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BUBBLING AND EXTINCTION FOR SOME FAST DIFFUSION EQUATIONS IN BOUNDED DOMAINS

TIANLING JIN AND JINGANG XIONG

ABSTRACT. We study a Sobolev critical fast diffusion equation in bounded domains with the Brézis-Nirenberg effect. We obtain extinction profiles of its positive solutions, and show that the convergence rates of the relative error in regular norms are at least polynomial. Exponential decay rates are proved for generic domains. Our proof makes use of its regularity estimates, a curvature type evolution equation, as well as blow up analysis. Results for Sobolev subcritical fast diffusion equations are also obtained.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with smooth boundary $\partial\Omega$. We consider the Cauchy-Dirichlet problem for the fast diffusion equation with the Sobolev critical exponent

$$\begin{aligned} \frac{\partial}{\partial t} u^{\frac{n+2}{n-2}} &= \Delta u + bu \quad \text{in } \Omega \times (0, \infty), \\ u &= 0 \quad \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) &= u_0 \geq 0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, u_0 is not identically zero, and

$$b \in [0, \lambda_1) \text{ is a constant} \tag{2}$$

with λ_1 being the first eigenvalue of $-\Delta$ in Ω with zero Dirichlet boundary condition. Hence, the operator $-\Delta - b$ is coercive on the Sobolev space $H_0^1(\Omega)$. The fast diffusion equations arise in the modelling of gas-kinetics, plasmas, thin liquid film dynamics driven by Van der Waals forces, and etc. If $b = 0$, this Sobolev critical equation (1) can be viewed a unnormalized Yamabe flow with metrics degenerate on the boundary.

The theory of existence and uniqueness of solutions to (1) is well understood, see Vázquez [41, 42]. If $u_0 \in L^q(\Omega)$ for some $q > \frac{2n}{n-2}$, then the solution will become instantaneously positive in Ω and globally bounded. Moreover, the solution will vanish in a finite time $T^* > 0$. If we assume that $u_0 \in H_0^1(\Omega) \cap L^q(\Omega)$ for some $q > \frac{2n}{n-2}$, then it follows from the work of Chen-DiBenedetto [15], DiBenedetto-Kwong-Vespri [23] and Jin-Xiong [29] that the solution is of $C^{3,2}_{x,t}(\overline{\Omega} \times (0, T^*))$. In particular,

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the solutions are classical. Therefore, when we investigate the asymptotic behavior of nonnegative solutions to (1) as t approaching to the extinction time T^*, there is no loss of generality to consider classical (up to the boundary) solutions to (1).

When $\frac{n+2}{n-2}$ is replaced by $p \in (1, \frac{n+2}{n-2})$ if $n \geq 3$, or $p \in (1, \infty)$ if $n = 1, 2$, which is a Sobolev subcritical exponent, the extinction behavior of solutions to the fast diffusion equation

$$\tag{3} \begin{aligned} \frac{\partial}{\partial t} u^p &= \Delta u \quad \text{in } \Omega \times (0, \infty), \\ u &= 0 \qquad \text{on } \partial\Omega \times (0, \infty) \end{aligned}$$

has been well-studied. By the scaling

$$\tag{4} v(x,t) = \Big(\frac{p}{(p-1)(T^* - \tau)}\Big)^{\frac{1}{p-1}} u(x, \tau), \quad t = \frac{p}{p-1} \ln\Big(\frac{T^*}{T^* - \tau}\Big),$$

where T^* is the extinction time, the equation (3) becomes

$$\tag{5} \begin{aligned} \frac{\partial}{\partial t} v^p &= \Delta v + v^p \quad \text{in } \Omega \times (0, \infty), \\ v &= 0 \quad \text{on } \partial\Omega \times (0, \infty). \end{aligned}$$

Berryman-Holland [5] proved that the solution of (5) converges to a stationary solution v_∞ in $H_0^1(\Omega)$ along a sequence of times. Feireisl-Simondon [26] proved the full convergence in the $C^0(\overline{\Omega})$ topology. Bonforte-Grillo-Vazquez [8] proved that the relative error $v(\cdot, t)/v_\infty$ converges to 1 in $L^\infty(\Omega)$. Recently, Bonforte-Figalli [7] proved the sharp exponential convergence of the relative error for generic domains Ω, which means that the domains Ω satisfy

(6) For every nonnegative H_0^1 solution v of $-\Delta v - v^p = 0$ in Ω, the linearized operator at v, that is $L_v := -\Delta - pv^{p-1}$, has a trivial kernel in $H_0^1(\Omega)$.

See Akagi [1] for another proof. The set of smooth domains satisfying (6) has generic properties, see Saut-Temam [35].

The main advantage of the subcritical regime is the upper bound of solutions u to (3) proved in DiBenedetto-Kwong-Vespri [23]

$$\tag{7} u(x,t) \leq C d(x)(T^* - t)^{\frac{1}{p-1}} \quad \text{for } t < T^*,$$

where $d(x) = dist(x, \partial\Omega)$. The estimate (7) implies that the function v defined by (4), which satisfies (5), is uniformly bounded as $t \to \infty$, and consequently, has uniform regularity estimates up to the boundary $\partial\Omega$ by the work of [15, 23, 29].

However, this uniform bound in general does not hold for (5) if $p = \frac{n+2}{n-2}$. For instance, it is the case if Ω is star-shaped, since there is no stationary solution of (5) due to the Pohozaev identity. In this paper, we will show that the uniform boundedness still holds for the equation (1) assuming $b > 0$ and $n \geq 4$. The role of the positivity of b when $n \geq 4$ was first discovered in the seminal paper Brézis-Nirenberg [12], and is similar to the role that the non-vanishing Weyl tensor and the positive mass theorem play in the resolution of the Yamabe problem on compact manifolds by Aubin [2] and Schoen [36].

Under the scaling

$$\tag{8} v(x,t) = \left(\frac{n+2}{4(T^* - \tau)}\right)^{\frac{n-2}{4}} u(x, \tau), \quad t = \frac{n+2}{4} \ln\left(\frac{T^*}{T^* - \tau}\right),$$

the equation (1) becomes

$$\begin{aligned} \frac{\partial}{\partial t} v^{\frac{n+2}{n-2}} &= \Delta v + bv + v^{\frac{n+2}{n-2}} \quad \text{in } \Omega \times (0,\infty), \\ v &= 0 \quad \text{on } \partial\Omega \times (0,\infty). \end{aligned} \tag{9}$$

We will show that every solution of (9) converges to a stationary solution, that is a solution of

$$\Delta v + bv + v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \tag{10}$$

with at least polynomial rates. Moreover, the convergence rate will be exponential if the domain Ω satisfies the following condition:

$$\begin{aligned} &\text{For every nonnegative } H_0^1 \text{ solution } v \text{ of } -\Delta v - bv - v^{\frac{n+2}{n-2}} = 0 \text{ in } \Omega, \\ &\text{the linearized operator at } v, \text{that is } L_v := -\Delta - b - \frac{n+2}{n-2} v^{\frac{4}{n-2}}, \\ &\text{has a trivial kernel in } H_0^1(\Omega). \end{aligned} \tag{11}$$

The set of smooth domains satisfying (11) also has generic properties, see Saut-Temam [35].

Theorem 1.1. *Let $n \geq 4$, and $b > 0$ satisfy* (2). *Let u be a classical nonnegative solution of* (1) *with extinction time $T^* > 0$. Let v be defined by* (8). *Then there is a nonzero stationary solution v_∞ of* (9), *and two positive constants θ and C such that*

$$\left\| \frac{v(\cdot,t)}{v_\infty} - 1 \right\|_{C^2(\overline{\Omega})} \leq Ct^{-\theta} \quad \textit{for all } t \geq 1.$$

If Ω satisfies (11), *then there exist two positive constants γ and C such that*

$$\left\| \frac{v(\cdot,t)}{v_\infty} - 1 \right\|_{C^2(\overline{\Omega})} \leq Ce^{-\gamma t} \quad \textit{for all } t \geq 1.$$

All the constants θ, γ and C depend only on n, b, Ω and u_0.

When $n = 3$, it was shown in Brézis-Nirenberg [12] that the situation for the stationary equation (10) changes drastically from dimensions $n \geq 4$. The positivity of b is not sufficient to give a minimal energy solution of (10). Druet [24] showed that the necessary and sufficient condition is the positivity of the regular part of the Green's function of $-\Delta - b$ at a diagonal point. There should be similar changes for the parabolic equation (9) as well.

When $b = 0$, Sire-Wei-Zheng [38] recently proved the existence of some initial data such that the solution of (9) blows up at finitely many points with an explicit blow up rate as $t \to \infty$, using the gluing method for parabolic equations in the spirit of Cortázar-del Pino-Musso [18] and Dávila-del Pino-Wei [20]. This generalizes and provides rigorous proof of a result of Galaktionov-King [27] for the radially symmetric case, where the solution blows up at one point. A class of type II ancient solutions to the Yamabe flow, which are rotationally symmetric and converge to a tower of spheres as $t \to -\infty$, was constructed by Daskalopoulos-del Pino-Sesum [19]. Bubble tower solutions for the energy critical heat equation were constructed in del Pino-Musso-Wei [21]. It is conjectured in Sire-Wei-Zheng [38] that bubble tower solutions to (9) with $b = 0$ also exist. Nevertheless, if it is the global case (Ω replaced by $\mathbb{R}^n$), then it has been proved by del Pino-Sáez [22] that the solution of (9) for $b = 0$ with fast decay initial data will converge to a nontrivial stationary solution, which is in fact a standard bubble.

To prove Theorem 1.1, we will adapt the blow up analysis of Struwe [39], Bahri-Coron [3], Schwetlick-Struwe [37] and Brendle [9]. See also Chen-Xu [16] and Mayer [32] for similar analysis of scalar curvature flows. Here, we define a curvature type quantity $\mathcal{R}$, and derive its equation along the parabolic equation (9). Due to the lack of information of $\mathcal{R}$ on the boundary $\partial\Omega$, extra work is needed to obtain estimates for $\mathcal{R}$. Here the optimal boundary regularity proved in our previous paper [29] is crucial. Part of the blow up analysis in this paper remains valid when $b = 0$ or $n = 3$. The condition $n \geq 4$ and $b > 0$ is used in the final step (i.e., Corollaries 4.6 and 4.17) to rule out bubbles, which is in the same spirit of Brézis-Nirenberg [12] in obtaining compactness of minimizing sequences.

Our proof of the polynomial decay rates in Theorem 1.1 can be applied to prove the polynomial rate of the convergence of the relative error for the Sobolev subcritical fast diffusion equation (5) in all smooth domains. We also provide an alternative proof the exponential convergence result of Bonforte-Figalli [7] for Ω satisfying (6).

Theorem 1.2. *Suppose $p \in (1, \frac{n+2}{n-2})$ if $n \geq 3$, and $p \in (1, \infty)$ if $n = 1, 2$. Let u be a classical nonnegative solution of* (3) *with extinction time $T^* > 0$. Let v be defined by* (4)*. Then there is a stationary solution v_∞ of* (5)*, and two positive constants θ and C such that*

$$\left\| \frac{v(\cdot,t)}{v_\infty} - 1 \right\|_{C^2(\overline{\Omega})} \leq Ct^{-\theta} \quad \text{for all } t \geq 1.$$

If Ω satisfies (6)*, then there exist two positive constants γ and C such that*

$$\left\| \frac{v(\cdot,t)}{v_\infty} - 1 \right\|_{C^2(\overline{\Omega})} \leq Ce^{-\gamma t} \quad \text{for all } t \geq 1.$$

All the constants θ, γ and C depend only on n, p, Ω and u_0.

All the rates θ and γ in both Theorem 1.1 and Theorem 1.2 are not explicit. As mentioned earlier, the sharp exponential convergence rate of the relative error for (5) under the condition (6) was obtained by Bonforte-Figalli [7] (see also Akagi [1] for a different method). We do not pursuit the sharpness of γ in this paper.

We know from the work of Carlotto-Chodosh-Rubinstein [14] that there exists a Yamabe flow on $\mathbb{S}^1(1/\sqrt{n-2}) \times \mathbb{S}^{n-1}(1)$ such that it converges exactly at a polynomial rate. Recently, Choi-McCann-Seis [17] proved that for the solutions of the fast diffusion equation (5), the relative error either decays exponentially with the sharp rate or else decays algebraically at a rate $1/t$ or slower.

This paper is organized as follows. Sections 2–5 deal with the critical equation (9). We first obtain certain integral bounds for solutions of this critical equation in Section 2. Sections 3 is for the possible concentration phenomenon for its solutions. In Section 4, we use blow up analysis to rule out such possible concentration phenomenon. Section 5 is devoted to the proof of the uniform boundedness and convergence results in Theorem 1.1. In Section 6, we consider the subcritical equation (5) and prove Theorem 1.2.

2. Integral bounds

For an open set Ω, let $H_0^1(\Omega)$ be the closure of $C_c^\infty(\Omega)$ under the norm

$$\|u\|_{H_0^1(\Omega)} := \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)^{1/2}.$$

For convenience, we define

$$\|u\| := \left(\int_\Omega (|\nabla u|^2 - bu^2)\,\mathrm{d}x\right)^{1/2} \tag{12}$$

and

$$\langle u, v\rangle = \int_\Omega (\nabla u \nabla v - buv)\,\mathrm{d}x \tag{13}$$

be the associated inner product. Since

$$\left(1 - \frac{b}{\lambda_1}\right)\int_\Omega |\nabla u|^2\,\mathrm{d}x \le \int_\Omega (|\nabla u|^2 - bu^2)\,\mathrm{d}x$$

and we assumed $b < \lambda_1$, by the Sobolev inequality, there exists a constant $K_b > 0$ such that

$$\|u\|_{L^{\frac{2n}{n-2}}(\Omega)} \le K_b^{1/2}\|u\| \quad \text{for any } u \in H_0^1(\Omega). \tag{14}$$

Recall that from [29], we know that the solution $u(x,t)$ of (1) is smooth in $t \in (0, T^*)$ for every $x \in \overline{\Omega}$, and $\partial_t^l u(\cdot, t) \in C^{\frac{3n-2}{n-2}}(\overline{\Omega})$ for all $l \ge 0$ and all $t \in (0, T^*)$.

Lemma 2.1. *Let u be a solution of* (1)*, and T^* be the extinction time of u. Then for every $0 < t < T^*$,*

$$\frac{1}{C}(T^* - t)^{\frac{n}{2}} \le \int_\Omega u(x,t)^{\frac{2n}{n-2}}\,\mathrm{d}x \le C(T^*-t)^{\frac{n}{2}},$$

where C is a positive constant depending only on n, b, Ω and u_0.

Proof. If $b = 0$, the lemma was proved by [5] (noting that by our regularity result in [29], the regularity assumptions in [5] are satisfied, and thus the calculations in [4] are justified). The same proof applies if $b \in (0, \lambda_1)$ by using (14). We sketch it in the below for reader's convenience.

Let

$$\xi(t) = \left(\int_\Omega u(x,t)^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{2}{n}} \quad \text{and} \quad S(t) = \frac{\|u(\cdot,t)\|^2}{\|u(\cdot,t)\|^2_{L^{\frac{2n}{n-2}}(\Omega)}}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t) = -\frac{4}{n+2}S(t) \le -\frac{4}{n+2}K_b^{-1},$$

where we used (14) in the last inequality. Then the first inequality of this lemma follows by integrating the above inequality from t to T^*.

Making use of the equation (1) and the same arguments in [5], we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|^2 \le 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}S(t) \le 0.$$

Hence, both $\|u(\cdot,t)\|$ and $S(t)$ are non-increasing in t. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,t)^{\frac{2n}{n-2}}\,\mathrm{d}x = -\frac{2n}{n+2}\|u(\cdot,t)\|^2,$$

then by integrating the above inequality from t to T^* and using the monotonicity of $\|u(\cdot,t)\|$ and $S(t)$, we have

$$\int_\Omega u(x,t)^{\frac{2n}{n-2}}\,\mathrm{d}x \le \frac{2n}{n+2}(T^*-t)\|u(\cdot,t)\|^2 \le \frac{2n}{n+2}(T^*-t)S(0)\|u(\cdot,t)\|^2_{L^{\frac{2n}{n-2}}(\Omega)}.$$

This leads to the second inequality of this lemma. □

Let v be as in (8). By Lemma 2.1, we have

$$\frac{1}{C} \le \int_\Omega v(x,t)^{\frac{2n}{n-2}}\,\mathrm{d}x \le C \tag{15}$$

for all $t>0$, where C is a positive constant depending only on n, b, Ω and u_0. Define

$$F(v(t)) = \int_\Omega \Big(|\nabla v(x,t)|^2 - bv(x,t)^2 - \frac{n-2}{n}v(x,t)^{\frac{2n}{n-2}}\Big)\,\mathrm{d}x. \tag{16}$$

It follows that $F(v(t))$ is bounded from below for all $t>0$. By the equation of v and integrating by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t}F(v(t)) = -2\int_\Omega (\Delta v + bv + v^{\frac{n+2}{n-2}})\partial_t v\,\mathrm{d}x = -\frac{2(n+2)}{n-2}\int_\Omega v^{\frac{4}{n-2}}|\partial_t v|^2\,\mathrm{d}x \le 0. \tag{17}$$

Hence, $F(v(t))$ is non-increasing in t. Together with (15), we have $\|v(\cdot,t)\|_{H_0^1(\Omega)}$ is uniformly bounded. Moreover, there exists some constant F_∞ such that

$$\lim_{t\to\infty} F(v(t)) = F_\infty. \tag{18}$$

Define

$$\mathcal{R} = v^{-\frac{n+2}{n-2}}(-\Delta v - bv) \tag{19}$$

and

$$M_q(t) = \int_\Omega |\mathcal{R}-1|^q v^{\frac{2n}{n-2}}\,\mathrm{d}x, \quad q\ge 1. \tag{20}$$

In [29], we proved that $\mathcal{R} = 1 - \frac{n+2}{n-2}\frac{\partial_t v}{v}$ is C^2 up to the boundary $\partial\Omega$. However, all the estimates there for solutions of (9) are only locally uniform in $t\in(0,\infty)$. We shall prove some uniform estimates for all $t\in[1,\infty)$ and $M_q(t)\to 0$ as $t\to\infty$.

