\delta)^2)\omega^2. \tag{89}
$$

Proof. We first observe that

$$\omega\mathcal{I}\wedge e_0\wedge\begin{pmatrix}0 & e^{\mathrm{T}} & Je^{\mathrm{T}}\\ -e & 0 & 0\\ -Je & 0 & 0\end{pmatrix}+e_0\wedge\begin{pmatrix}0 & e^{\mathrm{T}} & Je^{\mathrm{T}}\\ -e & 0 & 0\\ -Je & 0 & 0\end{pmatrix}\wedge\omega\mathcal{I}$$
$$=e_0\wedge\omega\wedge\begin{pmatrix}0 & Je^{\mathrm{T}} & -e^{\mathrm{T}}\\ Je & 0 & 0\\ -e & 0 & 0\end{pmatrix}$$

and

$$\omega\mathcal{I}\wedge e_0\wedge\begin{pmatrix}0 & 0 & 0\\ 0 & [Je] & [e]\\ 0 & [e] & -[Je]\end{pmatrix}+e_0\wedge\begin{pmatrix}0 & 0 & 0\\ 0 & [Je] & [e]\\ 0 & [e] & -[Je]\end{pmatrix}\wedge\omega\mathcal{I}=0.$$

Given the formula (56) for Q^{δ}_{-} we deduce (86).

Similarly, we observe that

$$\mathrm{tr}\Bigg(\omega\mathcal{I}\wedge e_0\wedge\begin{pmatrix}0 & 2(e\times Je)^{\mathrm{T}} & (e\times e-Je\times Je)^{\mathrm{T}}\\ -2(e\times Je) & 0 & 0\\ -(e\times e-Je\times Je) & 0 & 0\end{pmatrix}$$
$$+e_0\wedge\begin{pmatrix}0 & 2(e\times Je)^{\mathrm{T}} & (e\times e-Je\times Je)^{\mathrm{T}}\\ -2(e\times Je) & 0 & 0\\ -(e\times e-Je\times Je) & 0 & 0\end{pmatrix}\wedge\omega\mathcal{I}\Bigg)=0.$$

However,

$$\omega\mathcal{I}\wedge\begin{pmatrix}0 & 0 & 0\\ 0 & -Je\wedge Je^{\mathrm{T}} & Je\wedge e^{\mathrm{T}}\\ 0 & e\wedge Je^{\mathrm{T}} & -e\wedge e^{\mathrm{T}}\end{pmatrix}+\begin{pmatrix}0 & 0 & 0\\ 0 & -Je\wedge Je^{\mathrm{T}} & Je\wedge e^{\mathrm{T}}\\ 0 & e\wedge Je^{\mathrm{T}} & -e\wedge e^{\mathrm{T}}\end{pmatrix}\wedge\omega\mathcal{I}$$
$$=\omega\wedge\begin{pmatrix}0 & 0 & 0\\ 0 & Je\wedge e^{\mathrm{T}}-e\wedge Je^{\mathrm{T}} & e\wedge e^{\mathrm{T}}+Je\wedge Je^{\mathrm{T}}\\ 0 & -e\wedge e^{\mathrm{T}}-Je\wedge Je^{\mathrm{T}} & Je\wedge e^{\mathrm{T}}-e\wedge Je^{\mathrm{T}}\end{pmatrix}.$$

Taking the trace of this equation yields

$$2\omega\wedge(-2e_1\wedge Je_1-2e_2\wedge Je_2-2e_3\wedge Je_3)=-4\omega^2.$$

The equation (57) for Q^{δ}_{+} then gives (87).

Finally, we calculate

$$\frac{1}{2}\,\mathrm{tr}\left(\omega\mathcal{I}\wedge\begin{pmatrix}0 & 0 & 0\\ 0 & -[e\times e+Je\times Je] & -2([e]\wedge[Je]-[Je]\wedge[e])\\ 0 & 2([e]\wedge[Je]-[Je]\wedge[e]) & -[e\times e+Je\times Je]\end{pmatrix}\right)$$
$$+\frac{1}{2}\,\mathrm{tr}\left(\begin{pmatrix}0 & 0 & 0\\ 0 & -[e\times e+Je\times Je] & -2([e]\wedge[Je]-[Je]\wedge[e])\\ 0 & 2([e]\wedge[Je]-[Je]\wedge[e]) & -[e\times e+Je\times Je]\end{pmatrix}\wedge\omega\mathcal{I}\right)$$
$$=\mathrm{tr}\begin{pmatrix}0 & 0 & 0\\ 0 & -2\omega\wedge([e]\wedge[Je]-[Je]\wedge[e]) & \omega\wedge[e\times e+Je\times Je]\\ 0 & -\omega\wedge[e\times e+Je\times Je] & -2\omega\wedge([e]\wedge[Je]-[Je]\wedge[e])\end{pmatrix}$$
$$=-4\omega\wedge\mathrm{tr}([e]\wedge[Je]-[Je]\wedge[e])=-4\omega\wedge(-4\omega)=16\omega^2$$

by (31). Hence, (88) holds, and equation (89) then immediately follows from (71) and (86)–(88). $\square$

Inserting (89) in (85), we obtain:

$$\begin{aligned}\mathrm{tr}(R^2_{\theta^{\delta,k}_{\varepsilon,m}} - F_A^2) &= \frac{k^2\varepsilon^4(1-\delta+m)\big(k(4\delta^2-(1+\delta)^2)-3\big)}{2}\omega^2 \\ &\quad + \frac{k^2\varepsilon^2}{4}\,\mathrm{tr}(F_A\wedge Q^\delta_m + Q^\delta_m\wedge F_A) + \frac{k^4\varepsilon^4}{16}\,\mathrm{tr}(Q^\delta_m)^2.\end{aligned} \tag{90}$$

4.2. **Linear contribution from the G_2 field strength.** In this subsection, we wish to analyse the term $\mathrm{tr}(F_A\wedge Q^\delta_m + Q^\delta_m\wedge F_A)$ from (90).