To do this, we will first use Moser's iteration to obtain a uniform lower bound of $\mathcal{R}$ as an intermediate step. So we need the following evolution equation of $\mathcal{R}$ and integration by parts formula.

Lemma 2.2. *Let $g = v^{\frac{4}{n-2}}g_{flat}$. Then*

(i)

$$\partial_t v^{\frac{2n}{n-2}} = -\frac{2n}{n+2}(\mathcal{R}-1)v^{\frac{2n}{n-2}}. \tag{21}$$

(ii)

$$\partial_t(\mathcal{R}-1) = \frac{n-2}{n+2}\Delta_g(\mathcal{R}-1) + \frac{4}{n+2}(\mathcal{R}-1)^2 + \frac{4}{n+2}(\mathcal{R}-1), \tag{22}$$

where Δ_g is the Laplace-Beltrami operator of g.

(iii) *For any $f\in H^2(\Omega)$ and $h\in H^1(\Omega)$,*

$$\int_\Omega h\Delta_g f\,\mathrm{d}vol_g = -\int_\Omega \langle\nabla_g f, \nabla_g h\rangle_g\,\mathrm{d}vol_g. \tag{23}$$

Proof. The equation (21) follows immediately from (9) and (19). We also have $\partial_t v = \frac{n-2}{n+2} v(1-\mathcal{R})$.

By the definition of $\mathcal{R}$, we have

$$\begin{aligned}\partial_t(\mathcal{R}-1) &= \frac{n+2}{n-2} v^{-\frac{2n}{n-2}} \partial_t v(\Delta+b)v - v^{-\frac{n+2}{n-2}}(\Delta+b)\partial_t v\\ &= v^{-\frac{n+2}{n-2}}(1-\mathcal{R})(\Delta+b)v - \frac{n-2}{n+2} v^{-\frac{n+2}{n-2}}(\Delta+b)(v(1-\mathcal{R}))\\ &= (\mathcal{R}-1)\mathcal{R} - \frac{n-2}{n+2} v^{-\frac{n+2}{n-2}}(\Delta+b)(v(1-\mathcal{R})).\end{aligned}$$

Let $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ be the conformal Laplacian of g, where Δ_g is the Laplace–Beltrami operator of the metric g and R_g is the the scalar curvature of g. By the conformal transformation law

$$L_g(v^{-1}\varphi) = v^{-\frac{n+2}{n-2}} \Delta\varphi, \quad \forall \varphi \in C^2(\Omega),$$

we have

$$\frac{n-2}{4(n-1)} R_g = -L_g(1) = -v^{-\frac{n+2}{n-2}} \Delta v = \mathcal{R} + b v^{-\frac{4}{n-2}}$$

and

$$\begin{aligned}v^{-\frac{n+2}{n-2}}(\Delta+b)(v(1-\mathcal{R})) &= L_g(1-\mathcal{R}) + b v^{-\frac{4}{n-2}}(1-\mathcal{R})\\ &= \Delta_g(1-\mathcal{R}) - \frac{n-2}{4(n-1)} R_g(1-\mathcal{R}) + b v^{-\frac{4}{n-2}}(1-\mathcal{R})\\ &= \Delta_g(1-\mathcal{R}) - \mathcal{R}(1-\mathcal{R}).\end{aligned}$$

Then, (22) follows.

Finally,

$$\begin{aligned}\int_\Omega h\Delta_g f \,\mathrm{d}vol_g &= \int_\Omega h v^{-\frac{2n}{n-2}} \partial_i (v^{\frac{2n}{n-2}} v^{-\frac{4}{n-2}} \partial_i f) v^{\frac{2n}{n-2}} \,\mathrm{d}x\\ &= \int_\Omega h \partial_i (v^2 \partial_i f) \,\mathrm{d}x = -\int_\Omega v^2 \partial_i f \partial_i h \,\mathrm{d}x = -\int_\Omega \langle \nabla_g f, \nabla_g h\rangle_g \,\mathrm{d}vol_g,\end{aligned}$$

where we used $v=0$ on $\partial\Omega$ in the third equality. □

We have the following Sobolev inequality regarding the metric $g = v^{\frac{4}{n-2}} g_{flat}$:

Lemma 2.3. *There holds*

$$\left(\int_\Omega |f|^{\frac{2n}{n-2}} \,\mathrm{d}vol_g\right)^{\frac{n-2}{n}} \le K_b \int_\Omega (|\nabla_g f|_g^2 + \mathcal{R} f^2) \,\mathrm{d}vol_g$$

for any $f \in H^1(\overline{\Omega})$, where K_b is the constant in (14).

Proof. Note that

$$|\nabla(fv)|^2 = v^2|\nabla f|^2 + f^2|\nabla v|^2 + 2vf\nabla v\cdot\nabla f,$$

$$\begin{aligned}\int_\Omega (f^2|\nabla v|^2 + 2v\nabla v f\nabla f) \,\mathrm{d}x &= \int_\Omega (f^2|\nabla v|^2 + v\nabla v\nabla f^2) \,\mathrm{d}x\\ &= -\int_\Omega v f^2 \Delta v \,\mathrm{d}x\\ &= \int_\Omega (\mathcal{R} f^2 v^{\frac{2n}{n-2}} + b v^2 f^2) \,\mathrm{d}x.\end{aligned}$$

Hence,

$$\int_\Omega (|\nabla_g f|_g^2+\mathcal{R}f^2)\,\mathrm{d}vol_g=\int_\Omega (v^2|\nabla f|^2+\mathcal{R}f^2v^{\frac{2n}{n-2}})\,\mathrm{d}x=\int_\Omega(|\nabla(fv)|^2-b(fv)^2)\,\mathrm{d}x.$$

Therefore, the lemma follows from (14). □

For any $t_0\ge 0$ and $T>0$, let

$$V^1(\Omega\times(t_0,t_0+T))=C^0((t_0,t_0+T);L^2(\Omega))\cap L^2((t_0,t_0+T);H^1(\Omega)),$$

equipped with the norm

$$\|f\|^2_{V^1(\Omega\times(t_0,t_0+T))}=\sup_{t_0<t<t_0+T}\int_\Omega f(x,t)^2\,\mathrm{d}vol_g+\int_{t_0}^{t_0+T}\int_\Omega(|\nabla_g f|_g^2+\mathcal{R}f^2)\,\mathrm{d}vol_g\mathrm{d}t.$$

We have the following parabolic version of Sobolev inequality.

Lemma 2.4. *For any $f\in V^1(\Omega\times(t_0,t_0+T))$, we have*

$$\left(\int_{t_0}^{t_0+T}\int_\Omega |f|^{\frac{2(n+2)}{n}}\,\mathrm{d}vol_g\mathrm{d}t\right)^{\frac{n}{n+2}}\le K_b^{\frac{n}{n+2}}\|f\|^2_{V^1(\Omega\times(t_0,t_0+T))}.$$

Proof. By Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}\int_\Omega |f|^{\frac{2(n+2)}{n}}\,\mathrm{d}vol_g&=\int_\Omega |f|^2|f|^{\frac4n}\,\mathrm{d}vol_g\\&\le\Big(\int_\Omega|f|^{\frac{2n}{n-2}}\,\mathrm{d}vol_g\Big)^{\frac{n-2}{n}}\Big(\int_\Omega|f|^2\,\mathrm{d}vol_g\Big)^{\frac2n}\\&\le K_b\int_\Omega(|\nabla_g f|_g^2+\mathcal{R}f^2)\,\mathrm{d}vol_g\Big(\int_\Omega|f|^2\,\mathrm{d}vol_g\Big)^{\frac2n}.\end{aligned}$$

Hence, by Young's inequality

$$\begin{aligned}&\left(\int_{t_0}^{t_0+T}\int_\Omega |f|^{\frac{2(n+2)}{n}}\,\mathrm{d}vol_g\mathrm{d}t\right)^{\frac{n}{n+2}}\\&\le K_b^{\frac{n}{n+2}}\left(\int_\Omega(|\nabla_g f|_g^2+\mathcal{R}f^2)\,\mathrm{d}vol_g\right)^{\frac{n}{n+2}}\left(\sup_{t_0<t<t_0+T}\int_\Omega f(x,t)^2\,\mathrm{d}vol_g\right)^{\frac{2}{n+2}}\\&\le K_b^{\frac{n}{n+2}}\|f\|^2_{V^1(\Omega\times(t_0,t_0+T))}.\end{aligned}$$

Therefore, the proof is completed. □

With the Sobolev inequality in Lemma 2.4, we will apply Moser's iterations to the equation (22) to obtain a uniform lower bound of $\mathcal{R}$.

Lemma 2.5. *For $t\ge 1$, we have*

$$\mathcal{R}-1\ge -C,$$

where C is a constant depending only on Ω, n, b and v_0.

Proof. Let $T>2$, $\frac12\le T_2<T_1\le 1$, $\eta(t)$ be a smooth cut-off function so that $\eta(t)=0$ for all $t<T_2$, $0\le\eta(t)\le 1$ for $t\in[T_2,T_1]$, $\eta(t)=1$ for all $t>T_1$, and $|\eta'(t)|\le\frac{2}{T_1-T_2}$. Denote $\phi=(1-\mathcal{R})^+$. By (22), we have

$$\partial_t(1-\mathcal{R})=\frac{n-2}{n+2}\Delta_g(1-\mathcal{R})-\frac{4}{n+2}(1-\mathcal{R})^2+\frac{4}{n+2}(1-\mathcal{R}).$$

Let $k \geq \frac{n}{2} - 1$ be a real number. Multiplying both sides of the inequality by $\eta^2\phi^{1+k}$ and integrating by parts, we see that, for any $0 < s < T$,

$$\begin{aligned}&\frac{1}{2+k}\int_0^s\int_\Omega \eta^2\partial_t\phi^{2+k}\,\mathrm{d}vol_g\mathrm{d}t + \frac{4(n-2)(k+1)}{(n+2)(k+2)^2}\int_0^s\int_\Omega \eta^2|\nabla_g\phi^{\frac{k+2}{2}}|_g^2\,\mathrm{d}vol_g\mathrm{d}t\\&\leq -\frac{4}{n+2}\int_0^s\int_\Omega \phi^{3+k}\eta^2\,\mathrm{d}vol_g\mathrm{d}t + \frac{4}{n+2}\int_0^s\int_\Omega \phi^{2+k}\eta^2\,\mathrm{d}vol_g\mathrm{d}t.\end{aligned}$$

Note that using (21), we have

$$\begin{aligned}&\frac{1}{2+k}\int_0^s\int_\Omega \eta^2\partial_t\phi^{2+k}\,\mathrm{d}vol_g\mathrm{d}t\\&= \frac{1}{2+k}\int_\Omega \phi^{2+k}\eta^2\,\mathrm{d}vol_g\Big|_{t=s}\\&\quad -\frac{1}{2+k}\int_0^s\int_\Omega \phi^{2+k}\left(2\eta\partial_t\eta + \frac{2n}{n+2}(1-\mathcal{R})\eta^2\right)\mathrm{d}vol_g\mathrm{d}t\\&= \frac{1}{2+k}\int_\Omega \phi^{2+k}\eta^2\,\mathrm{d}vol_g\Big|_{t=s}\\&\quad -\frac{1}{2+k}\int_0^s\int_\Omega \left(2\phi^{2+k}\eta\partial_t\eta + \frac{2n}{n+2}\phi^{3+k}\eta^2\right)\mathrm{d}vol_g\mathrm{d}t.\end{aligned}$$

Note that the term $\frac{2n}{n+2}(1-\mathcal{R})\eta^2$ in the above comes from the derivative of the volume form $\mathrm{d}vol_g$ in t. Since $k \geq \frac{n}{2} - 1$, $\frac{1}{2+k}\frac{2n}{n+2} < \frac{4}{n+2}$. Furthermore,

$$\int_\Omega \mathcal{R}\phi^{2+k}\eta^2\,\mathrm{d}vol_g = -\int_\Omega (1-\mathcal{R})\phi^{2+k}\eta^2\,\mathrm{d}vol_g + \int_\Omega \phi^{2+k}\eta^2\,\mathrm{d}vol_g \leq \int_\Omega \phi^{2+k}\eta^2\,\mathrm{d}vol_g.$$

It follows that

$$\|\eta\phi^{\frac{2+k}{2}}\|^2_{V^1(\Omega\times(0,T))} \leq C(2+k)\int_0^T\int_\Omega \phi^{2+k}(\eta^2 + |\partial_t\eta|\eta)\,\mathrm{d}vol_g\mathrm{d}t,$$

where $C > 0$ depends only on n. Making use of Lemma 2.4, we have for all $\gamma := k+2 \geq \frac{n+2}{2}$ that

$$\left(\int_{T_1}^T\int_\Omega \phi^{\frac{\gamma(n+2)}{n}}\,\mathrm{d}vol_g\mathrm{d}t\right)^{\frac{n}{\gamma(n+2)}} \leq \left(\frac{C\gamma}{T_1 - T_2}\right)^{\frac{1}{\gamma}}\left(\int_{T_2}^T\int_\Omega \phi^\gamma\,\mathrm{d}vol_g\mathrm{d}t\right)^{\frac{1}{\gamma}}.$$

By the standard Moser's iteration argument, we have

$$\sup_{\Omega\times[1,T]}\phi \leq C(n,K_b)\Big(\int_{1/2}^T\int_\Omega \phi^{\frac{n+2}{2}}\,\mathrm{d}vol_g\mathrm{d}t\Big)^{\frac{2}{n+2}},$$

where $C(n,K_b) > 0$ depending only on n and K_b. Thus,

$$\begin{aligned}\sup_{\Omega\times[1,T]}\phi &\leq C(n,K_b)\Big(\int_{1/2}^1\int_\Omega \phi^{\frac{n+2}{2}}\,\mathrm{d}vol_g\mathrm{d}t\Big)^{\frac{2}{n+2}}\\&\quad + C(n,K_b)\Big(\int_1^T\int_\Omega \phi^{\frac{n+2}{2}}\,\mathrm{d}vol_g\mathrm{d}t\Big)^{\frac{2}{n+2}}\\&\leq C(n,K_b)\|\mathcal{R}-1\|_{L^\infty(\Omega\times(1/2,1))} + C(n,K_b)\Big(\int_1^T\int_\Omega \phi^{\frac{n+2}{2}}\,\mathrm{d}vol_g\mathrm{d}t\Big)^{\frac{2}{n+2}}.\end{aligned}$$

By Young's inequality, we have

$$\Big(\int_1^T\int_\Omega \phi^{\frac{n+2}{2}}\,\mathrm{d}vol_g\mathrm{d}t\Big)^{\frac{2}{n+2}} \le (\sup_{\Omega\times[1,T]}\phi)^{\frac{n-2}{n+2}}\Big(\int_1^T\int_\Omega \phi^2\,\mathrm{d}vol_g\mathrm{d}t\Big)^{\frac{2}{n+2}}$$
$$\le \varepsilon \sup_{\Omega\times[1,T]}\phi + C(\varepsilon)\Big(\int_1^T\int_\Omega \phi^2\,\mathrm{d}vol_g\mathrm{d}t\Big)^{\frac{1}{2}},$$

for any small constant ε. Therefore, by choosing a small ε, we have

$$\sup_{\Omega\times[1,T]}\phi \le C(n,K_b)\left\{\|\mathcal{R}-1\|_{L^\infty(\Omega\times(1/2,1))} + \Big(\int_1^T M_2\,\mathrm{d}t\Big)^{\frac{1}{2}}\right\}. \tag{24}$$

By (17) and the definition of $\mathcal{R}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F(v(t)) = -\frac{2(n-2)}{n+2}M_2(t).$$

It follows that

$$\int_0^\infty M_2(t)\,\mathrm{d}t \le \frac{n+2}{2(n-2)}(F(v(0))-F_\infty) < \infty. \tag{25}$$

Moreover, it was proved in [29] that $\|\mathcal{R}-1\|_{L^\infty(\Omega\times(1/2,1))} \le C$. Sending $T\to\infty$ in (24), we have

$$\sup_{\Omega\times[1,\infty)}(1-\mathcal{R})^+ \le C.$$

Therefore, the proof is completed. □

Using this uniform lower bound of $\mathcal{R}$, we can derive some useful differential inequalities for M_q defined in (20).

For $q>1$, using Lemma 2.2, we have

$$\frac{\mathrm{d}M_q}{\mathrm{d}t} = \int_\Omega q|\mathcal{R}-1|^{q-2}(\mathcal{R}-1)\frac{\partial}{\partial t}(\mathcal{R}-1)\,\mathrm{d}vol_g - \int_\Omega(\mathcal{R}-1)^q\frac{\partial}{\partial t}v^{\frac{2n}{n-2}}\,\mathrm{d}x$$
$$= q\frac{n-2}{n+2}\int_\Omega|\mathcal{R}-1|^{q-2}(\mathcal{R}-1)\Delta_g(\mathcal{R}-1)\,\mathrm{d}vol_g$$
$$+\frac{4q}{n+2}\int_\Omega|\mathcal{R}-1|^q\,\mathrm{d}vol_g + \frac{4}{n+2}(q-\frac{n}{2})\int_\Omega|\mathcal{R}-1|^q(\mathcal{R}-1)\,\mathrm{d}vol_g.$$

Using Lemma 2.5, we have for $t\ge1$ that

$$\left|\int_\Omega|\mathcal{R}-1|^q(\mathcal{R}-1)\,\mathrm{d}vol_g - \int_\Omega|\mathcal{R}-1|^{q+1}\,\mathrm{d}vol_g\right| = 2\int_\Omega|\mathcal{R}-1|^q(\mathcal{R}-1)^-\,\mathrm{d}vol_g$$
$$\le CM_q.$$

Using Lemma 2.2, we have

$$\int_\Omega|\mathcal{R}-1|^{q-2}(\mathcal{R}-1)\Delta_g(\mathcal{R}-1)\,\mathrm{d}vol_g = -\frac{4(q-1)}{q^2}\int_\Omega|\nabla_g|\mathcal{R}-1|^{\frac{q}{2}}|_g^2\,\mathrm{d}vol_g \le 0.$$

Therefore, for $q\le\frac{n}{2}$ we have,

$$\frac{\mathrm{d}M_q}{\mathrm{d}t} + \frac{4}{n+2}\Big(\frac{n}{2}-q\Big)M_{q+1} \le CM_q \quad \text{for } t\ge1, \tag{26}$$

where $C>0$ is a constant depending on q.

For $q \geq \frac{n}{2}$, we first obtain from Lemma 2.3 that

$$\begin{aligned}
&\int_\Omega |\mathcal{R}-1|^{q-2}(\mathcal{R}-1)\Delta_g(\mathcal{R}-1)\,\mathrm{d}vol_g \\
&= -\frac{4(q-1)}{q^2}\int_\Omega |\nabla_g|\mathcal{R}-1|^{\frac{q}{2}}|_g^2\,\mathrm{d}vol_g \\
&\leq -\beta M_{\frac{qn}{n-2}}^{\frac{n-2}{n}} + \frac{4(q-1)}{q^2}\int_\Omega \mathcal{R}|\mathcal{R}-1|^q\,\mathrm{d}vol_g \\
&\leq -\beta M_{\frac{qn}{n-2}}^{\frac{n-2}{n}} + \frac{4(q-1)}{q^2}\int_\Omega (\mathcal{R}-1)|\mathcal{R}-1|^q\,\mathrm{d}vol_g + \frac{4(q-1)}{q^2}M_q,
\end{aligned}$$

where $\beta > 0$ is a constant depending on K_b and q. Then, we have

$$\frac{\mathrm{d}M_q}{\mathrm{d}t} + \beta M_{\frac{qn}{n-2}}^{\frac{n-2}{n}} \leq \frac{4}{n+2}\left(q - \frac{n}{2} + \frac{(n-2)(q-1)}{q}\right)M_{q+1} + CM_q.$$

By the interpolation inequality and Young's inequality we have

$$M_{q+1} \leq M_{\frac{qn}{n-2}}^{\frac{n-2}{2q}} M_q^{\frac{2(q+1)-n}{2q}} \leq \varepsilon M_{q(p+1)/2}^{\frac{n-2}{n}} + C(\varepsilon)M_q^{\frac{2(q+1)-n}{2q-n}}.$$

By choosing a small ε, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}M_q(t) + \beta M_{\frac{qn}{n-2}}(t)^{\frac{n-2}{n}} \leq C\Big(M_q(t) + M_q(t)^{1+\frac{2}{2q-n}}\Big) \quad \text{for } t \geq 1 \tag{27}$$

for $q > \frac{n}{2}$, where β and C are positive constants depending on q.

The differential inequalities (26) and (27) will be used recursively to prove the decay of M_q for all $q \geq 1$.

Proposition 2.6. *For every $1 \leq q < \infty$, we have*

$$\lim_{t\to\infty} M_q(t) = 0.$$

Proof. By Hölder's inequality and (15), we only need to consider $q \geq 2$.

The idea of the proof will go recursively as follows. Note that if the right hand sides of (26) and (27) are integrable in $[1,\infty)$, then by integrating both sides, and noticing that $\frac{4}{n+2}\left(\frac{n}{2}-q\right)M_{q+1}$ with $q \leq n/2$ and $\beta M_{\frac{qn}{n-2}}(t)^{\frac{n-2}{n}}$ are nonnegative and thus can be dropped, we will have $M_q(t) \to 0$ as $t \to \infty$. Integrating again including these two nonnegative terms will in return show that they are integrable. This iteration shows the integrability and the limit of M_{q+1} or $M_{\frac{qn}{n-2}}$ from M_q. The starting point of this iteration is $q = 2$, because of (25). This gives us a desired sequence $\{q_k\}$ for which the proposition holds. The conclusion for all q is then followed by Hölder's inequality and (15). The details of the proof are given in the below.

Let us assume $n \geq 4$ first.

Case 1. $2 \leq q \leq \frac{n}{2}$.

Since $M_2 \in L^1(0,\infty)$, we can pick $t_j \to \infty$ such that $M_2(t_j) \to 0$ as $j \to \infty$. By (26) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}M_2(t) \leq CM_2(t).$$

Integrating the above inequality we have

$$M_2(t) \leq M_2(t_j) + C\int_{t_j}^\infty M_2(s)\,\mathrm{d}s \quad \text{for } t \geq t_j.$$

Hence, $\lim_{t\to\infty} M_2(t) = 0$. If $2 < q \le \frac{n}{2}$, (26) we have

$$\int_1^\infty M_3(t)\,\mathrm{d}t \le C\Big(\int_1^\infty M_2(t)\,\mathrm{d}t + M_2(1)\Big) < \infty.$$

For any $2 < q \le \min\{3, \frac{n}{2}\}$, we have $M_q(t) \le M_2(t) + M_3(t)$. Hence, $\int_1^\infty M_q(t)\,\mathrm{d}t < \infty$. We can repeat the argument for M_2 to show that $M_q(t) \to 0$ as $t \to \infty$. If $3 < \frac{n}{2}$, we can show that $\int_1^\infty M_4 < \infty$ and $M_q(t) \to 0$ as $t \to \infty$ for all $3 < q \le \min\{4, \frac{n}{2}\}$. Repeating this argument in finite times, and using Hölder's inequality with (15), we then have $M_q \in L^1(1,\infty)$ and $M_q(t) \to 0$ as $t \to \infty$ for all $2 \le q \le \frac{n}{2}$.

Case 2. $q > \max\{2, \frac{n}{2}\}$.

By (26) with $q = n/2$, we have

$$\int_1^\infty M_{\frac{n^2}{2(n-2)}}(t)^{\frac{n-2}{n}}\,\mathrm{d}t < \infty. \tag{28}$$

Using (27) to have

$$\frac{\mathrm{d}}{\mathrm{d}t} M_q(t) \le C M_q(t)^{\frac{n-2}{n}}\Big(M_q(t)^{\frac{2}{n}} + M_q(t)^{\frac{2}{n}+\frac{2}{2q-n}}\Big) \quad \text{for } t \ge 1.$$

Hence,

$$H(M_q(t)) \le H(M_q(T)) + C\int_T^\infty M_q^{\frac{n-2}{n}}\,\mathrm{d}t \quad \text{for } 1 \le T < t < \infty$$

where

$$H(\rho) = \int_0^\rho \frac{1}{s^{\frac{2}{n}} + s^{\frac{2}{n}+\frac{2}{2q-n}}}\,\mathrm{d}s.$$

Let

$$q_0 = \frac{n^2}{2(n-2)}, \quad q_k = \frac{n}{n-2} q_{k-1}, \quad k = 1, 2, \cdots.$$

Note that $\frac{2}{n} + \frac{2}{2q_0-n} = 1$ and $\frac{2}{n} + \frac{2}{2q_k-n} < 1$ for all $k \ge 1$. Hence, starting with (28) that $\int_1^\infty M_{q_0}(t)^{\frac{n-2}{n}}\,\mathrm{d}t < \infty$, using similar arguments to those in Case 1, we can recursively prove in the order of $k = 0, 1, 2, \cdots$ that $M_{q_k}(t) \to 0$ as $t \to \infty$, $\int_1^\infty \Big(M_{q_k}(t) + M_{q_k}(t)^{1+\frac{2}{2q-n}}\Big) < \infty$, $\int_1^\infty M_{q_{k+1}}(t)^{\frac{n-2}{n}}\,\mathrm{d}t < \infty$, and $M_{q_{k+1}}(t) \to 0$ as $t \to \infty$. Hence, using Hölder's inequality with (15), $M_q(t) \to 0$ for any $q \ge \frac{n^2}{2(n-2)}$.

Finally, let us consider $n = 3$. By (27), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} M_2(t) \le C M_2(t)(1 + M_2(t)^2).$$

Using (25), we can pick $t_i \to \infty$ such that $M_2(t_i) \to 0$. Hence,

$$\arctan M_2(t) \le \arctan M_2(t_i) + C\int_{t_i}^\infty M_2(t)\,\mathrm{d}t.$$

It follows that $\lim_{t\to\infty} \arctan M_2(t) = 0$ and thus $\lim_{t\to\infty} M_2(t) = 0$. Hence, $\int_1^\infty M_6(t)^{\frac{1}{3}}\,\mathrm{d}t < \infty$. Since $6 > q_0$ when $n = 3$, we can use the argument of those in Case 2 to show that $M_q(t) \to 0$ for all $q \ge 6$. By Hölder inequality, we conclude that $M_q(t) \to 0$ for all $q \ge 1$. □

Corollary 2.7. *We have*

$$\lim_{t\to\infty} \|\mathcal{R}-1\|_{L^\infty(\Omega)} = 0.$$

Proof. Consider the equation of $1-\mathcal{R}$ as in the proof of Lemma 2.5:

$$\partial_t(1-\mathcal{R}) = \frac{n-2}{n+2}\Delta_g(1-\mathcal{R}) + c(x,t)(1-\mathcal{R}) + \frac{4}{n+2}(1-\mathcal{R}),$$

where $c(x,t) = -\frac{4}{n+2}(1-\mathcal{R})$. This is a linear equation of $1-\mathcal{R}$. We know from the proof of Proposition 2.6 that there exists a sufficiently large $q>1$ such that

$$\int_1^\infty M_q(t)\,\mathrm{d}t < \infty.$$

This means that $c(x,t)$ has very high integrability against $\mathrm{d}vol_g$ in space-time. Then we can apply the Moser's iteration as in the proof of Lemma 2.5 to obtain

$$\|\mathcal{R}-1\|_{L^\infty(\Omega\times(T,\infty))} \le C\left(\int_{T-1}^\infty M_q(t)\,\mathrm{d}t\right)^{\frac{1}{q}}$$

for all large T. Hence, the corollary follows. □

3. Concentration compactness

The solution of (9) may blow up as $t\to\infty$ because of the critical exponent $\frac{n+2}{n-2}$. Nevertheless, we also know how the solutions may blow up.

Proposition 3.1. *Let v be a solution of (9). For any $t_\nu\to\infty$, $\nu\to\infty$, $v_\nu = v(\cdot,t_\nu)$ is a Palais-Smale sequence of the functional F given by (16) in $H_0^1(\Omega)$.*

Proof. We have already proved that v_ν is bounded in $H_0^1(\Omega)$ and $F(v_\nu)\to F_\infty$ as $\nu\to\infty$. It remains to show the derivative of F at v_ν tends to zero. Indeed, for any $\varphi\in H_0^1(\Omega)$, we have

$$\begin{aligned}\langle \mathrm{d}F(v_\nu),\varphi\rangle &= 2\int_\Omega(-\Delta v_\nu - bv_\nu - v_\nu^{\frac{n+2}{n-2}})\varphi\,\mathrm{d}x\\ &= 2\int_\Omega(\mathcal{R}-1)v_\nu^{\frac{n+2}{n-2}}\varphi\,\mathrm{d}x\\ &\le 2\left(\int_\Omega|\mathcal{R}-1|^{\frac{2n}{n+2}}v_\nu^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{n+2}{2n}}\left(\int_\Omega|\varphi|^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{\frac{n-2}{2n}}\\ &\le C(n)M_{\frac{2n}{n+2}}(t_\nu)^{\frac{n+2}{2n}}\|\varphi\|_{H_0^1(\Omega)},\end{aligned}$$

where we used Hölder's inequality and the Sobolev inequality. It follows from Proposition 2.6 that $\mathrm{d}F(v_\nu)$ strongly converges to 0 in $H^{-1}(\Omega)$.

Therefore, the proof is completed. □

Proposition 3.2 shows that the blow up points, if exist, will stay uniformly away from the boundary $\partial\Omega$.

Proposition 3.2. *There exist two positive constants δ_0 and C, depending on $v(\cdot,1)$, such that for all $x\in\Omega$ with $d(x) := \mathrm{dist}(x,\partial\Omega) < \delta_0$ and $t\ge 1$,*

$$v(x,t)\le Cd(x).$$

Proof. We are going to use the moving plane method as Han [28] did for the elliptic case. By the Hopf Lemma, there exist $\rho_0 > 0$ and $\alpha_0 > 0$ such that $v(z-\rho e, 1)$ is nondecreasing for $0 < \rho < \rho_0$, where $z \in \partial\Omega$, $e \in \mathbb{R}^n$ with $|e| = 1$, and $(e, \nu(z)) \geq \alpha_0$ with $\nu(z)$ the unit out normal to $\partial\Omega$ at z. If Ω is strictly convex, using the moving plane method we can conclude that $v(z-\rho e, t)$ is nondecreasing for $0 < \rho < \rho_0$, and for all $t \geq 1$. Therefore, we can find $\gamma > 0$ and $\delta > 0$ such that for any fixed $t \geq 1$, and any $x \in \Omega$ satisfying $0 < d(x) < \delta$, there exists a measurable set Γ_x with (i) $meas(\Gamma_x) \geq \gamma$, (ii) $\Gamma_x \subset \{z : d(z) \geq \delta/2\}$, and (iii) $v(y,t) \geq v(x,t)$ for any $y \in \Gamma_x$. Actually, Γ_x can be taken to a piece of cone with vertex at x. It follows that for any $x \in \{z : 0 < d(z) < \delta\}$, we have

$$v(x,t) \leq \frac{1}{meas(\Gamma_x)} \int_{\Gamma_x} v(y,t)\, \mathrm{d}y \leq \frac{C}{\gamma},$$

where we used (15) and Hölder's inequality. Namely, $v(x,t) \leq C$ for $(x,t) \in \{z : 0 < d(z) < \delta\} \times [1,\infty)$. By the proof of Theorem 4.1 in [23], we have $v(x,t) \leq Cd(x)$ for $(x,t) \in \{z : 0 < d(z) < \delta\} \times [1,\infty)$.

For a general domain, one can first use a Kelvin transform near each boundary point, and then apply the moving plane method. Pick any point $P \in \partial\Omega$ for instance. Since we assume the boundary of the domain Ω is smooth, we may assume, without loss of generality, that the unit ball B_1 contacts P from the exterior of Ω (i.e., $B_1 \subset \Omega^c$ and $P \in \partial B_1$). Let $w(x,t)$ be the Kelvin transform of v:

$$w(x,t) = |x|^{2-n} v\left(\frac{x}{|x|^2}, t\right).$$

Then

$$\begin{cases} \partial_t w^{\frac{n+2}{n-2}} = \Delta w + b|x|^{-4} w + w^{\frac{n+2}{n-2}} & \text{in } \Omega_P \times (0,\infty) \\ w = 0 \quad \text{on } \partial\Omega_P \times (0,\infty), \end{cases}$$

where Ω_P is the image of Ω under the Kelvin transform. Since $b \geq 0$, $b|x|^{-4}$ is nondecreasing along the $-P$ direction. Applying the moving plane method we have that $w(\cdot,t)$ is nondecreasing along the $-P$ direction in a neighborhood (uniform in t) of P. Since the $L^{\frac{2n}{n-2}}$ norm is invariant under the Kelvin transform, using the above argument in the case of strictly convex domains, we conclude that $w(\cdot,t)$ is bounded in a neighborhood of P independent of t and so is $v(\cdot,t)$. It follows that $v(x,t) \leq Cd(x)$ for $(x,t) \in \{z : 0 < d(z) < \delta\} \times [1,\infty)$ for some $\delta > 0$.