Lemma 4.3. *For Q^δ_- in* (56) *and Q^δ_+ in* (57) *we have*

$$\mathrm{tr}(F_A\wedge Q^\delta_- + Q^\delta_-\wedge F_A) = 0 \quad \textit{and} \quad \mathrm{tr}(F_A\wedge Q^\delta_+ + Q^\delta_+\wedge F_A) = 0.$$

Proof. We see, from (45), that

$$\begin{aligned}&F_A\wedge e_0\wedge\begin{pmatrix}0 & e^{\mathrm{T}} & Je^{\mathrm{T}}\\ -e & 0 & 0\\ -Je & 0 & 0\end{pmatrix} + e_0\wedge\begin{pmatrix}0 & e^{\mathrm{T}} & Je^{\mathrm{T}}\\ -e & 0 & 0\\ -Je & 0 & 0\end{pmatrix}\wedge F_A\\ &\quad = e_0\wedge\begin{pmatrix}0 & e^{\mathrm{T}}\wedge\alpha - Je^{\mathrm{T}}\wedge\beta & e^{\mathrm{T}}\wedge\beta + Je^{\mathrm{T}}\wedge\alpha\\ -\alpha\wedge e-\beta\wedge Je & 0 & 0\\ \beta\wedge e-\alpha\wedge Je & 0 & 0\end{pmatrix} = 0.\end{aligned}$$

We may also compute

$$\begin{aligned}&\mathrm{tr}\left(F_A\wedge e_0\wedge\begin{pmatrix}0&0&0\\0&[Je]&[e]\\0&[e]&-[Je]\end{pmatrix} + e_0\wedge\begin{pmatrix}0&0&0\\0&[Je]&[e]\\0&[e]&-[Je]\end{pmatrix}\wedge F_A\right)\\ &= e_0\wedge\mathrm{tr}\begin{pmatrix}0 & 0 & 0\\ 0 & \alpha\wedge[Je]+\beta\wedge[e]+[Je]\wedge\alpha-[e]\wedge\beta & \alpha\wedge[e]-\beta\wedge[Je]+[Je]\wedge\beta+[e]\wedge\alpha\\ 0 & -\beta\wedge[Je]+\alpha\wedge[e]+[e]\wedge\alpha+[Je]\wedge\beta & -\beta\wedge[e]-\alpha\wedge[Je]+[e]\wedge\beta-[Je]\wedge\alpha\end{pmatrix}\\ &= 0.\end{aligned}$$

The first result now follows from (56).

For the second equation, we clearly have

$$\begin{aligned}&\mathrm{tr}\left(F_A\wedge\begin{pmatrix}0 & 2(e\times Je)^{\mathrm{T}} & (e\times e-Je\times Je)^{\mathrm{T}}\\ -2(e\times Je) & 0 & 0\\ -(e\times e-Je\times Je) & 0 & 0\end{pmatrix}\right.\\ &\quad\left.+\begin{pmatrix}0 & 2(e\times Je)^{\mathrm{T}} & (e\times e-Je\times Je)^{\mathrm{T}}\\ -2(e\times Je) & 0 & 0\\ -(e\times e-Je\times Je) & 0 & 0\end{pmatrix}\wedge F_A\right) = 0\end{aligned}$$

since the matrix the trace of which we are taking has no entries along the diagonal. On the other hand, if we consider

$$F_A\wedge\begin{pmatrix}0&0&0\\0&-Je\wedge Je^{\mathrm{T}}&Je\wedge e^{\mathrm{T}}\\0&e\wedge Je^{\mathrm{T}}&-e\wedge e^{\mathrm{T}}\end{pmatrix} + \begin{pmatrix}0&0&0\\0&-Je\wedge Je^{\mathrm{T}}&Je\wedge e^{\mathrm{T}}\\0&e\wedge Je^{\mathrm{T}}&-e\wedge e^{\mathrm{T}}\end{pmatrix}\wedge F_A,$$

we find that the only entries which are not trivially zero are

$$\begin{gathered}(-\alpha\wedge Je+\beta\wedge e)\wedge Je^{\mathrm{T}}-Je\wedge(Je^{\mathrm{T}}\wedge\alpha+e^{\mathrm{T}}\wedge\beta),\\ (\alpha\wedge Je-\beta\wedge e)\wedge e^{\mathrm{T}}-Je\wedge(Je^{\mathrm{T}}\wedge\beta-e^{\mathrm{T}}\wedge\alpha),\\ (\beta\wedge Je+\alpha\wedge e)\wedge Je^{\mathrm{T}}+e\wedge(Je^{\mathrm{T}}\wedge\alpha+e^{\mathrm{T}}\wedge\beta),\\ -(\beta\wedge Je-\alpha\wedge e)\wedge e^{\mathrm{T}}+e\wedge(Je^{\mathrm{T}}\wedge\beta-e^{\mathrm{T}}\wedge\alpha),\end{gathered}$$

yet these also vanish, by (45). Using (57) completes the result. □

From Lemma 4.3 we deduce that

$$\operatorname{tr}(F_A\wedge Q^\delta_m+Q^\delta_m\wedge F_A)=\delta^2\operatorname{tr}(F_A\wedge Q_0+Q_0\wedge F_A).$$

We conclude this section by studying this final term.

Lemma 4.4. *For Q_0 in* (58)*, we have*

$$\operatorname{tr}(F_A\wedge Q_0+Q_0\wedge F_A)=0.$$

Proof. We first see that

$$\begin{aligned}\operatorname{tr}(F_A\wedge Q_0)&=\operatorname{tr}(F_A\wedge\begin{pmatrix}0&0&0\\0&-[e\times e+Je\times Je]&-2([e]\wedge[Je]-[Je]\wedge[e])\\0&2([e]\wedge[Je]-[Je]\wedge[e])&-[e\times e+Je\times Je]\end{pmatrix})\\&=2\operatorname{tr}(-\alpha\wedge[e\times e+Je\times Je]+2\beta\wedge([e]\wedge[Je]-[Je]\wedge[e]),\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}(Q_0\wedge F_A)&=\operatorname{tr}(\begin{pmatrix}0&0&0\\0&-[e\times e+Je\times Je]&-2([e]\wedge[Je]-[Je]\wedge[e])\\0&2([e]\wedge[Je]-[Je]\wedge[e])&-[e\times e+Je\times Je]\end{pmatrix}\wedge F_A)\\&=2\operatorname{tr}(-[e\times e+Je\times Je]\wedge\alpha+2([e]\wedge[Je]-[Je]\wedge[e])\wedge\beta).\end{aligned}$$

Hence,

$$\operatorname{tr}(F_A\wedge Q_0+Q_0\wedge F_A)=4\operatorname{tr}(-\alpha\wedge[e\times e+Je\times Je]+2\beta([e]\wedge[Je]-[Je]\wedge[e]).$$

Using Lemma A.3 we find that

$$\begin{gathered}[e\times e+Je\times Je]=2e\wedge e^{\mathrm{T}}+2Je\wedge Je^{\mathrm{T}},\\ [e]\wedge[Je]-[Je]\wedge[e]=e\wedge Je^{\mathrm{T}}-Je\wedge e^{\mathrm{T}}-2\omega I.\end{gathered}$$

Therefore,

$$\begin{aligned}&\operatorname{tr}(F_A\wedge Q_0+Q_0\wedge F_A)\\&\quad=8\operatorname{tr}\big(-(\alpha\wedge e+\beta\wedge Je)\wedge e^{\mathrm{T}}-(\alpha\wedge Je-\beta\wedge e)\wedge Je^{\mathrm{T}}\big)-16\omega\wedge\operatorname{tr}\beta=0\end{aligned}$$

by (45) and the fact that β is traceless. □

By (90) and Lemmas 4.3 and 4.4 we obtain, for $\theta=\theta^{\delta,k}_{\varepsilon,m}$,

$$\operatorname{tr}(R^2_\theta-F^2_A)=\frac{k^2\varepsilon^4(1-\delta+m)\big(k(4\delta^2-(1+\delta)^2)-3\big)}{2}\omega^2+\frac{k^4\varepsilon^4}{16}\operatorname{tr}(Q^\delta_m)^2. \tag{91}$$

4.3. **The nonlinear contribution** $\mathrm{tr}(Q^\delta_m)^2$**.** We now wish to compute the term $\mathrm{tr}(Q^\delta_m)^2$ in (91), to complete our analysis of the difference in the traces of the squares of the curvatures of $\theta^{\delta,k}_{\varepsilon,m}$ and A. We begin with the 'square terms' in $(Q^\delta_m)^2$.

Lemma 4.5. *For* Q^δ_-, Q^δ_+, Q_0 *in* (56)–(58) *we have*

$$\mathrm{tr}(Q^\delta_-)^2 = 0, \quad \mathrm{tr}(Q^\delta_+)^2 = -8\delta^2\omega^2, \quad \mathrm{tr}(Q_0)^2 = 0.$$

Proof. Since $Q^\delta_- = e_0 \wedge Q$ for some matrix of 1-forms, we see immediately that $(Q^\delta_-)^2 = 0$.

For Q^δ_+, we note that

$$\begin{aligned} Q^\delta_+ = \delta &\begin{pmatrix} 0 & 2(e\times Je)^{\mathrm{T}} & (e\times e - Je\times Je)^{\mathrm{T}} \\ -2(e\times Je) & 0 & 0 \\ -(e\times e - Je\times Je) & 0 & 0 \end{pmatrix} \\ &+ (1+\delta)\begin{pmatrix} 0 & 0 & 0 \\ 0 & -Je\wedge Je^{\mathrm{T}} & Je\wedge e^{\mathrm{T}} \\ 0 & e\wedge Je^{\mathrm{T}} & -e\wedge e^{\mathrm{T}} \end{pmatrix}. \end{aligned} \tag{92}$$

We see that, in $(Q^\delta_+)^2$, the cross-terms coming from the pair of matrices above will be obviously traceless, so it suffices to compute the trace of each square. We see that

$$\begin{aligned} \mathrm{tr} &\begin{pmatrix} 0 & 2(e\times Je)^{\mathrm{T}} & (e\times e - Je\times Je)^{\mathrm{T}} \\ -2(e\times Je) & 0 & 0 \\ -(e\times e - Je\times Je) & 0 & 0 \end{pmatrix}^2 \\ &= -4(e\times Je)^{\mathrm{T}}\wedge(e\times Je) - (e\times e - Je\times Je)^{\mathrm{T}}\wedge(e\times e - Je\times Je). \end{aligned}$$

We observe that

$$\begin{aligned} 4(e_2\wedge Je_3 - e_3\wedge Je_2)\wedge(e_2\wedge Je_3 - e_3\wedge Je_2) &= 8e_2\wedge Je_2\wedge e_3\wedge Je_3 \\ 2(e_2\wedge e_3 - Je_2\wedge Je_3)\wedge 2(e_2\wedge e_3 - Je_2\wedge Je_3) &= 8e_2\wedge Je_2\wedge e_3\wedge Je_3 \end{aligned}$$

and hence

$$\mathrm{tr}\begin{pmatrix} 0 & 2(e\times Je)^{\mathrm{T}} & (e\times e - Je\times Je)^{\mathrm{T}} \\ -2(e\times Je) & 0 & 0 \\ -(e\times e - Je\times Je) & 0 & 0 \end{pmatrix}^2 = -8\omega^2.$$

On the other hand,

$$\mathrm{tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -Je\wedge Je^{\mathrm{T}} & Je\wedge e^{\mathrm{T}} \\ 0 & e\wedge Je^{\mathrm{T}} & -e\wedge e^{\mathrm{T}} \end{pmatrix}^2 = Je\wedge e^{\mathrm{T}}\wedge e\wedge Je^{\mathrm{T}} + e\wedge Je^{\mathrm{T}}\wedge Je\wedge e^{\mathrm{T}} = 0.$$

This gives the result for $\mathrm{tr}(Q^\delta_+)^2$.