Therefore, the proof is completed. □

For $a \in \mathbb{R}^n$ and $\lambda \in (0,\infty)$, let

$$\bar{\xi}_{a,\lambda}(x) = c_0 \left(\frac{\lambda}{1+\lambda^2|x-a|^2}\right)^{\frac{n-2}{2}} \tag{29}$$

with $c_0 = (n(n-2))^{\frac{n-2}{4}}$. Then we have

$$-\Delta\bar{\xi}_{a,\lambda} = \bar{\xi}_{a,\lambda}^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n$$

and

$$\int_{\mathbb{R}^n} \bar{\xi}_{a,\lambda}^{\frac{2n}{n-2}} = Y(\mathbb{S}^n)^{\frac{n}{2}},$$

where $\mathbb{S}^n$ is the standard unit sphere in $\mathbb{R}^{n+1}$,

$$Y(\mathbb{S}^n) = \frac{n(n-2)}{4}|\mathbb{S}^n|^{\frac{2}{n}} = \inf_{u\in H^1(\mathbb{S}^n)} \frac{\int_{\mathbb{S}^n} |\nabla u|^2 + \frac{n(n-2)}{4}u^2 \mathrm{d}vol_{g_{\mathbb{S}^n}}}{\left(\int_{\mathbb{S}^n} |u|^{\frac{2n}{n-2}} vol_{g_{\mathbb{S}^n}}\right)^{\frac{n-2}{n}}}$$

and $|\mathbb{S}^n|$ is the area of $\mathbb{S}^n$. Define

$$\xi_{a,\lambda}(x) = \bar{\xi}_{a,\lambda}(x) - h_{a,\lambda}(x), \tag{30}$$

where $\Delta h_{a,\lambda}(x) = 0$ in Ω and $h_{a,\lambda} = \bar{\xi}_{a,\lambda}$ on $\partial\Omega$. By the maximum principle, $\xi_{a,\lambda} > 0$ in Ω and $h_{a,\lambda} > 0$ in $\overline{\Omega}$.

Proposition 3.3. *Let v be a solution of (9). For any $t_\nu \to \infty$, $\nu \to \infty$, after passing to a subsequence if necessary, v_ν weakly converges to v_∞ in $H^1_0(\Omega)$ and we can find a nonnegative integer m and a sequence of m-tuplets $(x^*_{k,\nu}, \lambda^*_{k,\nu})_{1\le k\le m}$, $(x^*_{k,\nu}, \lambda^*_{k,\nu}) \in \Omega \times (0,\infty)$, with the following properities.*

1. *The function $v_\infty \in H^1_0(\Omega)$ satisfies the equation $-\Delta v_\infty - bv_\infty = v_\infty^{\frac{n+2}{n-2}}$ in Ω.*
2. *There hold, for all $i \neq j$,*
$$\frac{\lambda^*_{i,\nu}}{\lambda^*_{j,\nu}} + \frac{\lambda^*_{j,\nu}}{\lambda^*_{i,\nu}} + \lambda^*_{i,\nu}\lambda^*_{j,\nu}|x^*_{i,\nu} - x^*_{j,\nu}|^2 \to \infty,$$
and for all k, $d(x^*_{k,\nu}) \ge \delta_0/2$ with the constant $\delta_0 > 0$ in Proposition 3.2,
$$\lambda^*_{k,\nu} d(x^*_{k,\nu}) \to \infty$$
as $\nu \to \infty$.
3. *We have*
$$\left\| v_\nu - v_\infty - \sum_{k=1}^m \xi_{x^*_{k,\nu},\lambda^*_{k,\nu}} \right\| \to 0$$
as $\nu \to \infty$.
4. *We have*
$$F(v_\nu) = F(v_\infty) + \frac{2m}{n} Y(\mathbb{S}^n)^{n/2} + o(1),$$
where $o(1) \to 0$ as $\nu \to \infty$.

Proof. This proposition follows from Propositions 3.1, and the compactness result of Brézis-Coron [10] and Struwe [39]. More precisely, the proposition except item 2 follows from Proposition 2.1 in Struwe [39]. By Proposition 3.2, $d(x^*_{k,\nu}) \ge \delta_0/2$ with the same $\delta_0/2 > 0$. Namely, the energy cannot concentrate at a fixed neighborhood of the boundary. By Theorem 2 in Brézis-Coron [10] or Proposition 4 in Bahri-Coron [3], we have, for all $i \neq j$,

$$\frac{\lambda^*_{i,\nu}}{\lambda^*_{j,\nu}} + \frac{\lambda^*_{j,\nu}}{\lambda^*_{i,\nu}} + \lambda^*_{i,\nu}\lambda^*_{j,\nu}|x^*_{i,\nu} - x^*_{j,\nu}|^2 \to \infty,$$

and for all k and $\lambda^*_{k,\nu} \to \infty$ as $\nu \to \infty$. This is item 2. ☐

A similar result for the harmonic map heat flow was proved by Qing-Tian [34]. The correction term $h_{a,\lambda}$ in (30) is small and can be controlled.

Lemma 3.4. *Let $\xi_{a,\lambda}$ and $h_{a,\lambda}$ be defined as in* (30)*. Suppose $a \in \Omega$ with $d(a) > \delta > 0$ and $\lambda > 1$. Then we have, for $x \in \Omega$,*

$$|h_{a,\lambda}(x)| + |\nabla_a h_{a,\lambda}(x)| + \lambda|\nabla_\lambda h_{a,\lambda}(x)| \le C(n,\Omega,\delta)\lambda^{-\frac{n-2}{2}},$$
$$\nabla_a \xi_{a,\lambda}(x) = (n-2)\xi_{a,\lambda}\frac{\lambda^2(x-a)}{1+\lambda^2|x-a|^2} + O(\lambda^{-\frac{n-2}{2}}),$$

and

$$\nabla_\lambda \xi_{a,\lambda}(x) = \frac{(n-2)}{2\lambda}\xi_{a,\lambda}\frac{1-\lambda^2|x-a|^2}{1+\lambda^2|x-a|^2} + O(\lambda^{-\frac{n}{2}}),$$

where $|O(\lambda^{-\frac{n-2}{2}})| \le C\lambda^{-\frac{n-2}{2}}$ for some C depending only on n, Ω and δ.

Proof. Since $\Delta h_{a,\lambda}(x) = 0$ in Ω and $h_{a,\lambda} = \bar{\xi}_{a,\lambda}$ on $\partial\Omega$, the estimate of $h_{a,\lambda}$ follows from the Poisson formula for the Laplace equation. Then,

$$\begin{aligned}\nabla_a \xi_{a,\lambda}(x) &= \nabla_a \bar{\xi}_{a,\lambda}(x) - \nabla_a h_{a,\lambda}\\ &= (n-2)\bar{\xi}_{a,\lambda}\frac{\lambda^2(x-a)}{1+\lambda^2|x-a|^2} + O(\lambda^{-\frac{n-2}{2}})\\ &= (n-2)\xi_{a,\lambda}\frac{\lambda^2(x-a)}{1+\lambda^2|x-a|^2} + O(\lambda^{-\frac{n-2}{2}}).\end{aligned}$$

The estimate $\nabla_\lambda \xi_{a,\lambda}(x)$ can be obtained similarly. □

4. Refined blow up analysis

We continue from Proposition 3.3. By the strong maximum principle, the non-negative limit v_∞ either is positive in Ω or identically equals to zero. We will treat these two cases separably in two subsections. We will adapt the refined blow up analysis in Brendle [9] to the equation (9).

4.1. **The case $v_\infty \equiv 0$.** First, we shall project v_ν to an $m(n+2)$-dimensional surface in $H_0^1(\Omega)$ generated by m-bubbles. For every ν, let $\mathcal{A}_\nu$ be the closed set of all m-tuplets $(x_k, \lambda_k, \alpha_k)_{1\le k\le m}$ satisfying $(x_k, \lambda_k, \alpha_k) \in \overline{B}_{\frac{1}{\lambda^*_{k,\nu}}}(x^*_{k,\nu}) \times [\frac{\lambda^*_{k,\nu}}{2}, \frac{3\lambda^*_{k,\nu}}{2}] \times [\frac{1}{2}, \frac{3}{2}]$. Choose an m-tuplet $(x_{k,\nu}, \lambda_{k,\nu}, \alpha_{k,\nu})_{1\le k\le m} \in \mathcal{A}_\nu$ such that

$$\left\| v_\nu - \sum_{k=1}^m \alpha_{k,\nu}\xi_{x_{k,\nu},\lambda_{k,\nu}} \right\| = \inf_{(x_k,\lambda_k,\alpha_k)_{1\le k\le m}\in\mathcal{A}_\nu} \left\| v_\nu - \sum_{k=1}^m \alpha_k \xi_{x_k,\lambda_k} \right\|. \tag{31}$$

By Proposition 3.3, Proposition 3.2 and the definition of $(x_{k,\nu}, \lambda_{k,\nu}, \alpha_{k,\nu})_{1\le k\le m}$, we have, for all $i \ne j$,

$$\frac{\lambda_{i,\nu}}{\lambda_{j,\nu}} + \frac{\lambda_{j,\nu}}{\lambda_{i,\nu}} + \lambda_{i,\nu}\lambda_{j,\nu}|x_{i,\nu} - x_{j,\nu}|^2 \to \infty, \tag{32}$$

and for all k

$$\lambda_{k,\nu} d(x_{k,\nu}) \to \infty \tag{33}$$

as $\nu \to \infty$. In addition, $d(x_{k,\nu}) > \delta_0/2$ with same δ_0 in Proposition 3.2, and

$$\left\| v_\nu - \sum_{k=1}^m \alpha_k \xi_{x_{k,\nu},\lambda_{k,\nu}} \right\| \to 0 \tag{34}$$

as $\nu \to \infty$.

By the triangle inequality,

$$\begin{aligned}&\Big\|\sum_{k=1}^m \alpha_k \xi_{x_{k,\nu},\lambda_{k,\nu}} - \sum_{k=1}^m \xi_{x^*_{k,\nu},\lambda^*_{k,\nu}}\Big\|\\ &\le \Big\|v_\nu - \sum_{k=1}^m \alpha_k \xi_{x_{k,\nu},\lambda_{k,\nu}}\Big\| + \Big\|v_\nu - \sum_{k=1}^m \xi_{x^*_{k,\nu},\lambda^*_{k,\nu}}\Big\| = o(1).\end{aligned}$$

It follows that, for all $1 \le k \le m$,

$$|x_{k,\nu} - x^*_{k,\nu}| = o(1)\frac{1}{\lambda^*_{k,\nu}}, \quad \frac{\lambda_{k,\nu}}{\lambda^*_{k,\nu}} = 1 + o(1), \quad \alpha_{k,\nu} = 1 + o(1). \tag{35}$$

In particular, $(x_{k,\nu}, \lambda_{k,\nu}, \alpha_{k,\nu})_{1\le k\le m}$ is an interior point of $\mathcal{A}_\nu$.

In the sequel, we assume

$$\lambda_{1,\nu} \ge \lambda_{2,\nu} \ge \cdots \ge \lambda_{m,\nu}. \tag{36}$$

Let

$$U_\nu = \sum_{k=1}^m \alpha_{k,\nu}\xi_{x_{k,\nu},\lambda_{k,\nu}}, \quad w_\nu = v_\nu - U_\nu. \tag{37}$$

Next, we shall estimate the orthogonal part w_ν of the above projection.

Lemma 4.1. *We have for $1 \le k \le m$,*

$$\begin{aligned}&\Big|\int_\Omega \xi^{\frac{n+2}{n-2}}_{x_{k,\nu},\lambda_{k,\nu}} w_\nu \,\mathrm{d}x\Big| + \Big|\int_\Omega \xi^{\frac{n+2}{n-2}}_{x_{k,\nu},\lambda_{k,\nu}} \frac{1-\lambda^2|x-x_{k,\nu}|^2}{1+\lambda^2|x-x_{k,\nu}|^2} w_\nu \,\mathrm{d}x\Big|\\ &\quad + \Big|\int_\Omega \xi^{\frac{n+2}{n-2}}_{x_{k,\nu},\lambda_{k,\nu}} \frac{\lambda^2(x-x_{k,\nu})}{1+\lambda^2|x-x_{k,\nu}|^2} w_\nu \,\mathrm{d}x\Big| \le o(1)\Big(\int_\Omega |w_\nu|^{\frac{2n}{n-2}}\,\mathrm{d}x\Big)^{\frac{n-2}{2n}}.\end{aligned}$$

Proof. By the finite dimensional variational problem (31) and (35), taking derivatives in $\mathcal{A}_\nu$, we have

$$\int_\Omega \Big[\nabla(\nabla_{a,\lambda}\xi_{x_{k,\nu},\lambda_{k,\nu}})\nabla w_\nu - b\nabla_{a,\lambda}\xi_{x_{k,\nu},\lambda_{k,\nu}} w_\nu\Big]\,\mathrm{d}x = 0$$

and

$$\int_\Omega \Big[\nabla\xi_{x_{k,\nu},\lambda_{k,\nu}}\nabla w_\nu - b\xi_{x_{k,\nu},\lambda_{k,\nu}} w_\nu\Big]\,\mathrm{d}x = 0,$$

where $\nabla_{a,\lambda}\xi_{x_{k,\nu},\lambda_{k,\nu}} = \nabla_{a,\lambda}\xi_{a,\lambda}\Big|_{(a,\lambda)=(x_{k,\nu},\lambda_{k,\nu})}$. Integrating by parts, using the equation of $\bar\xi_{a,\lambda}$, Hölder's inequality and Lemma 3.4, the lemma follows. □

Note that the bubbles are non-degenerate, since we have the following well known lemma (see (3.14) in Rey [33]).

Lemma 4.2. *Let $\bar\xi_{a,\lambda}$ be defined in* (29). *Then there exists a constant $c_1 > 0$ depending only on n such that*

$$(1-c_1)\int_{\mathbb{R}^n} |\nabla\varphi|^2 \ge \frac{n+2}{n-2}\int_{\mathbb{R}^n} \bar\xi_{0,1}^{\frac{4}{n-2}}\varphi^2$$

for any $\varphi \in H^1_0(\mathbb{R}^n)$ satisfying

$$\int_{\mathbb{R}^n} \bar\xi_{0,1}^{\frac{4}{n-2}}(\nabla_{a,\lambda}\bar\xi_{0,1})\varphi\,\mathrm{d}x = 0.$$

We have the following non-degeneracy estimates of the second variation of F for w_ν.

Lemma 4.3. *For large ν, we have*

$$\frac{n+2}{n-2}\int_\Omega \sum_{k=1}^m \xi_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{4}{n-2}} w_\nu^2 \le (1-c)\int_\Omega (|\nabla w_\nu|^2 - b w_\nu^2)\,\mathrm{d}x,$$

where $c>0$ is independent of ν.

Proof. We assume w_ν is not zero, otherwise there is nothing to prove. Define $\tilde w_\nu = \frac{w_\nu}{\|w_\nu\|}$. Suppose the lemma is not true. Then we can find a subsequence of $\{\tilde w_\nu\}$ (still denoted by $\{\tilde w_\nu\}$) satisfying

$$\lim_{\nu\to\infty}\frac{n+2}{n-2}\int_\Omega \sum_{k=1}^m \xi_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{4}{n-2}} \tilde w_\nu^2 \ge 1. \tag{38}$$

By (14),

$$\int_\Omega |\tilde w_\nu|^{\frac{2n}{n-2}} \le K_b^{\frac{n}{n-2}}\|\tilde w_\nu\| = K_b^{\frac{n}{n-2}}. \tag{39}$$

By (32) and (36), we can find $R_\nu\to\infty$, $R_\nu\lambda_{j,\nu}^{-1}\to 0$ for all $1\le j\le m$, and

$$\frac{\lambda_{i,\nu}}{R_\nu}(\lambda_{j,\nu}^{-1} + |x_{i,\nu}-x_{j,\nu}|)\to\infty \tag{40}$$

for all $i<j$. Set

$$\Omega_{j,\nu} = B_{R_\nu\lambda_{j,\nu}^{-1}}(x_{j,\nu})\setminus \bigcup_{i=1}^{j-1} B_{R_\nu\lambda_{i,\nu}^{-1}}(x_{i,\nu}).$$

By (38) and $\|\tilde w_\nu\|=1$, we can find $1\le j\le m$ such that

$$\lim_{\nu\to\infty}\int_\Omega \xi_{x_{j,\nu},\lambda_{j,\nu}}^{\frac{4}{n-2}}\tilde w_\nu^2 > 0$$

and

$$\lim_{\nu\to\infty}\int_{\Omega_{j,\nu}} (|\nabla\tilde w_\nu|^2 - b\tilde w_\nu^2) \le \lim_{\nu\to\infty}\frac{n+2}{n-2}\int_\Omega \xi_{x_{j,\nu},\lambda_{j,\nu}}^{\frac{4}{n-2}}\tilde w_\nu^2.$$

Let $\hat w_\nu(x) = \lambda_{j,\nu}^{-\frac{n-2}{2}}\tilde w_\nu(x_{j,\nu}+\lambda_{j,\nu}^{-1}x)$. Under this scaling, by using (40), we know that either $B_{R_\nu\lambda_{j,\nu}^{-1}}(x_{j,\nu})$ will be disjoint with $\bigcup_{i=1}^{j-1} B_{R_\nu\lambda_{i,\nu}^{-1}}(x_{i,\nu})$ or $B_{R_\nu\lambda_{i,\nu}^{-1}}(x_{i,\nu})$ will shrink to a point for every $1\le i\le j-1$. By passing to a weak limit in $H^1_{loc}(\mathbb{R}^n)$, and using the above two inequalities and Lemma 4.1, we then obtain a contradiction to Lemma 4.2.