From the formula (58) for Q_0 we see that

$$\mathrm{tr}(Q_0)^2 = \frac{1}{2}\,\mathrm{tr}[e\times e + Je\times Je]^2 - 2\,\mathrm{tr}([e]\wedge[Je] - [Je]\wedge[e])^2.$$

We then calculate

$$\begin{aligned}
&\operatorname{tr}[e\times e+Je\times Je]^2\\
&=\operatorname{tr}\begin{pmatrix} 0 & 2e_1\wedge e_2+2Je_1\wedge Je_2 & -2e_3\wedge e_1-2Je_3\wedge Je_1\\ -2e_1\wedge e_2-2Je_1\wedge Je_2 & 0 & 2e_2\wedge e_3+2Je_2\wedge Je_3\\ 2e_3\wedge e_1+2Je_3\wedge Je_1 & -2e_2\wedge e_3-2Je_2\wedge Je_3 & 0\end{pmatrix}^2\\
&=16(e_1\wedge Je_1\wedge e_2\wedge Je_2+e_3\wedge Je_3\wedge e_1\wedge Je_1+e_2\wedge Je_2\wedge e_3\wedge Je_3)=8\omega^2
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{tr}([e]\wedge[Je]-[Je]\wedge[e])^2\\
&=\operatorname{tr}\begin{pmatrix} -2e_2\wedge Je_2-2e_3\wedge Je_3 & e_2\wedge Je_1+e_1\wedge Je_2 & e_3\wedge Je_1+e_1\wedge Je_3\\ e_1\wedge Je_2+e_2\wedge Je_1 & -2e_3\wedge Je_3-2e_1\wedge Je_1 & e_3\wedge Je_2+e_2\wedge Je_3\\ e_1\wedge Je_3+e_3\wedge Je_1 & e_2\wedge Je_3+e_3\wedge Je_2 & -2e_1\wedge Je_1-2e_2\wedge Je_2\end{pmatrix}^2\\
&=8(e_2\wedge Je_2\wedge e_3\wedge Je_3+e_3\wedge Je_3\wedge e_1\wedge Je_1+e_1\wedge Je_1\wedge e_2\wedge Je_2)\\
&\quad+4(e_1\wedge Je_2\wedge e_2\wedge Je_1+e_3\wedge Je_1\wedge e_1\wedge Je_3+e_2\wedge Je_3\wedge e_3\wedge Je_2)\\
&=4\omega^2-2\omega^2=2\omega^2.
\end{aligned}$$

The formula for $\operatorname{tr}(Q_0)^2$ then follows. □

We now look at the 'cross terms' in $(Q_m^\delta)^2$.

Lemma 4.6. *For Q_-^δ, Q_+^δ in (56)–(57), we have*

$$\operatorname{tr}(Q_-^\delta\wedge Q_+^\delta+Q_+^\delta\wedge Q_-^\delta)=0.$$

Proof. Just as for Q_+^δ in (92) we can split Q_-^δ as

$$Q_-^\delta=(1+\delta)e_0\wedge\begin{pmatrix}0 & e^{\mathrm T} & Je^{\mathrm T}\\ -e & 0 & 0\\ -Je & 0 & 0\end{pmatrix}-2\delta e_0\wedge\begin{pmatrix}0&0&0\\0&[Je]&[e]\\0&[e]&-[Je]\end{pmatrix}. \tag{93}$$

Hence, we can break down the calculation of $\operatorname{tr}(Q_-^\delta\wedge Q_+^\delta+Q_+^\delta\wedge Q_-^\delta)$ into more manageable steps. First, we see that

$$\begin{aligned}
&\operatorname{tr}\left(e_0\wedge\begin{pmatrix}0 & e^{\mathrm T} & Je^{\mathrm T}\\ -e & 0 & 0\\ -Je & 0 & 0\end{pmatrix}\wedge\begin{pmatrix}0 & 2(e\times Je)^{\mathrm T} & (e\times e-Je\times Je)^{\mathrm T}\\ -2(e\times Je) & 0 & 0\\ -(e\times e-Je\times Je) & 0 & 0\end{pmatrix}\right)\\
&+\operatorname{tr}\left(\begin{pmatrix}0 & 2(e\times Je)^{\mathrm T} & (e\times e-Je\times Je)^{\mathrm T}\\ -2(e\times Je) & 0 & 0\\ -(e\times e-Je\times Je) & 0 & 0\end{pmatrix}\wedge e_0\wedge\begin{pmatrix}0 & e^{\mathrm T} & Je^{\mathrm T}\\ -e & 0 & 0\\ -Je & 0 & 0\end{pmatrix}\right)\\
&=2e_0\wedge\big(-2e^{\mathrm T}\wedge(e\times Je)-Je^{\mathrm T}\wedge(e\times e-Je\times Je)-2\operatorname{tr}(e\wedge(e\times Je)^{\mathrm T})\\
&\qquad-\operatorname{tr}(Je\wedge(e\times e-Je\times Je)^{\mathrm T})\big)\\
&=4e_0\wedge\big(-2e^{\mathrm T}\wedge(e\times Je)-Je^{\mathrm T}\wedge(e\times e-Je\times Je)\big).
\end{aligned}$$

We observe that

$$\begin{aligned}
2e^{\mathrm{T}} \wedge (e \times Je) &= 2e_1 \wedge (e_2 \wedge Je_3 - e_3 \wedge Je_2) \\
&\quad + 2e_2 \wedge (e_3 \wedge Je_1 - e_1 \wedge Je_3) \\
&\quad + 2e_3 \wedge (e_1 \wedge Je_2 - e_2 \wedge Je_1) \\
&= 4 \operatorname{Im} \Omega + 4Je_1 \wedge Je_2 \wedge Je_3, \\
Je^{\mathrm{T}} \wedge (e \times e - Je \times Je) &= 2Je_1 \wedge (e_2 \wedge e_3 - Je_2 \wedge Je_3) \\
&\quad + 2Je_2 \wedge (e_3 \wedge e_1 - Je_3 \wedge Je_1) \\
&\quad + 2Je_3 \wedge (e_1 \wedge e_2 - Je_3 \wedge Je_1), \\
&= 2 \operatorname{Im} \Omega - 4Je_1 \wedge Je_2 \wedge Je_3
\end{aligned}$$

and thus

$$4e_0 \wedge \big(- 2e^{\mathrm{T}} \wedge (e \times Je) - Je^{\mathrm{T}} \wedge (e \times e - Je \times Je\big) = -24 e_0 \wedge \operatorname{Im} \Omega.$$

Now, clearly,

$$\operatorname{tr}\left(e_0 \wedge \begin{pmatrix} 0 & e^{\mathrm{T}} & Je^{\mathrm{T}} \\ -e & 0 & 0 \\ -Je & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & -Je \wedge Je^{\mathrm{T}} & Je \wedge e^{\mathrm{T}} \\ 0 & e \wedge Je^{\mathrm{T}} & -e \wedge e^{\mathrm{T}} \end{pmatrix} \right) = 0,$$

$$\operatorname{tr}\left(e_0 \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & [Je] & [e] \\ 0 & [e] & -[Je] \end{pmatrix} \wedge \begin{pmatrix} 0 & 2(e \times Je)^{\mathrm{T}} & (e \times e - Je \times Je)^{\mathrm{T}} \\ -2(e \times Je) & 0 & 0 \\ -(e \times e - Je \times Je) & 0 & 0 \end{pmatrix} \right) = 0,$$

so for $\operatorname{tr}(Q^{\delta}_{-} \wedge Q^{\delta}_{+})$ we are simply left with computing

$$\begin{aligned}
&\operatorname{tr}\left(e_0 \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & [Je] & [e] \\ 0 & [e] & -[Je] \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & -Je \wedge Je^{\mathrm{T}} & Je \wedge e^{\mathrm{T}} \\ 0 & e \wedge Je^{\mathrm{T}} & -e \wedge e^{\mathrm{T}} \end{pmatrix} \right) \\
&+ \operatorname{tr}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & -Je \wedge Je^{\mathrm{T}} & Je \wedge e^{\mathrm{T}} \\ 0 & e \wedge Je^{\mathrm{T}} & -e \wedge e^{\mathrm{T}} \end{pmatrix} \wedge e_0 \wedge \begin{pmatrix} 0 & 0 & 0 \\ 0 & [Je] & [e] \\ 0 & [e] & -[Je] \end{pmatrix} \right) \\
&= 2e_0 \wedge \operatorname{tr}\big([Je] \wedge (e \wedge e^{\mathrm{T}} - Je \wedge Je^{\mathrm{T}}) + [e] \wedge (e \wedge Je^{\mathrm{T}} + Je \wedge e^{\mathrm{T}}) \big).
\end{aligned}$$

To conclude, we notice that

$$\begin{aligned}
&\operatorname{tr}\big([Je] \wedge (e \wedge e^{\mathrm{T}} - Je \wedge Je^{\mathrm{T}}) \big) \\
&\quad = -2Je_3 \wedge e_1 \wedge e_2 - 2Je_2 \wedge e_3 \wedge e_1 - 2Je_1 \wedge e_2 \wedge e_3 + 6Je_1 \wedge Je_2 \wedge Je_3 \\
&\quad = -2 \operatorname{Im} \Omega + 4Je_1 \wedge Je_2 \wedge Je_3, \\
&\operatorname{tr}\big([e] \wedge (e \wedge Je^{\mathrm{T}} + Je \wedge e^{\mathrm{T}}) \big) \\
&\quad = 2e_3 \wedge (e_2 \wedge Je_1 + Je_2 \wedge e_1) + 2e_2 \wedge (e_3 \wedge Je_1 + Je_3 \wedge e_1) \\
&\qquad + 2e_1 \wedge (e_3 \wedge Je_2 + Je_3 \wedge e_2) \\
&\quad = -4 \operatorname{Im} \Omega - 4Je_1 \wedge Je_2 \wedge Je_3,
\end{aligned}$$

which gives

$$2e_0 \wedge \operatorname{tr}\big([Je] \wedge (e \wedge e^{\mathrm{T}} - Je \wedge Je^{\mathrm{T}}) + [e] \wedge (e \wedge Je^{\mathrm{T}} + Je \wedge e^{\mathrm{T}}) \big) = -12 e_0 \wedge \operatorname{Im} \Omega.$$

Hence, as claimed,

$$\begin{aligned}&\operatorname{tr}(Q^\delta_- \wedge Q^\delta_+ + Q^\delta_+ \wedge Q^\delta_-)\\ &\quad = (1+\delta)\delta(-24 e_0 \wedge \operatorname{Im}\Omega) - 2\delta(1+\delta)(-12 e_0 \wedge \operatorname{Im}\Omega) = 0.\end{aligned}$$