Therefore, Lemma 4.3 is proved. □

Corollary 4.4. *For large ν, we have*

$$\frac{n+2}{n-2}\int_\Omega U_\nu^{\frac{4}{n-2}} w_\nu^2 \le (1-c)\int_\Omega (|\nabla w_\nu|^2 - b w_\nu^2)\,\mathrm{d}x,$$

where $c>0$ is independent of ν.

Proof. It follows from Lemma 4.3, Hölder's inequality, the Sobolev inequality (14) and the fact that

$$\int_\Omega \Big| U_\nu^{\frac{4}{n-2}} - \sum_{k=1}^m \xi_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{4}{n-2}} \Big|^{\frac{n}{2}} = o(1).$$

□

Now we can have an expansion of the Hamitonian F defined in (16).

Proposition 4.5. *When $n \geq 4$ and ν is sufficiently large, we have*

$$F(U_\nu) \leq \sum_{k=1}^m F(\xi_{x_{k,\nu},\lambda_{k,\nu}}) + o(1) \sum_{k=1}^m \int_\Omega \xi_{x_{k,\nu},\lambda_{k,\nu}}^2 + C \sum_{k=1}^m \lambda_{k,\nu}^{2-n}.$$

Proof. We shall need the following inequality

$$\left(\sum_{k=1}^m a_k\right)^{\frac{2n}{n-2}} \geq \sum_{k=1}^m a_k^{\frac{2n}{n-2}} + \frac{2n}{n-2} \sum_{k<l} a_k^{\frac{n+2}{n-2}} a_l + c_{n,m} \sum_{k<l} (a_k \vee a_l)^{\frac{4}{n-2}} (a_k \wedge a_l)^2 \tag{41}$$

for any $a_1, \ldots, a_m \geq 0$, where $c_{n,m} > 0$ is a constant, and $a_k \vee a_l = \max(a_k, a_l)$ and $a_k \wedge a_l = \min(a_k, a_l)$. This inequality can be proved using Lemma A.1 and induction.

Using the inequality (41), we have

$$\begin{aligned}
&\int_\Omega (|\nabla U_\nu|^2 - b U_\nu^2) - \frac{n-2}{n} \int_\Omega U_\nu^{\frac{2n}{n-2}} \\
&\leq \sum_k \alpha_{k,\nu}^2 \int_\Omega (|\nabla \xi_{x_{k,\nu},\lambda_{k,\nu}}|^2 - b \xi_{x_{k,\nu},\lambda_{k,\nu}}^2) - \sum_k \alpha_{k,\nu}^{\frac{2n}{n-2}} \frac{n-2}{n} \int_\Omega \xi_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{2n}{n-2}} \\
&\quad + 2 \sum_{i<j} \alpha_{j,\nu} \Big[\alpha_{i,\nu} \int_\Omega (\nabla \xi_{x_{i,\nu},\lambda_{i,\nu}} \nabla \xi_{x_{j,\nu},\lambda_{j,\nu}} - b \xi_{x_{i,\nu},\lambda_{i,\nu}} \xi_{x_{j,\nu},\lambda_{j,\nu}}) \\
&\qquad - \alpha_{i,\nu}^{\frac{n+2}{n-2}} \int_\Omega \xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}} \xi_{x_{j,\nu},\lambda_{j,\nu}} \Big] \\
&\quad - c_{n,m} \sum_{i<j} \int_\Omega (\xi_{x_{i,\nu},\lambda_{i,\nu}} \vee \xi_{x_{j,\nu},\lambda_{j,\nu}})^{\frac{4}{n-2}} (\xi_{x_{i,\nu},\lambda_{i,\nu}} \wedge \xi_{x_{j,\nu},\lambda_{j,\nu}})^2.
\end{aligned} \tag{42}$$

By the equation of $\bar{\xi}_{x_{k,\nu},\lambda_{k,\nu}}$ and the definition of $\xi_{x_{k,\nu},\lambda_{k,\nu}}$, we have

$$\begin{aligned}
&\alpha_{k,\nu}^2 \int_\Omega (|\nabla \xi_{x_{k,\nu},\lambda_{k,\nu}}|^2 - b \xi_{x_{k,\nu},\lambda_{k,\nu}}^2) - \alpha_{k,\nu}^{\frac{2n}{n-2}} \frac{n-2}{n} \int_\Omega \xi_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{2n}{n-2}} \\
&\leq \alpha_{k,\nu}^2 \int_{\mathbb{R}^n} |\nabla \bar{\xi}_{x_{k,\nu},\lambda_{k,\nu}}|^2 - \alpha_{k,\nu}^{\frac{2n}{n-2}} \frac{n-2}{n} \int_{\mathbb{R}^n} \bar{\xi}_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{2n}{n-2}} - b \alpha_{k,\nu}^2 \int_\Omega \xi_{x_{k,\nu},\lambda_{k,\nu}}^2 + C \lambda_{k,\nu}^{2-n} \\
&\leq \int_{\mathbb{R}^n} |\nabla \bar{\xi}_{x_{k,\nu},\lambda_{k,\nu}}|^2 - \frac{n-2}{n} \int_{\mathbb{R}^n} \bar{\xi}_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{2n}{n-2}} - \frac{4}{n-2} (\alpha_{k,\nu} - 1)^2 Y(\mathbb{S}^n)^{\frac{n}{2}} \\
&\quad - b \alpha_{k,\nu}^2 \int_\Omega \xi_{x_{k,\nu},\lambda_{k,\nu}}^2 + C \lambda_{k,\nu}^{2-n} \\
&\leq F(\xi_{x_{k,\nu},\lambda_{k,\nu}}) - \frac{4}{n-2} (\alpha_{k,\nu} - 1)^2 Y(\mathbb{S}^n)^{\frac{n}{2}} + o(1) \int_\Omega \xi_{x_{k,\nu},\lambda_{k,\nu}}^2 + C \lambda_{k,\nu}^{2-n},
\end{aligned} \tag{43}$$

where we used $\alpha_{k,\nu}=1+o(1)$, $\alpha_{k,\nu}^2-\frac{n-2}{n}\alpha_{k,\nu}^{\frac{2n}{n-2}}\le\frac{2}{n}-\frac{4}{n-2}(\alpha_{k,\nu}-1)^2$ and

$$\int_{\mathbb{R}^n}|\nabla\bar\xi_{x_{k,\nu},\lambda_{k,\nu}}|^2=\int_{\mathbb{R}^n}\bar\xi_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{2n}{n-2}}=Y(\mathbb{S}^n)^{\frac{n}{2}}.$$

In addition,

$$\begin{aligned}
&\alpha_{i,\nu}\int_\Omega(\nabla\xi_{x_{i,\nu},\lambda_{i,\nu}}\nabla\xi_{x_{j,\nu},\lambda_{j,\nu}}-b\xi_{x_{i,\nu},\lambda_{i,\nu}}\xi_{x_{j,\nu},\lambda_{j,\nu}})-\alpha_{i,\nu}^{\frac{n+2}{n-2}}\int_\Omega\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}\xi_{x_{j,\nu},\lambda_{j,\nu}}\\
&=\alpha_{i,\nu}\int_\Omega(-\Delta\xi_{x_{i,\nu},\lambda_{i,\nu}}-b\xi_{x_{i,\nu},\lambda_{i,\nu}}-\alpha_{i,\nu}^{\frac{4}{n-2}}\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}})\xi_{x_{j,\nu},\lambda_{j,\nu}}\\
&\le C|\alpha_{i,\nu}-1|\int_\Omega\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}\xi_{x_{j,\nu},\lambda_{j,\nu}}+\int_\Omega\Big(\bar\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}-\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}\Big)\xi_{x_{j,\nu},\lambda_{j,\nu}}\\
&\le C|\alpha_{i,\nu}-1|\int_\Omega\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}\xi_{x_{j,\nu},\lambda_{j,\nu}}+O(\lambda_{i,\nu}^{2-n}+\lambda_{j,\nu}^{2-n})
\end{aligned}$$

(44)

$$\le\frac{2}{n-2}(\alpha_{k,\nu}-1)^2Y(\mathbb{S}^n)^{\frac{n}{2}}+C\Big(\int_\Omega\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}\xi_{x_{j,\nu},\lambda_{j,\nu}}\Big)^2+O(\lambda_{i,\nu}^{2-n}+\lambda_{j,\nu}^{2-n}),$$

where $C>0$ is independent of ν, and in the second inequality we used

$$\begin{aligned}
\int_\Omega\Big(\bar\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}-\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}\Big)\xi_{x_{j,\nu},\lambda_{j,\nu}}&\le C\int_\Omega\bar\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{4}{n-2}}|h_{x_{i,\nu},\lambda_{i,\nu}}|\bar\xi_{x_{j,\nu},\lambda_{j,\nu}}\\
&\le C\lambda_{i,\nu}^{\frac{2-n}{2}}\Big(\int_\Omega\bar\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}\Big)^{\frac{4}{n+2}}\Big(\int_\Omega\bar\xi_{x_{j,\nu},\lambda_{j,\nu}}^{\frac{n+2}{n-2}}\Big)^{\frac{n-2}{n+2}}\\
&\le C\lambda_{i,\nu}^{\frac{2-n}{2}}\lambda_{i,\nu}^{\frac{2-n}{2}\frac{4}{n+2}}\lambda_{j,\nu}^{\frac{2-n}{2}\frac{n-2}{n+2}}\\
&\le C\lambda_{i,\nu}^{\frac{2-n}{2}}(\lambda_{i,\nu}^{\frac{2-n}{2}}+\lambda_{j,\nu}^{\frac{2-n}{2}})\\
&\le C(\lambda_{i,\nu}^{2-n}+\lambda_{j,\nu}^{2-n}).
\end{aligned}$$

Combining (42), (43) and (44), we have

$$\begin{aligned}
F(U_\nu)\le\sum_{k=1}^mF(\xi_{x_{k,\nu},\lambda_{k,\nu}})&+o(1)\sum_{k=1}^m\int_\Omega\xi_{x_{k,\nu},\lambda_{k,\nu}}^2+C\sum_{k=1}^m\lambda_{k,\nu}^{2-n}\\
&+\sum_{i<j}\Big[C\Big(\int_\Omega\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}\xi_{x_{j,\nu},\lambda_{j,\nu}}\Big)^2\\
&\quad-c_{n,m}\int_\Omega(\xi_{x_{i,\nu},\lambda_{i,\nu}}\vee\xi_{x_{j,\nu},\lambda_{j,\nu}})^{\frac{4}{n-2}}(\xi_{x_{i,\nu},\lambda_{i,\nu}}\wedge\xi_{x_{j,\nu},\lambda_{j,\nu}})^2\Big].
\end{aligned}$$

Meanwhile, we have

$$\begin{aligned}
&C\Big(\int_\Omega\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}\xi_{x_{j,\nu},\lambda_{j,\nu}}\Big)^2-c_{n,m}\int_\Omega(\xi_{x_{i,\nu},\lambda_{i,\nu}}\vee\xi_{x_{j,\nu},\lambda_{j,\nu}})^{\frac{4}{n-2}}(\xi_{x_{i,\nu},\lambda_{i,\nu}}\wedge\xi_{x_{j,\nu},\lambda_{j,\nu}})^2\\
&\le C\Big(\int_{\mathbb{R}^n}\bar\xi_{x_{i,\nu},\lambda_{i,\nu}}^{\frac{n+2}{n-2}}\bar\xi_{x_{j,\nu},\lambda_{j,\nu}}\Big)^2\\
&\quad-c_{n,m}\int_{\mathbb{R}^n}(\bar\xi_{x_{i,\nu},\lambda_{i,\nu}}\vee\bar\xi_{x_{j,\nu},\lambda_{j,\nu}})^{\frac{4}{n-2}}(\bar\xi_{x_{i,\nu},\lambda_{i,\nu}}\wedge\bar\xi_{x_{j,\nu},\lambda_{j,\nu}})^2\\
&\quad+C(\lambda_{i,\nu}^{2-n}+\lambda_{j,\nu}^{2-n})\\
&\le C(\lambda_{i,\nu}^{2-n}+\lambda_{j,\nu}^{2-n})\quad\text{for all large }\nu,
\end{aligned}$$

where we used (78) in the last inequality.

Therefore, the proof is completed. □

Corollary 4.6. *If $n \geq 4$ and $b > 0$ satisfying* (2), *we have, for large ν,*

$$F(U_\nu) \leq \frac{2m}{n} Y(\mathbb{S}^n)^{\frac{n}{2}}.$$

Proof. By Proposition 3.2 and Lemma 3.4,

$$\begin{aligned} F(\xi_{x_{k,\nu},\lambda_{k,\nu}}) &= \int_\Omega \left(|\nabla \xi_{x_{k,\nu},\lambda_{k,\nu}}|^2 - \frac{n-2}{n} \xi_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{2n}{n-2}} \right) \mathrm{d}x - b \int_\Omega \xi_{x_{k,\nu},\lambda_{k,\nu}}^2 \,\mathrm{d}x \\ &\leq \frac{2}{n} Y(\mathbb{S}^n)^{\frac{n}{2}} + C\lambda_{k,\nu}^{2-n} - b \int_\Omega \xi_{x_{k,\nu},\lambda_{k,\nu}}^2 \,\mathrm{d}x, \end{aligned}$$

where $C > 0$ is independent of ν. Note that

$$\int_\Omega \xi_{x_{j,\nu},\lambda_{j,\nu}}^2 \geq \begin{cases} \frac{1}{C} \lambda_{j,\nu}^{-\frac{4}{n-2}}, & \text{if } n \neq 4, \\ \frac{1}{C} \lambda_{j,\nu}^{-2} \ln \lambda_{j,\nu}, & \text{if } n = 4. \end{cases}$$

Hence, if $n \geq 4$ and $b > 0$, for any large constant N we can find $j_N > 0$ such that for all $j \geq j_N$ there holds $b \int_\Omega \xi_{x_{j,\nu},\lambda_{j,\nu}}^2 \geq N \lambda_{j,\nu}^{2-n}$. The corollary follows immediately from Proposition 4.5. □

4.2. **The case $v_\infty > 0$.** In this case, we shall also project v_ν to a finite-dimensional surface in $H_0^1(\Omega)$ generated by v_∞ and m-bubbles. In order to understand the new contribution from v_∞, we need to perform spectral analysis of the linearized operator at v_∞ as Brendle [9] did for the Yamabe flow on compact manifolds. Our current $H_0^1(\Omega)$ setting is more close to that in Section 2.1 of Bonforte-Figalli [7]. Indeed, the analysis of [7] applies here with little change and the election of L below is the same as k_p in [7].

Let $\mathcal{L}^2(\Omega) := \left\{ f : \int_\Omega f^2 v_\infty^{\frac{4}{n-2}} < \infty \right\}$ be the Hilbert space with the inner product $\langle f, g \rangle = \int_\Omega f g v_\infty^{\frac{4}{n-2}} \,\mathrm{d}x$. Then the operator

$$f \longmapsto \left[v_\infty^{-\frac{4}{n-2}} (-\Delta - b) \right]^{-1} f$$

is a bounded linear compact symmetric operator mapping $\mathcal{L}^2(\Omega)$ into itself. Using the spectral theorem, there exists a sequence of $H_0^1(\Omega)$ functions $\{\phi_l : l \in \mathbb{N}\}$ and a sequence of positive real numbers $\{\mu_l : l \in \mathbb{N}\}$ such that $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \to \infty$,

$$-\Delta \phi_l - b\phi_l = \mu_l v_\infty^{\frac{4}{n-2}} \phi_l \quad \text{in } \Omega, \quad \phi_l = 0 \quad \text{on } \partial\Omega,$$

and $\{\phi_l : l \in \mathbb{N}\}$ forms an orthonormal basis of $\mathcal{L}^2(\Omega)$. In particular,

$$\int_\Omega v_\infty^{\frac{4}{n-2}} \phi_i \phi_j = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

By the regularity theory of linear elliptic equations, $\phi_l \in C^{2+\frac{4}{n-2}}(\bar{\Omega}) \cap C^\infty(\Omega)$ for every l. By the equation of v_∞ and the positivity of v_∞, we know that $\mu_1 = 1$ and $\phi_1 = v_\infty (\int_\Omega v_\infty^{\frac{2n}{n-2}})^{-1/2}$. It is easy to check that $\{\frac{1}{\sqrt{\mu_l}} \phi_l\}$ is also an orthonormal basis of $H_0^1(\Omega)$ with respect to the inner product (13).

Let L be the largest number such that

$$\mu_l \le \frac{n+2}{n-2} \quad \text{for all } l \le L.$$

For $f \in L^p(\Omega)$, $p \ge 1$, we denote by Π the projection operator

$$\Pi f = f - \sum_{i=1}^{L} \left(\int_\Omega f \phi_i \,\mathrm{d}x \right) v_\infty^{\frac{4}{n-2}} \phi_i.$$

It is clear that $\Pi(L^p(\Omega)) = \{f \in L^p(\mathbb{R}^n) : \int_\Omega f\phi_i = 0,\ i = 1, 2, \cdots, L\}$. Hence, $\Pi(L^p(\Omega))$ is a closed subspace of $L^p(\Omega)$, and thus, is a Banach space with the inherited L^p norm.

We have several estimates regarding this projection.

Lemma 4.7. *For every $1 \le p < \infty$, we can find a constant C depending only on n, b, Ω, p and v_∞ such that*

$$\|f\|_{L^p(\Omega)} \le C \Big\| \Delta f + bf + \frac{n+2}{n-2} v_\infty^{\frac{4}{n-2}} f \Big\|_{L^p(\Omega)} + C \sup_{1 \le l \le L} \Big| \int_\Omega v_\infty^{\frac{4}{n-2}} \phi_l f \Big|$$

for all $f \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Proof. Suppose that this is not true. Then there exists a sequence of functions $f_k \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $\|f_k\|_{L^p(\Omega)} = 1$ for all k, and

$$\lim_{k\to\infty} \Big\| \Delta f_k + bf_k + \frac{n+2}{n-2} v_\infty^{\frac{4}{n-2}} f_k \Big\|_{L^p(\Omega)} + \lim_{k\to\infty} \sup_{1 \le l \le L} \Big| \int_\Omega v_\infty^{\frac{4}{n-2}} \phi_l f_k \Big| = 0.$$

If $p > 1$, then by the $W^{2,p}$ estimates, we have $\|f_k\|_{W^{2,p}(\Omega)} \le C$. If $p = 1$, by the estimates of Brézis-Strauss [13], $\|f_k\|_{W^{1,q}(\Omega)} \le C$ for some $q > 1$. Therefore, by the compactness, we obtain an f such that $\|f\|_{L^p(\Omega)} = 1$, $\int_\Omega v_\infty^{\frac{4}{n-2}} \phi_l f_k = 0$ for all $1 \le l \le L$, and

$$\Delta f + bf + \frac{n+2}{n-2} v_\infty^{\frac{4}{n-2}} f = 0$$

in the distribution sense. Multiplying ϕ_l and integrating by parts, we have

$$\left(\mu_l - \frac{n+2}{n-2} \right) \int_\Omega v_\infty^{\frac{4}{n-2}} \phi_l f_k = 0.$$

Hence, $\int_\Omega v_\infty^{\frac{4}{n-2}} \phi_l f_k = 0$ for all $l > L$. Meanwhile, from the elliptic regularity, we know that $f \in L^\infty(\Omega)$. Hence, $f \in \mathcal{L}^2(\Omega)$, and thus, $f \equiv 0$, which is a contradiction. □

Lemma 4.8. *There exists a constant C depending only on n, b, Ω, p and v_∞ such that*

(i)

$$\|f\|_{L^{\frac{n+2}{n-2}}(\Omega)} \le C \Big\| \Pi(\Delta f + bf + \frac{n+2}{n-2} v_\infty^{\frac{4}{n-2}} f) \Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)} + C \sup_{1 \le l \le L} \Big| \int_\Omega v_\infty^{\frac{4}{n-2}} \phi_l f \Big|$$

for all $f \in W^{2,\frac{n(n+2)}{n^2+4}}(\Omega) \cap W_0^{1,\frac{n(n+2)}{n^2+4}}(\Omega)$.

(ii)

$$\|f\|_{L^1(\Omega)} \le C\Big\|\Pi(\Delta f + bf + \frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}} f)\Big\|_{L^1(\Omega)} + C\sup_{1\le l\le L}\Big|\int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l f\Big|$$

for all $f \in W^{2,1}(\Omega)\cap W_0^{1,1}(\Omega)$.

Proof. Given Lemma 4.7, the proof is the same as that of Lemma 6.3 in [9]. We include it for reader's convenience. By the definition of Π, we have for $f \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ that

$$\begin{aligned}&\Delta f + bf + \frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}} f\\ &= \Pi(\Delta f + bf + \frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}} f) + \sum_{i=1}^{L}\left(\frac{n+2}{n-2}-\mu_i\right)\left(\int_\Omega f\phi_i v_\infty^{\frac{4}{n-2}}\,\mathrm{d}x\right)v_\infty^{\frac{4}{n-2}}\phi_i.\end{aligned}$$

Hence,

$$\begin{aligned}&\|\Delta f + bf + \frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}} f\|_{L^p(\Omega)}\\ &\le \|\Pi(\Delta f + bf + \frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}} f)\|_{L^p(\Omega)} + C\sup_{1\le l\le L}\Big|\int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l f\Big|.\end{aligned}$$

The assertion (ii) follows from the above inequality with $p=1$ and Lemma 4.7.

For the assertion (i), by choosing $p = \frac{n(n+2)}{n^2+4}$ in the above inequality and using Lemma 4.7, we have

$$\begin{aligned}&\|\Delta f + bf + \frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}} f\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)}\\ &\le \|\Pi(\Delta f + bf + \frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}} f)\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)} + C\sup_{1\le l\le L}\Big|\int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l f\Big|\end{aligned}$$

and

$$\|f\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)} \le C\Big\|\Delta f + bf + \frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}} f\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)} + C\sup_{1\le l\le L}\Big|\int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l f\Big|.$$

By the $W^{2,p}$ regularity theory for the Laplace equation and the Sobolev embedding $W^{2,\frac{n(n+2)}{n^2+4}} \hookrightarrow L^{\frac{n+2}{n-2}}$, we have

$$\begin{aligned}\|f\|_{L^{\frac{n+2}{n-2}}(\Omega)} &\le C\|f\|_{W^{2,\frac{n(n+2)}{n^2+4}}(\Omega)}\\ &\le C\Big\|\Delta f + bf + \frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}} f\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)} + C\Big\|f\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)}.\end{aligned}$$

Then the assertion (i) is followed by combining these three inequalities. ☐

Lemma 4.9. *There exists* $\delta_1 > 0$ *such that for every* $z = (z_1, \dots, z_L) \in \mathbb{R}^L$ *with* $|z| \le \delta_1$, *there exists* $\xi_z \in C_0^{\frac{3n-2}{n-2}}(\overline{\Omega})$ *satisfying* $1/2 \le \xi_z/v_\infty \le 2$ *in* Ω,

$$\int_\Omega v_\infty^{\frac{4}{n-2}}(\xi_z - v_\infty)\phi_l\,\mathrm{d}x = z_l, \quad l = 1, \dots, L,$$

and

$$\Pi(\Delta\xi_z + b\xi_z + \xi_z^{\frac{n+2}{n-2}}) = 0. \tag{45}$$

Furthermore, the map $z \mapsto \xi_z$ is real analytic and $\frac{\partial}{\partial z_1}\xi_z(0) = v_\infty$, $\frac{\partial}{\partial z_l}\xi_z(0) = \phi_l$ for $2 \le l \le L$.

Proof. Let $\xi_z = (1+z_1)v_\infty + \sum_{l=2}^{L} z_l\phi_l + h$, where

$$h \in \mathcal{H} := span\{\phi_1, \dots, \phi_L\}^\perp$$

and the "$\perp$" is with respect to the inner product (13). By a direct computation,

$$\begin{aligned}
&\Pi(\Delta\xi_z + b\xi_z + \xi_z^{\frac{n+2}{n-2}}) \\
&= (\Delta+b)\xi_z - \sum_{l=1}^{L}\Big(\int_\Omega (\Delta+b)\xi_z\phi_l\Big)v_\infty^{\frac{4}{n-2}}\phi_l + \xi_z^{\frac{n+2}{n-2}} - \sum_{l=1}^{L}\Big(\int_\Omega \xi_z^{\frac{n+2}{n-2}}\phi_l\Big)v_\infty^{\frac{4}{n-2}}\phi_l \\
&= (\Delta+b)h + \xi_z^{\frac{n+2}{n-2}} - \sum_{l=1}^{L}\Big(\int_\Omega \xi_z^{\frac{n+2}{n-2}}\phi_l\Big)v_\infty^{\frac{4}{n-2}}\phi_l =: G(z,h).
\end{aligned}$$

For any $p > n$, we claim that there exists a small constant $\delta > 0$ such that

$$G : \{|z| < \delta\} \times \{\|h\|_{W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)} < \delta\} \to \Pi(L^p(\Omega))$$

is analytic. Indeed, let $\Phi(z,h) = \xi_z$, $\mathcal{L}u = \Delta u + bu + u^{\frac{n+2}{n-2}}$. Then we have $G = \Pi \circ \mathcal{L} \circ \Phi$. Obviously, the linear maps Φ and Π are analytic. By Lemma 5.3 of Feireisl-Simondon [26], $\mathcal{L}$ is also analytic in some small neighborhood of v_∞ in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Note that $G(0,0) = 0$ and

$$G_h(0,0)\varphi = (\Delta+b)\varphi + \frac{n+2}{n-2}\left(v_\infty^{\frac{4}{n-2}}\varphi - \sum_{l=1}^{L}\Big(\int_\Omega v_\infty^{\frac{4}{n-2}}\varphi\phi_l\Big)v_\infty^{\frac{4}{n-2}}\phi_l\right).$$

Since $G_h(0,0)$ is coercive on $H_0^1(\Omega)\cap\mathcal{H}$, then $G_h(0,0) : W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\cap\mathcal{H} \to \Pi(L^p(\Omega))$ is invertible, and both $G_h(0,0)$ and $(G_h(0,0))^{-1}$ are continuous. By the Implicit Function Theorem we can find $h(z) \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\cap\mathcal{H}$ such that $G(z,h(z)) = 0$ and h is analytic in z, see, e.g., Section 3.3B of Berger [4]. The regularity of $h(z)(\cdot)$ follows from elliptic regularity theory for the linear elliptic equation $G(z,h) = 0$ in Ω and $h = 0$ on $\partial\Omega$. Since $h(0) = 0$, and $0 = G_z(0,0) + G_h(0,0)\partial_z h(0) = G_h(0,0)\partial_z h(0)$, we have $\partial_z h(0) = 0$. It follows that $\frac{\partial}{\partial z_1}\xi_z(0) = v_\infty$ and $\frac{\partial}{\partial z_l}\xi_z(0) = \phi_l$ for $2 \le l \le L$. Therefore, the proof is completed. □

The difference of the energy at ξ_z and v_∞ can be controlled as follows.

Lemma 4.10. *There exists a real number $\gamma \in (0,1)$ depending only on n, b, Ω and v_∞ such that*

$$F(\xi_z) - F(v_\infty) \le 2\sup_{1\le l\le L}\Big|\int_\Omega (\Delta\xi_z + b\xi_z + \xi_z^{\frac{n+2}{n-2}})\phi_l\,\mathrm{d}x\Big|^{1+\gamma}$$

if z is sufficiently small.

Proof. Since $z \mapsto \xi_z$ is real analytic by Lemma 4.9, and $F(\cdot)$ is also real analytic by Lemma 5.3 of [26], then the function $z \mapsto F(\xi_z)$ is real analytic. Using the Łojasiewicz inequality (see Théorème 4 of [30] or Proposition 1 of [31] on page 92), we have

$$|F(\xi_z) - F(v_\infty)| \le \sup_l\Big|\frac{\partial}{\partial z_l}F(\xi_z)\Big|^{1+\gamma}$$

if z is sufficiently small, where $\gamma \in (0,1)$ depends only on n, b, Ω and v_∞, but is not explicit. By a direct computation, we have

$$\frac{\partial}{\partial z_l} F(\xi_z) = -2\int_\Omega (\Delta \xi_z + b\xi_z + \xi_z^{\frac{n+2}{n-2}}) \frac{\partial \xi_z}{\partial z_l}\,\mathrm{d}x = -2\int_\Omega (\Delta \xi_z + b\xi_z + \xi_z^{\frac{n+2}{n-2}})\phi_l\,\mathrm{d}x.$$

Therefore, the proof is completed. □

For every ν, as in the beginning of Section 4.1, let $\mathcal{A}_\nu$ be the closed set of all m-tuplets $(x_k, \lambda_k, \alpha_k)_{1\le k\le m}$ satisfying $(x_k, \lambda_k, \alpha_k) \in \overline{B}_{\frac{1}{\lambda^*_{k,\nu}}}(x^*_{k,\nu}) \times [\frac{\lambda^*_{k,\nu}}{2}, \frac{3\lambda^*_{k,\nu}}{2}] \times [\frac{1}{2}, \frac{3}{2}]$. Let $\delta_1 > 0$ be the constant in Lemma 4.9 and $\overline{B}^L_{\delta_1}$ is the open ball in $\mathbb{R}^L$ centered at origin with radius δ_1. Choose an element $(z_\nu, (x_{k,\nu}, \lambda_{k,\nu}, \alpha_{k,\nu})_{1\le k\le m}) \in \overline{B}^L_{\delta_1} \times \mathcal{A}_\nu$ such that

$$\left\| v_\nu - \xi_{z_\nu} - \sum_{k=0}^m \alpha_{k,\nu}\xi_{x_{k,\nu},\lambda_{k,\nu}} \right\| = \inf_{(z,(x_k,\lambda_k,\alpha_k)_{1\le k\le m})\in \overline{B}^L_{\delta_1}\times\mathcal{A}_\nu} \left\| v_\nu - \xi_z - \sum_{k=0}^m \alpha_k \xi_{x_k,\lambda_k}\right\|. \tag{46}$$

Similar to (32)–(34), we have

$$\frac{\lambda_{i,\nu}}{\lambda_{j,\nu}} + \frac{\lambda_{j,\nu}}{\lambda_{i,\nu}} + \lambda_{i,\nu}\lambda_{j,\nu}|x_{i,\nu} - x_{j,\nu}|^2 \to \infty, \tag{47}$$

and for all k

$$\lambda_{k,\nu} d(x_{k,\nu}) \to \infty \tag{48}$$

as $\nu \to \infty$. In addition, $d(x_{k,\nu}) > \delta/2$, and

$$\left\| v_\nu - \xi_{z_\nu} - \sum_{k=1}^m \alpha_k \xi_{x_{k,\nu},\lambda_{k,\nu}} \right\| \to 0 \tag{49}$$

as $\nu \to \infty$.

By the triangle inequality,

$$\begin{aligned} &\left\| \xi_{z_\nu} - v_\infty + \sum_{k=1}^m \alpha_k \xi_{x_{k,\nu},\lambda_{k,\nu}} - \sum_{k=1}^m \xi_{x^*_{k,\nu},\lambda^*_{k,\nu}} \right\| \\ &\le \left\| v_\nu - \xi_{z_\nu} - \sum_{k=1}^m \alpha_k \xi_{x_{k,\nu},\lambda_{k,\nu}} \right\| + \left\| v_\nu - v_\infty - \sum_{k=1}^m \xi_{x^*_{k,\nu},\lambda^*_{k,\nu}} \right\| = o(1). \end{aligned}$$

It follows that, for all $1 \le k \le m$,

$$|z_\nu| = o(1), \quad |x_{k,\nu} - x^*_{k,\nu}| = o(1)\frac{1}{\lambda^*_{k,\nu}}, \quad \frac{\lambda_{k,\nu}}{\lambda^*_{k,\nu}} = 1 + o(1), \quad \alpha_{k,\nu} = 1 + o(1). \tag{50}$$

In particular, $(z_\nu, (x_{k,\nu}, \lambda_{k,\nu}, \alpha_{k,\nu})_{1\le k\le m}) \in \overline{B}^L_{\delta_1} \times \mathcal{A}_\nu$ is an interior point.