□

Lemma 4.7. *For Q^δ_-, Q_0, respectively in* (56), (58), *we have*

$$\operatorname{tr}(Q^\delta_- \wedge Q_0 + Q_0 \wedge Q^\delta_-) = 0.$$

Proof. Recall the splitting (93). Since we have

$$\begin{aligned}&\operatorname{tr}\left(e_0\wedge\begin{pmatrix}0 & 0 & 0\\ 0 & [Je] & [e]\\ 0 & [e] & -[Je]\end{pmatrix}\wedge\begin{pmatrix}0 & 0 & 0\\ 0 & -[e\times e+Je\times Je] & -2([e]\wedge[Je]-[Je]\wedge[e])\\ 0 & 2([e]\wedge[Je]-[Je]\wedge[e]) & -[e\times e+Je\times Je]\end{pmatrix}\right)\\
&= e_0 \wedge \operatorname{tr}(-[Je]\wedge[e\times e + Je\times Je] + 2[e]\wedge([e]\wedge[Je]-[Je]\wedge[e]))\\
&\quad + e_0\wedge\operatorname{tr}(-2[e]\wedge([e]\wedge[Je]-[Je]\wedge[e]) + [Je]\wedge[e\times e+Je\times Je])\\
&= 0,\end{aligned}$$

the result then follows from (93) and (58). □

Lemma 4.8. *For Q^δ_+, Q_0, respectively in* (57), (58), *we have*

$$\operatorname{tr}(Q^\delta_+ \wedge Q_0 + Q_0 \wedge Q^\delta_+) = 16(1+\delta)\omega^2.$$

Proof. Recall the splitting (92). We see that to calculate $\operatorname{tr}(Q^\delta_+\wedge Q_0)$ it suffices to compute the following:

$$\begin{aligned}&\operatorname{tr}\left(\begin{pmatrix}0 & 0 & 0\\ 0 & -Je\wedge Je^{\mathrm T} & Je\wedge e^{\mathrm T}\\ 0 & e\wedge Je^{\mathrm T} & -e\wedge e^{\mathrm T}\end{pmatrix}\wedge\begin{pmatrix}0 & 0 & 0\\ 0 & -[e\times e+Je\times Je] & -2([e]\wedge[Je]-[Je]\wedge[e])\\ 0 & 2([e]\wedge[Je]-[Je]\wedge[e]) & -[e\times e+Je\times Je]\end{pmatrix}\right)\\
&= \operatorname{tr}\big((Je\wedge Je^{\mathrm T} + e\wedge e^{\mathrm T})\wedge[e\times e+Je\times Je]\\
&\qquad + 2(Je\wedge e^{\mathrm T} - e\wedge Je^{\mathrm T})\wedge([e]\wedge[Je]-[Je]\wedge[e])\big)\\
&= 2\operatorname{tr}(Je\wedge Je^{\mathrm T}+e\wedge e^{\mathrm T})^2 - 2\operatorname{tr}(Je\wedge e^{\mathrm T} - e\wedge Je^{\mathrm T})^2\\
&\qquad - 4\omega\wedge\operatorname{tr}(Je\wedge e^{\mathrm T} - e\wedge Je^{\mathrm T})\end{aligned}$$

by Lemma A.3.

We first see that

$$\begin{aligned}&2\operatorname{tr}(Je\wedge Je^{\mathrm T}+e\wedge e^{\mathrm T})^2\\
&\quad = 2(4e_1\wedge e_2\wedge Je_2\wedge Je_1 + 4e_3\wedge e_1\wedge Je_1\wedge Je_3 + 4e_2\wedge e_3\wedge Je_3\wedge Je_2)\\
&\quad = 4\omega^2.\end{aligned}$$

We also see that

$$\begin{aligned}&-2\operatorname{tr}(Je\wedge e^{\mathrm T} - e\wedge Je^{\mathrm T})^2\\
&\quad = -2\operatorname{tr}(Je\wedge e^{\mathrm T})^2 - 2\operatorname{tr}(e\wedge Je^{\mathrm T})^2\\
&\quad = -2(2Je_1\wedge e_2\wedge Je_2\wedge e_1 + 2Je_3\wedge e_1\wedge Je_1\wedge e_3 + 2Je_2\wedge e_3\wedge Je_3\wedge e_2)\\
&\qquad - 2(2e_1\wedge Je_2\wedge e_2\wedge Je_1 + 2e_3\wedge Je_1\wedge e_1\wedge Je_3 + 2e_2\wedge Je_3\wedge e_3\wedge Je_2)\\
&\quad = 4\omega^2\end{aligned}$$

and

$$-4\omega\wedge\operatorname{tr}(Je\wedge e^{\mathrm T} - e\wedge Je^{\mathrm T}) = -4\omega\wedge(-2\omega) = 8\omega^2.$$

Hence,

$$\begin{aligned}
&\operatorname{tr}\left(\begin{pmatrix}0 & 0 & 0\\ 0 & -Je\wedge Je^{\mathrm{T}} & Je\wedge e^{\mathrm{T}}\\ 0 & e\wedge Je^{\mathrm{T}} & -e\wedge e^{\mathrm{T}}\end{pmatrix}\right.\\
&\qquad\left.\wedge\frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & -[e\times e+Je\times Je] & -2([e]\wedge[Je]-[Je]\wedge[e])\\ 0 & 2([e]\wedge[Je]-[Je]\wedge[e]) & -[e\times e+Je\times Je]\end{pmatrix}\right)\\
&=\frac{1}{2}(4\omega^2+4\omega^2+8\omega^2)=8\omega^2.
\end{aligned}$$

The result then follows from (92) and (58). □

Corollary 4.9. *For Q^δ_m in* (71)*, we have*

$$\operatorname{tr}(Q^\delta_m)^2 = 8\delta^2(1+\delta)^2\omega^2.$$

Proof. From the definition of Q^δ_m in (71), using Lemmas 4.5–4.8, we compute:

$$\begin{aligned}
\operatorname{tr}(Q^\delta_m)^2 &= \operatorname{tr}\big((1-\delta+m)Q^\delta_- + (1+\delta)Q^\delta_+ + \delta^2 Q_0\big)^2\\
&= (1-\delta+m)^2\operatorname{tr}(Q^\delta_-)^2 + (1+\delta)^2\operatorname{tr}(Q^\delta_+)^2 + \delta^4\operatorname{tr}(Q_0^2)\\
&\quad + (1-\delta+m)(1+\delta)\operatorname{tr}(Q^\delta_-\wedge Q^\delta_+ + Q^\delta_+\wedge Q^\delta_-)\\
&\quad + (1-\delta+m)\delta^2\operatorname{tr}(Q^\delta_-\wedge Q_0 + Q_0\wedge Q^\delta_-)\\
&\quad + (1+\delta)\delta^2\operatorname{tr}(Q^\delta_+\wedge Q_0 + Q_0\wedge Q^\delta_+)\\
&= -8(1+\delta)^2\delta^2\omega^2 + 16(1+\delta)^2\delta^2\omega^2\\
&= 8\delta^2(1+\delta)^2\omega^2.
\end{aligned}$$ □