In the sequel, we assume

$$\lambda_{1,\nu} \ge \lambda_{2,\nu} \ge \cdots \ge \lambda_{m,\nu}. \tag{51}$$

Let

$$U_\nu = \xi_{z_\nu} + \sum_{k=1}^m \alpha_{k,\nu}\xi_{x_{k,\nu},\lambda_{k,\nu}}, \quad w_\nu = v_\nu - U_\nu. \tag{52}$$

Lemma 4.11. *We have for* $1 \le l \le L$,

$$\Big|\int_\Omega v_\infty^{\frac{4}{n-2}} \phi_l w_\nu \,\mathrm{d}x\Big| \le o(1)\int_\Omega |w_\nu|\,\mathrm{d}x, \tag{53}$$

and for $1 \le k \le m$,

$$\begin{aligned}&\Big|\int_\Omega \xi_{(x_{k,\nu},\lambda_{k,\nu})}^{\frac{n+2}{n-2}} w_\nu \,\mathrm{d}x\Big| + \Big|\int_\Omega \xi_{(x_{k,\nu},\lambda_{k,\nu})}^{\frac{n+2}{n-2}} \frac{1-\lambda^2|x-x_{k,\nu}|^2}{1+\lambda^2|x-x_{k,\nu}|^2} w_\nu \,\mathrm{d}x\Big| \\ &\quad + \Big|\int_\Omega \xi_{(x_{k,\nu},\lambda_{k,\nu})}^{\frac{n+2}{n-2}} \frac{\lambda^2(x-x_{k,\nu})}{1+\lambda^2|x-x_{k,\nu}|^2} w_\nu \,\mathrm{d}x\Big| \le o(1)\Big(\int_\Omega |w_\nu|^{\frac{2n}{n-2}}\,\mathrm{d}x\Big)^{\frac{n-2}{2n}}.\end{aligned} \tag{54}$$

Proof. Let $\tilde\phi_l = \frac{\partial}{\partial z_l}\xi_z$. By (50), we have $\|\tilde\phi_1 - v_\infty\|_{C^2(\Omega)} = o(1)$ and $\|\tilde\phi_l - \phi_l\|_{C^2(\Omega)} = o(1)$ for $l = 2,\dots,L$. By the definition of $(z_\nu,(x_{k,\nu},\lambda_{k,\nu},\alpha_{k,\nu})_{1\le k\le m})$, we have

$$\int \nabla\tilde\phi_l \nabla w_\nu - b\tilde\phi_l w_\nu = 0.$$

Hence,

$$\begin{aligned}\mu_l \int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l w_\nu\,\mathrm{d}x &= \int_\Omega \Big(-\Delta\phi_l - b\phi_l\Big) w_\nu\,\mathrm{d}x \\ &= \int_\Omega \Big(\Delta(\tilde\phi_l - \phi_l) + b(\tilde\phi_l-\phi_l)\Big) w_\nu\,\mathrm{d}x.\end{aligned}$$

Since $\mu_l > 0$, then we can conclude (53). The proof of (54) is the same as that of Lemma 4.1. ∎

Now we can show the non-degeneracy estimates of the second variation of F for w_ν.

Lemma 4.12. *For large* ν, *we have*

$$\frac{n+2}{n-2}\int_\Omega \big(v_\infty^{\frac{4}{n-2}} + \sum_{k=1}^m \xi_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{4}{n-2}}\big) w_\nu^2 \le (1-c)\int_\Omega (|\nabla w_\nu|^2 - b w_\nu^2)\,\mathrm{d}x,$$

where $c > 0$ *is independent of* ν.

Proof. We assume w_ν is not zero, otherwise there is nothing to prove. Define $\tilde w_\nu = \frac{w_\nu}{\|w_\nu\|}$. Suppose the lemma is not true. Then we can find a subsequence of $\{\tilde w_\nu\}$ (still denoted by $\{\tilde w_\nu\}$) satisfying

$$\lim_{\nu\to\infty}\frac{n+2}{n-2}\int_\Omega \big(v_\infty^{\frac{4}{n-2}} + \sum_{k=1}^m \xi_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{4}{n-2}}\big)\tilde w_\nu^2 \ge 1. \tag{55}$$

By (14),

$$\int_\Omega |\tilde w_\nu|^{\frac{2n}{n-2}} \le K_b^{\frac{n}{n-2}}\|\tilde w_\nu\| = K_b^{\frac{n}{n-2}}. \tag{56}$$

By (47) and (51), we can find $R_\nu\to\infty$, $R_\nu\lambda_{j,\nu}^{-1}\to 0$ for all $1\le j\le m$, and

$$\frac{\lambda_{i,\nu}}{R_\nu}(\lambda_{j,\nu}^{-1} + |x_{i,\nu}-x_{j,\nu}|)\to\infty \tag{57}$$

for all $i<j$. Set

$$\Omega_{j,\nu} = B_{R_\nu\lambda_{j,\nu}^{-1}}(x_{j,\nu}) \setminus \bigcup_{i=1}^{j-1} B_{R_\nu\lambda_{i,\nu}^{-1}}(x_{i,\nu}).$$

By (55) and $\|\tilde{w}_\nu\| = 1$, there are two cases:

(i) We can find $1 \le j \le m$ such that

$$\lim_{\nu\to\infty} \int_\Omega \xi_{x_{j,\nu},\lambda_{j,\nu}}^{\frac{4}{n-2}} \tilde{w}_\nu^2 > 0$$

and

$$\lim_{\nu\to\infty} \int_{\Omega_{j,\nu}} (|\nabla \tilde{w}_\nu|^2 - b\tilde{w}_\nu^2) \le \frac{n+2}{n-2} \int_\Omega \xi_{x_{j,\nu},\lambda_{j,\nu}}^{\frac{4}{n-2}} \tilde{w}_\nu^2.$$

(ii)

$$\lim_{\nu\to\infty} \int_\Omega v_\infty^{\frac{4}{n-2}} \tilde{w}_\nu^2 > 0$$

and

$$\lim_{\nu\to\infty} \int_{\Omega\setminus\cup_j \Omega_{j,\nu}} (|\nabla \tilde{w}_\nu|^2 - b\tilde{w}_\nu^2) \le \frac{n+2}{n-2} \int_\Omega v_\infty^{\frac{4}{n-2}} \tilde{w}_\nu^2.$$

In the first case, we can obtain a contradiction similar to that in the proof of Lemma 4.3.

In the latter case, after passing to subsequence we suppose $\tilde{w}_\nu \rightharpoonup \tilde{w}$ in H_0^1 as $\nu \to \infty$. It follows that

$$\int_\Omega v_\infty^{\frac{4}{n-2}} \tilde{w}^2 > 0 \tag{58}$$

and

$$\int_\Omega (|\nabla \tilde{w}|^2 - b\tilde{w}^2) \le \frac{n+2}{n-2} \int_\Omega v_\infty^{\frac{4}{n-2}} \tilde{w}^2. \tag{59}$$

By (53), we further have

$$\int_\Omega v_\infty^{\frac{4}{n-2}} \tilde{w}\phi_l = 0 \quad \text{for } l = 1, \dots, L. \tag{60}$$

Combining (59) and (60), $\tilde{w}$ has to be identically zero, which contradicts (58).

Therefore, Lemma 4.12 is proved. ☐

Corollary 4.13. *For large ν, we have*

$$\frac{n+2}{n-2} \int_\Omega U_\nu^{\frac{4}{n-2}} w_\nu^2 \le (1-c) \int_\Omega (|\nabla w_\nu|^2 - b w_\nu^2)\, \mathrm{d}x,$$

where $c > 0$ is independent of ν.

Proof. It follows from Lemma 4.12, Hölder's inequality, the Sobolev inequality (14) and the fact that

$$\int_\Omega \left| U_\nu^{\frac{4}{n-2}} - v_\infty^{\frac{4}{n-2}} - \sum_{k=1}^m \xi_{x_{k,\nu},\lambda_{k,\nu}}^{\frac{4}{n-2}} \right|^{\frac{n}{2}} = o(1).$$

☐

The following two lemmas are estimates of $v_\nu - \xi_{z_\nu}$ in $L^{\frac{n+2}{n-2}}(\Omega)$ and $L^1(\Omega)$, respectively.

Lemma 4.14. *For large ν, we have*

$$\|v_\nu - \xi_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(\Omega)}^{\frac{n+2}{n-2}} \le C \|v_\nu^{\frac{n+2}{n-2}} (\mathcal{R}(t_\nu) - 1)\|_{L^{\frac{2n}{n+2}}(\Omega)}^{\frac{n+2}{n-2}} + C \sum_{k=1}^m \lambda_{k,\nu}^{\frac{2-n}{2}},$$

where $C > 0$ is independent of ν.

Proof. From (19), we have

$$\Delta v_\nu + b v_\nu + v_\nu^{\frac{n+2}{n-2}} = (1-\mathcal{R}(t_\nu)) v_\nu^{\frac{n+2}{n-2}}.$$

Combining with (45), we obtain

$$\begin{aligned}
&\Pi\Big(\Delta(v_\nu-\xi_{z_\nu}) + b(v_\nu-\xi_{z_\nu}) + \frac{n+2}{n-2} v_\infty^{\frac{4}{n-2}}(v_\nu-\xi_{z_\nu})\Big)\\
&= \Pi\Big((1-\mathcal{R}(t_\nu))v_\nu^{\frac{n+2}{n-2}} - \frac{n+2}{n-2}(\xi_{z_\nu}^{\frac{4}{n-2}} - v_\infty^{\frac{4}{n-2}})(v_\nu-\xi_{z_\nu})\\
&\quad + \xi_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\xi_{z_\nu}^{\frac{4}{n-2}}(v_\nu-\xi_{z_\nu}) - v_\nu^{\frac{n+2}{n-2}}\Big).
\end{aligned} \tag{61}$$

Apply (i) of Lemma 4.8 to $v_\nu - \xi_{z_\nu}$, we obtain

$$\begin{aligned}
&\|v_\nu-\xi_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(\Omega)}\\
&\le C\Big\|(1-\mathcal{R}(t_\nu))v_\nu^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)} + C\Big\|(\xi_{z_\nu}^{\frac{4}{n-2}} - v_\infty^{\frac{4}{n-2}})(v_\nu-\xi_{z_\nu})\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)}\\
&\quad + C\Big\|\xi_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\xi_{z_\nu}^{\frac{4}{n-2}}(v_\nu-\xi_{z_\nu}) - v_\nu^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)}\\
&\quad + C\sup_{1\le l\le L}\Big|\int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l(v_\nu-\xi_{z_\nu})\Big|.
\end{aligned}$$

Using the estimates for all $a, b \ge 0$ that

$$\Big|a^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}a^{\frac{4}{n-2}}(b-a) - b^{\frac{n+2}{n-2}}\Big| \le C a^{\max(0,\frac{4}{n-2}-1)}|b-a|^{\min(\frac{n+2}{n-2},2)} + C|b-a|^{\frac{n+2}{n-2}}, \tag{62}$$

we obtain

$$\begin{aligned}
&\Big\|\xi_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\xi_{z_\nu}^{\frac{4}{n-2}}(v_\nu-\xi_{z_\nu}) - v_\nu^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)}\\
&\le C\Big\||v_\nu-\xi_{z_\nu}|^{\min(\frac{n+2}{n-2},2)} + |v_\nu-\xi_{z_\nu}|^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)}\\
&\le C\Big\||v_\nu-\xi_{z_\nu}|^{\min(\frac{n+2}{n-2},2)} + |v_\nu-\xi_{z_\nu}|^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\cup_{k=1}^m B_{N/\lambda_{k,\nu}}(x_{k,\nu}))}\\
&\quad + C\Big\||v_\nu-\xi_{z_\nu}|^{\min(\frac{n+2}{n-2},2)} + |v_\nu-\xi_{z_\nu}|^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega\setminus\cup_{k=1}^m B_{N/\lambda_{k,\nu}}(x_{k,\nu}))},
\end{aligned}$$

where N is a large real number to be chosen later. Using Hölder's inequality, we have

$$\begin{aligned}
&\Big\||v_\nu-\xi_{z_\nu}|^{\min(\frac{n+2}{n-2},2)} + |v_\nu-\xi_{z_\nu}|^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{n(n+2)}{n^2+4}}(\cup_{k=1}^m B_{N/\lambda_{k,\nu}}(x_{k,\nu}))}\\
&\le C\sum_{k=1}^m (N/\lambda_{k,\nu})^{\frac{(n-2)^2}{2(n+2)}}\Big\||v_\nu-\xi_{z_\nu}|^{\min(\frac{n+2}{n-2},2)} + |v_\nu-\xi_{z_\nu}|^{\frac{n+2}{n-2}}\Big\|_{L^{\frac{2n}{n+2}}(\Omega)}\\
&\le C\sum_{k=1}^m (N/\lambda_{k,\nu})^{\frac{(n-2)^2}{2(n+2)}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| |v_\nu - \xi_{z_\nu}|^{\min(\frac{n+2}{n-2},2)} + |v_\nu - \xi_{z_\nu}|^{\frac{n+2}{n-2}} \right\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega\setminus\cup_{k=1}^m B_{N/\lambda_{k,\nu}}(x_{k,\nu}))} \\
&\le \left\| |v_\nu - \xi_{z_\nu}|^{\min(\frac{4}{n-2},1)} + |v_\nu - \xi_{z_\nu}|^{\frac{4}{n-2}} \right\|_{L^{\frac{n}{2}}(\Omega\setminus\cup_{k=1}^m B_{N/\lambda_{k,\nu}}(x_{k,\nu}))} \\
&\quad \cdot \|v_\nu - \xi_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(\Omega)}.
\end{aligned}$$

Since

$$\begin{aligned}
&\|v_\nu - \xi_{z_\nu}\|_{L^{\frac{2n}{n-2}}(\Omega\setminus\cup_{k=1}^m B_{N/\lambda_{k,\nu}}(x_{k,\nu}))} \\
&= \left\| \sum_{k=1}^m \alpha_{k,\nu}\xi_{x_{k,\nu},\lambda_{k,\nu}} + w_\nu \right\|_{L^{\frac{2n}{n-2}}(\Omega\setminus\cup_{k=1}^m B_{N/\lambda_{k,\nu}}(x_{k,\nu}))} \\
&\le \sum_{k=1}^m \alpha_{k,\nu} \left\|\xi_{x_{k,\nu},\lambda_{k,\nu}}\right\|_{L^{\frac{2n}{n-2}}(\Omega\setminus B_{N/\lambda_{k,\nu}}(x_{k,\nu}))} + \|w_\nu\|_{L^{\frac{2n}{n-2}}(\Omega)} \\
&\le CN^{-\frac{n-2}{2}} + o(1),
\end{aligned}$$

we have

$$\begin{aligned}
&\left\| \xi_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\xi_{z_\nu}^{\frac{4}{n-2}}(v_\nu - \xi_{z_\nu}) - v_\nu^{\frac{n+2}{n-2}} \right\|_{L^{\frac{n(n+2)}{n^2+4}}(\Omega)} \\
&\le C\sum_{k=1}^m (N/\lambda_{k,\nu})^{\frac{(n-2)^2}{2(n+2)}} + C(N^{-\frac{n-2}{2}} + N^{-2} + o(1))\,\|v_\nu - \xi_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(\Omega)}.
\end{aligned}$$

Also,

$$\begin{aligned}
&\sup_{1\le l\le L}\left|\int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l(v_\nu - \xi_{z_\nu})\right| \qquad (63)\\
&= \sup_{1\le l\le L}\left|\int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l\Big(\sum_{k=1}^m \alpha_{k,\nu}\xi_{x_{k,\nu},\lambda_{k,\nu}} + w_\nu\Big)\right| \\
&\le C\sum_{k=1}^m \lambda_{k,\nu}^{\frac{2-n}{2}} + o(1)\|w_\nu\|_{L^1(\Omega)} \\
&\le C\sum_{k=1}^m \lambda_{k,\nu}^{\frac{2-n}{2}} + o(1)\left\| v_\nu - \xi_{z_\nu} - \sum_{k=1}^m \alpha_{k,\nu}\xi_{x_{k,\nu},\lambda_{k,\nu}}\right\|_{L^1(\Omega)} \\
&\le C\sum_{k=1}^m \lambda_{k,\nu}^{\frac{2-n}{2}} + o(1)\,\|v_\nu - \xi_{z_\nu}\|_{L^1(\Omega)}. \qquad (64)
\end{aligned}$$

Putting these facts together, we have

$$\begin{aligned}
&\|v_\nu - \xi_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(\Omega)} \\
&\le C\left\|(1-\mathcal{R}(t_\nu))v_\nu^{\frac{n+2}{n-2}}\right\|_{L^{\frac{2n}{n+2}}(\Omega)} + C\sum_{k=1}^m (N/\lambda_{k,\nu})^{\frac{(n-2)^2}{2(n+2)}} \\
&+ C(N^{-\frac{n-2}{2}} + N^{-2} + o(1))\,\|v_\nu - \xi_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(\Omega)} + C\sum_{k=1}^m \lambda_{k,\nu}^{\frac{2-n}{2}}.
\end{aligned}$$

By choosing N sufficiently large, we obtain

$$\|v_\nu - \xi_{z_\nu}\|_{L^{\frac{n+2}{n-2}}(\Omega)} \le C \left\|(1-\mathcal{R}(t_\nu))v_\nu^{\frac{n+2}{n-2}}\right\|_{L^{\frac{2n}{n+2}}(\Omega)} + C\sum_{k=1}^m \lambda_{k,\nu}^{-\frac{(n-2)^2}{2(n+2)}},$$

from which the conclusion follows. ☐

Lemma 4.15. *For large ν, we have*

$$\|v_\nu - \xi_{z_\nu}\|_{L^1(\Omega)} \le C\|v_\nu^{\frac{n+2}{n-2}}(\mathcal{R}(t_\nu)-1)\|^{\frac{n+2}{n-2}}_{L^{\frac{2n}{n+2}}(\Omega)} + C\sum_{k=1}^m \lambda_{k,\nu}^{\frac{2-n}{2}},$$

where $C>0$ is independent of ν.