Combining Corollary 4.9 and (91), we conclude that

$$\operatorname{tr}(R_\theta^2 - F_A^2) = \frac{k^2\varepsilon^4\big(k^2\delta^2(1+\delta)^2 + (1-\delta+m)\big(k(4\delta^2-(1+\delta)^2)-3\big)\big)}{2}\omega^2, \quad \text{with}\quad \theta=\theta^{\delta,k}_{\varepsilon,m}. \tag{94}$$

4.4. **Proof of Theorem 1.** We are now in position to prove the final parts (iv) and (v) in Theorem 1. Replacing the Chern–Simons defect (91), between gauge fields A and θ, in the heterotic Bianchi identity (82), we obtain

$$-\varepsilon^2\omega^2 = -\frac{\alpha'}{4}\frac{k^2\varepsilon^4\big(k^2\delta^2(1+\delta)^2 + (1-\delta+m)\big(k(4\delta^2-(1+\delta)^2)-3\big)\big)}{2}\omega^2. \tag{95}$$

Hence, there is a solution for $\alpha'>0$ if, and only if,

$$k^2\big(k^2\delta^2(1+\delta)^2 + (1-\delta+m)\big(k(4\delta^2-(1+\delta)^2)-3\big)\big) > 0, \tag{96}$$

in which case

$$\alpha' = \frac{8}{k^2\varepsilon^2\big(k^2\delta^2(1+\delta)^2 + (1-\delta+m)\big(k(4\delta^2-(1+\delta)^2)-3\big)\big)}. \tag{97}$$

We deduce the following constraints to have an approximate solution to the heterotic G_2 system, in the sense that all of the conditions in Definition 1.2 for the heterotic G_2 system are satisfied except that we only require that θ be a G_2-instanton to order $O(\alpha')^2$ as in (11):

Proposition 4.10. *There is an approximate solution to the heterotic* G_2 *system if and only if*

$$\lambda_0 := k^2\varepsilon^2\big(k^2\delta^2(1+\delta)^2 + (1-\delta+m)\big(k(4\delta^2-(1+\delta)^2)-3\big)\big) > 0 \tag{98}$$

is large so that

$$\alpha' = \frac{8}{\lambda_0} > 0 \tag{99}$$

is small and the terms in the G_2*-instanton condition* (78),

$$\begin{aligned} \lambda_1 &:= \frac{k\varepsilon^2\big(6(1-\delta+m)+k(1-\delta)(1+3\delta)\big)}{4}, \\ \lambda_2 &:= \frac{k^2\varepsilon^2}{4}(1+m-5\delta)(1+\delta), \\ \lambda_3 &:= \frac{k^2\varepsilon^2}{4}(\delta^2-2(2+m)\delta-1) \end{aligned} \tag{100}$$

are all $O(\alpha')^2$, *so that* (11) *is satisfied.*

Inspecting (98), there are at least three manifest Ansätze for this asymptotic regime, all of which satisfy items (i)–(v) of Theorem 1:

Case 1. $1-\delta+m=0$ and $\delta\neq 0,-1$:

$$\begin{aligned} \alpha' &= \frac{8}{\delta^2(1+\delta)^2}\frac{1}{\varepsilon^2k^4}, \\ \lambda_1 &= \frac{(1-\delta)(1+3\delta)}{4}k^2\varepsilon^2, \\ \lambda_2 &= -\delta(1+\delta)k^2\varepsilon^2, \\ \lambda_3 &= -\frac{(\delta+1)^2}{4}k^2\varepsilon^2. \end{aligned}$$

In order to have $k^2\varepsilon^2 = O(\alpha')^2$, we may take, for instance,

$$k^2 = \frac{1}{(\alpha')^3} \quad \text{and} \quad \varepsilon^2 = \frac{8}{\delta^2(1+\delta)^2}(\alpha')^5, \quad \text{with} \quad \delta\neq 0,-1 \quad \text{and} \quad m=\delta-1,$$

which is physically meaningful with $\varepsilon \ll 1$ and $k \gg 1$.

Case 2. $\delta = 0$ and $(1+m)(k+3) < 0$:

$$\begin{aligned} \alpha' &= -\frac{8}{(1+m)(1+\frac{3}{k})}\frac{1}{\varepsilon^2k^3}, \\ \lambda_1 &= \frac{\big(1+\frac{6(1+m)}{k}\big)}{4}k^2\varepsilon^2, \\ \lambda_2 &= \frac{1+m}{4}k^2\varepsilon^2, \\ \lambda_3 &= -\frac{1}{4}k^2\varepsilon^2. \end{aligned}$$

In order to have $k\varepsilon^2 = O(\alpha')^2$ and $k^2\varepsilon^2 = O(\alpha')^2$, we may take, for instance,

$$k = \frac{1}{(\alpha')^3} \quad \text{and} \quad \varepsilon^2 = \frac{8}{(1+m)(1+3(\alpha')^3)}(\alpha')^8, \quad \text{with} \quad m < -1,$$

which is physically meaningful with $\varepsilon \ll 1$ and $k \gg 1$.

Case 3. $\delta = -1$ and $(2+m)(4k-3) > 0$:

$$\begin{aligned}
\alpha' &= \frac{8}{(2+m)(4-\frac{3}{k})}\frac{1}{\varepsilon^2 k^3},\\
\lambda_1 &= \left(\frac{3(2+m)}{2k} - 1\right) k^2\varepsilon^2,\\
\lambda_2 &= 0,\\
\lambda_3 &= -\frac{2+m}{2}k^2\varepsilon^2.
\end{aligned}$$

In order to have $k\varepsilon^2 = O(\alpha')^2$ and $k^2\varepsilon^2 = O(\alpha')^2$, we may take, for instance,

$$k = \frac{1}{(\alpha')^3} \quad \text{and} \quad \varepsilon^2 = \frac{8}{(2+m)(4-3(\alpha')^3)}(\alpha')^8, \quad \text{with} \quad m > -2,$$

which is physically meaningful with $\varepsilon \ll 1$ and $k \gg 1$.

NB.: Several other solution regimes are possible, in particular one may adjust the choices of m and δ to the string scale α' itself. Furthermore, it should be noted that the asymptotic properties of $\varepsilon(\alpha')$ and $k(\alpha')$ as $\alpha' \to 0$ are a *consequence* of the heterotic Bianchi identity (82) and the G_2-instanton condition (78) 'up to $O(\alpha')^2$ terms', and therefore *not a choice* imposed on the Ansatz.

Appendix A. Covariant matrix operations

Definition A.1. For a 3×1 vector a, we define $[a]$ by

$$\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}. \tag{101}$$

This leads us to the following definition and lemma.

Definition A.2. Let

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

be vectors of 1-forms and define

$$a \times b = \begin{pmatrix} a_2 \wedge b_3 - a_3 \wedge b_2 \\ a_3 \wedge b_1 - a_1 \wedge b_3 \\ a_1 \wedge b_2 - a_2 \wedge b_1 \end{pmatrix}. \tag{102}$$

Notice that

$$b \times a = a \times b. \tag{103}$$

Lemma A.3. *Let a and b be 3×1 vectors of 1-forms. Then*

$$[a] \wedge b = -a \times b, \tag{104}$$

$$a^{\mathrm{T}} \wedge [b] = -(a \times b)^{\mathrm{T}}, \tag{105}$$

$$[a] \wedge [b] + [b] \wedge [a] = -[a \times b], \tag{106}$$

$$[a] \wedge [b] - [b] \wedge [a] = a \wedge b^{\mathrm{T}} - b \wedge a^{\mathrm{T}} - 2I \otimes \sum_{j=1}^{3} a_j \wedge b_j. \tag{107}$$

In particular,

$$[a] \wedge [a] = -a \wedge a^{\mathrm{T}} = -\frac{1}{2}[a \times a]. \tag{108}$$

Proof. We first see that

$$\begin{aligned} [a] \wedge b &= \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_3 \wedge b_2 - a_2 \wedge b_3 \\ a_1 \wedge b_3 - a_3 \wedge b_1 \\ a_2 \wedge b_1 - a_1 \wedge b_2 \end{pmatrix} \\ &= -a \times b \end{aligned}$$

by Definition A.2. Similarly,

$$\begin{aligned} a^{\mathrm{T}} \wedge [b] &= \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \wedge \begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -a_2 \wedge b_3 + a_3 \wedge b_2 & -a_3 \wedge b_1 + a_1 \wedge b_3 & -a_1 \wedge b_2 + a_2 \wedge b_1 \end{pmatrix} \\ &= -(a \times b)^{\mathrm{T}}. \end{aligned}$$

From Definition A.2 we see that

$$[a \times b] = \begin{pmatrix} 0 & a_1 \wedge b_2 - a_2 \wedge b_1 & a_1 \wedge b_3 - a_3 \wedge b_1 \\ a_2 \wedge b_1 - a_1 \wedge b_2 & 0 & a_2 \wedge b_3 - a_3 \wedge b_2 \\ a_3 \wedge b_1 - a_1 \wedge b_3 & a_3 \wedge b_2 - a_2 \wedge b_3 & 0 \end{pmatrix}.$$

On the other hand,

$$\begin{aligned} [a] \wedge [b] &= \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -a_2 \wedge b_2 - a_3 \wedge b_3 & a_2 \wedge b_1 & a_3 \wedge b_1 \\ a_1 \wedge b_2 & -a_3 \wedge b_3 - a_1 \wedge b_1 & a_3 \wedge b_2 \\ a_1 \wedge b_3 & a_2 \wedge b_3 & -a_1 \wedge b_1 - a_2 \wedge b_2 \end{pmatrix} \\ &= -b \wedge a^{\mathrm{T}} - I \otimes \sum_{j=1}^{3} a_j \wedge b_j, \end{aligned}$$

and

$$\begin{aligned} [b] \wedge [a] &= \begin{pmatrix} -b_2 \wedge a_2 - b_3 \wedge a_3 & b_2 \wedge a_1 & b_3 \wedge a_1 \\ b_1 \wedge a_2 & -b_3 \wedge a_3 - b_1 \wedge a_1 & b_3 \wedge a_2 \\ b_1 \wedge a_3 & b_2 \wedge a_3 & -b_1 \wedge a_1 - b_2 \wedge a_2 \end{pmatrix} \\ &= \begin{pmatrix} a_2 \wedge b_2 + a_3 \wedge b_3 & -a_1 \wedge b_2 & -a_1 \wedge b_3 \\ -a_2 \wedge b_1 & a_3 \wedge b_3 + a_1 \wedge b_1 & -a_2 \wedge b_3 \\ -a_3 \wedge b_1 & -a_3 \wedge b_2 & a_1 \wedge b_1 + a_2 \wedge b_2 \end{pmatrix} \\ &= -a \wedge b^{\mathrm{T}} + I \otimes \sum_{j=1}^{3} a_j \wedge b_j. \end{aligned}$$

□

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