Proof. Using (61), and applying (ii) of Lemma 4.8 to $v_\nu - \xi_{z_\nu}$, we obtain

$$\begin{aligned}
&\|v_\nu - \xi_{z_\nu}\|_{L^1(\Omega)}\\
&\le C\left\|(1-\mathcal{R}(t_\nu))v_\nu^{\frac{n+2}{n-2}}\right\|_{L^1(\Omega)}\\
&\quad + C\left\|(\xi_{z_\nu}^{\frac{4}{n-2}} - v_\infty^{\frac{4}{n-2}})(v_\nu - \xi_{z_\nu})\right\|_{L^1(\Omega)}\\
&\quad + C\left\|\xi_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\xi_{z_\nu}^{\frac{4}{n-2}}(v_\nu - \xi_{z_\nu}) - v_\nu^{\frac{n+2}{n-2}}\right\|_{L^1(\Omega)}\\
&\quad + C\sup_{1\le l\le L}\left|\int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l(v_\nu - \xi_{z_\nu})\right|.
\end{aligned}$$

It follows from (62) that

$$\begin{aligned}
&\left\|\xi_{z_\nu}^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}\xi_{z_\nu}^{\frac{4}{n-2}}(v_\nu - \xi_{z_\nu}) - v_\nu^{\frac{n+2}{n-2}}\right\|_{L^1(\Omega)}\\
&\le C\left\||v_\nu - \xi_{z_\nu}|^{\min(\frac{n+2}{n-2},2)} + |v_\nu - \xi_{z_\nu}|^{\frac{n+2}{n-2}}\right\|_{L^1(\Omega)}\\
&\le C\|v_\nu - \xi_{z_\nu}\|_{L^1(\Omega)}^{\max(0,1-\frac{n-2}{4})}\left\||v_\nu - \xi_{z_\nu}|^{\frac{n+2}{n-2}}\right\|_{L^1(\Omega)}^{\min(1,\frac{n-2}{4})} + C\|v_\nu - \xi_{z_\nu}\|^{\frac{n+2}{n-2}}_{L^{\frac{n+2}{n-2}}(\Omega)}\\
&\le C\|v_\nu - \xi_{z_\nu}\|_{L^1(\Omega)}^{\max(0,1-\frac{n-2}{4})}\|v_\nu - \xi_{z_\nu}\|^{\frac{n+2}{n-2}\min(1,\frac{n-2}{4})}_{L^{\frac{n+2}{n-2}}(\Omega)} + C\|v_\nu - \xi_{z_\nu}\|^{\frac{n+2}{n-2}}_{L^{\frac{n+2}{n-2}}(\Omega)}\\
&\le \frac{1}{2C}\|v_\nu - \xi_{z_\nu}\|_{L^1(\Omega)} + C\|v_\nu - \xi_{z_\nu}\|^{\frac{n+2}{n-2}}_{L^{\frac{n+2}{n-2}}(\Omega)},
\end{aligned}$$

where we used Hölder's inequality in the second inequality and the Young inequality in the last inequality. Combining (64), we have

$$\begin{aligned}
&\|v_\nu - \xi_{z_\nu}\|_{L^1(\Omega)}\\
&\le C\left\|(1-\mathcal{R}(t_\nu))v_\nu^{\frac{n+2}{n-2}}\right\|_{L^1(\Omega)} + o(1)\|v_\nu - \xi_{z_\nu}\|_{L^1(\Omega)}\\
&\quad + \frac{1}{2}\|v_\nu - \xi_{z_\nu}\|_{L^1(\Omega)} + C\|v_\nu - \xi_{z_\nu}\|^{\frac{n+2}{n-2}}_{L^{\frac{n+2}{n-2}}(\Omega)} + C\sum_{k=1}^m \lambda_{k,\nu}^{\frac{2-n}{2}}.
\end{aligned}$$

Then the conclusion follows from Lemma 4.14. ☐

Using the above two lemmas, we can continue to estimate $F(\xi_{z_\nu}) - F(v_\infty)$ from Lemma 4.10.

Proposition 4.16. *For all large ν, we have*

$$F(\xi_{z_\nu}) - F(v_\infty) \le C\left(\int_\Omega |\mathcal{R}(t_\nu) - 1|^{\frac{2n}{n+2}} v_\nu^{\frac{2n}{n-2}}\right)^{\frac{n+2}{2n}(1+\gamma)} + C\sum_{k=1}^m \lambda_{k,\nu}^{\frac{2-n}{2}(1+\gamma)},$$

where $\gamma \in (0,1)$ is the one in Lemma 4.10.

Proof. It follows from integration by parts that

$$\begin{aligned}
&\int_\Omega (\Delta\xi_{z_\nu} + b\xi_{z_\nu} + \xi_{z_\nu}^{\frac{n+2}{n-2}})\phi_l \,\mathrm{d}x \\
&= \int_\Omega (\Delta v_\nu + bv_\nu + v_\nu^{\frac{n+2}{n-2}})\phi_l \,\mathrm{d}x + \mu_l \int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l(v_\nu - \xi_{z_\nu})\,\mathrm{d}x \\
&\quad - \int_\Omega \phi_l(v_\nu^{\frac{n+2}{n-2}} - \xi_{z_\nu}^{\frac{n+2}{n-2}})\,\mathrm{d}x \\
&= \int_\Omega (1-\mathcal{R}(t_\nu))v_\nu^{\frac{n+2}{n-2}}\phi_l \,\mathrm{d}x + \mu_l \int_\Omega v_\infty^{\frac{4}{n-2}}\phi_l(v_\nu - \xi_{z_\nu})\,\mathrm{d}x - \int_\Omega \phi_l(v_\nu^{\frac{n+2}{n-2}} - \xi_{z_\nu}^{\frac{n+2}{n-2}})\,\mathrm{d}x.
\end{aligned}$$

Using the pointwise estimate

$$|v_\nu^{\frac{n+2}{n-2}} - \xi_{z_\nu}^{\frac{n+2}{n-2}}| \le C\xi_{z_\nu}^{\frac{4}{n-2}}|v_\nu - \xi_{z_\nu}| + C|v_\nu - \xi_{z_\nu}|^{\frac{n+2}{n-2}},$$

we have

$$\begin{aligned}
&\sup_{1\le l\le L}\left|\int_\Omega (\Delta\xi_z + b\xi_z + \xi_z^{\frac{n+2}{n-2}})\phi_l \,\mathrm{d}x\right| \\
&\le C\left\|(1-\mathcal{R}(t_\nu))v_\nu^{\frac{n+2}{n-2}}\right\|_{L^{\frac{2n}{n+2}}(\Omega)} + C\left\|v_\nu - \xi_{z_\nu}\right\|_{L^1(\Omega)} + C\left\|v_\nu - \xi_{z_\nu}\right\|_{L^{\frac{n+2}{n-2}}(\Omega)}^{\frac{n+2}{n-2}} \\
&\le C\left\|(1-\mathcal{R}(t_\nu))v_\nu^{\frac{n+2}{n-2}}\right\|_{L^{\frac{2n}{n+2}}(\Omega)} + C\left\|v_\nu^{\frac{n+2}{n-2}}(\mathcal{R}(t_\nu)-1)\right\|_{L^{\frac{2n}{n+2}}(\Omega)}^{\frac{n+2}{n-2}} + C\sum_{k=1}^m \lambda_{k,\nu}^{\frac{2-n}{2}} \\
&\le C\left\|(1-\mathcal{R}(t_\nu))v_\nu^{\frac{n+2}{n-2}}\right\|_{L^{\frac{2n}{n+2}}(\Omega)} + C\sum_{k=1}^m \lambda_{k,\nu}^{\frac{2-n}{2}},
\end{aligned}$$

where we used Lemma 4.14 and Lemma 4.15 in the second inequality, and Proposition 2.6 in the last inequality.

Then the conclusion follows from Lemma 4.10. □

Corollary 4.17. *If $n \ge 4$ and $b > 0$ satisfying* (2), *we have*

$$F(U_\nu) \le F(v_\infty) + \frac{2m}{n}Y(\mathbb{S}^n)^{n/2} + C\left(\int_\Omega |\mathcal{R}(t_\nu) - 1|^{\frac{2n}{n+2}} v_\nu^{\frac{2n}{n-2}}\right)^{\frac{n+2}{2n}(1+\gamma)}.$$

Proof. Let $\widetilde{U}_\nu = \sum_{k=1}^m \alpha_k \xi_{x_{k,\nu},\lambda_{k,\nu}}$.

$$\begin{aligned}
F(U_\nu) &= \int_\Omega |\nabla(\xi_{z_\nu} + \widetilde{U}_\nu)|^2 - b(\xi_{z_\nu} + \widetilde{U}_\nu)^2 - \frac{n-2}{n}\int_\Omega (\xi_{z_\nu} + \widetilde{U}_\nu)^{\frac{2n}{n-2}} \\
&= F(\xi_{z_\nu}) + F(\widetilde{U}_\nu) + 2\int_\Omega (\nabla\xi_{z_\nu}\nabla\widetilde{U}_\nu - b\xi_{z_\nu}\widetilde{U}_\nu - \xi_{z_\nu}^{\frac{n+2}{n-2}}\widetilde{U}_\nu) \\
&\quad - \frac{n-2}{n}\int_\Omega \left((\xi_{z_\nu} + \widetilde{U}_\nu)^{\frac{2n}{n-2}} - \frac{2n}{n-2}\xi_{z_\nu}^{\frac{n+2}{n-2}}\widetilde{U}_\nu - \xi_{z_\nu}^{\frac{2n}{n-2}} - \widetilde{U}^{\frac{2n}{n-2}}\right).
\end{aligned}$$

We have

$$
\begin{aligned}
&\Big|\int_\Omega (\nabla \xi_{z_\nu}\nabla \widetilde U_\nu - b\xi_{z_\nu}\widetilde U_\nu - \xi_{z_\nu}^{\frac{n+2}{n-2}}\widetilde U_\nu)\Big|\\
&= \Big|\int_\Omega \Big(\Delta(\xi_{z_\nu}-v_\infty) + b(\xi_{z_\nu}-v_\infty) + \xi_{z_\nu}^{\frac{n+2}{n-2}} - v_\infty^{\frac{n+2}{n-2}}\Big)\widetilde U_\nu)\Big|\\
&\le o(1)\sum_{k=1}^m \lambda_k^{\frac{2-n}{2}}.
\end{aligned}
$$

By Lemma A.1, there exists $c>0$, depending only on n such that

$$
(\xi_{z_\nu}+\widetilde U_\nu)^{\frac{2n}{n-2}} - \frac{2n}{n-2}\xi_{z_\nu}^{\frac{n+2}{n-2}}\widetilde U_\nu - \xi_{z_\nu}^{\frac{2n}{n-2}} - \widetilde U^{\frac{2n}{n-2}} \ge \begin{cases} c\xi_{z_\nu}^{\frac{4}{n-2}}(\widetilde U_\nu)^2, & \text{if } \xi_{z_\nu}\ge U'_\nu,\\ c\xi_{z_\nu}(\widetilde U_\nu)^{\frac{n+2}{n-2}}, & \text{if } \xi_{z_\nu}< U'_\nu.\end{cases}
$$

Since $v_\infty/2\le \xi_{z_\nu}\le 2v_\infty$, and

$$
\int_{|x|<\sqrt{\lambda^{-1}}}\left(\frac{\lambda}{1+\lambda^2|x|^2}\right)^{\frac{n+2}{2}} \ge \lambda^{-\frac{n-2}{2}}\int_{|y|<1}\left(\frac{1}{1+|y|^2}\right)^{\frac{n+2}{2}} \ge c\lambda^{-\frac{n-2}{2}},
$$

we have

$$
\int_\Omega \left((\xi_{z_\nu}+\widetilde U_\nu)^{\frac{2n}{n-2}} - \frac{2n}{n-2}\xi_{z_\nu}^{\frac{n+2}{n-2}}\widetilde U_\nu - \xi_{z_\nu}^{\frac{2n}{n-2}} - \widetilde U^{\frac{2n}{n-2}}\right) \ge c\sum_{k=1}^m \lambda_k^{\frac{2-n}{2}}.
$$

Then, the conclusion follows from Proposition 4.16 and Corollary 4.6. □

5. Convergence

Using the estimates in Corollaries 4.4 and 4.13, we have the following estimate of $F(v_\nu)-F_\infty$ for any sequence of times $\{t_\nu:\nu\in\mathbb N\}$.

Proposition 5.1. *Let $n\ge 4$, and $b>0$ satisfy (2). Let $\{t_\nu:\nu\in\mathbb N\}$ be a sequence of times such that $t_\nu\to\infty$ as $\nu\to\infty$. Then, we can find a real number $\gamma\in(0,1)$ and a constant $C>0$ such that, after passing to a subsequence, we have*

$$
F(v_\nu)-F_\infty \le C\left(\int_\Omega |\mathcal R(t_\nu)-1|^{\frac{2n}{n+2}} v_\nu^{\frac{2n}{n-2}}\right)^{\frac{n+2}{2n}(1+\gamma)}
$$

for all integers ν in that subsequence, where F_∞ is the one defined in (18). Note that γ and C may depend on the sequence $\{t_\nu:\nu\in\mathbb N\}$.

Proof. It follows from (18) that $F(v_\infty)=F_\infty$. Recall that $U_\nu=v_\nu-w_\nu$. We have

$$
\begin{aligned}
&F(v_\nu)-F(U_\nu)\\
&= \frac{n-2}{n}\int_\Omega (U_\nu^{\frac{2n}{n-2}} - v_\nu^{\frac{2n}{n-2}}) + 2\int_\Omega(\nabla v_\nu\nabla w_\nu - bv_\nu w_\nu) - \int_\Omega(|\nabla w_\nu|^2 - bw_\nu^2)\\
&= \frac{n-2}{n}\int_\Omega (U_\nu^{\frac{2n}{n-2}} - v_\nu^{\frac{2n}{n-2}}) + 2\int_\Omega \mathcal R v_\nu^{\frac{n+2}{n-2}}w_\nu - \int_\Omega(|\nabla w_\nu|^2 - bw_\nu^2)\\
&= \frac{n-2}{n}\int_\Omega\left(U_\nu^{\frac{2n}{n-2}} - v_\nu^{\frac{2n}{n-2}} + \frac{2n}{n-2}v_\nu^{\frac{n+2}{n-2}}w_\nu - \frac{n(n+2)}{(n-2)^2}U_\nu^{\frac{4}{n-2}}w_\nu^2\right)\\
&\quad + 2\int_\Omega(\mathcal R-1)v_\nu^{\frac{n+2}{n-2}}w_\nu - \int_\Omega\left(|\nabla w_\nu|^2 - bw_\nu^2 - \frac{n+2}{n-2}U_\nu^{\frac{4}{n-2}}w_\nu^2\right).
\end{aligned}
$$

Using the pointwise estimate

$$\begin{aligned}
&\left|U_\nu^{\frac{2n}{n-2}} - v_\nu^{\frac{2n}{n-2}} + \frac{2n}{n-2} v_\nu^{\frac{n+2}{n-2}} w_\nu - \frac{n(n+2)}{(n-2)^2} U_\nu^{\frac{4}{n-2}} w_\nu^2\right| \\
&= \left|U_\nu^{\frac{2n}{n-2}} - (w_\nu + U_\nu)^{\frac{2n}{n-2}} + \frac{2n}{n-2}(w_\nu + U_\nu)^{\frac{n+2}{n-2}} w_\nu - \frac{n(n+2)}{(n-2)^2} U_\nu^{\frac{4}{n-2}} w_\nu^2\right| \\
&\le C U_\nu^{\max\{0, \frac{4}{N-2}-1\}} |w_\nu|^{\min\{\frac{2N}{N-2}, 3\}} + C|w_\nu|^{\frac{2N}{N-2}},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\int_\Omega \left|U_\nu^{\frac{2n}{n-2}} - v_\nu^{\frac{2n}{n-2}} + \frac{2n}{n-2} v_\nu^{\frac{n+2}{n-2}} w_\nu - \frac{n(n+2)}{(n-2)^2} U_\nu^{\frac{4}{n-2}} w_\nu^2\right| \\
&\le C \int_\Omega U_\nu^{\max\{0, \frac{4}{n-2}-1\}} |w_\nu|^{\min\{\frac{2n}{n-2}, 3\}} + C \int_\Omega |w_\nu|^{\frac{2n}{n-2}} \\
&\le C \left(\int_\Omega |w_\nu|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n} \min\{\frac{n}{n-2}, \frac{3}{2}\}}.
\end{aligned}$$

By Hölder inequality and Cauchy inequality, we have

$$\begin{aligned}
\left|\int_\Omega (\mathcal{R}-1) v_\nu^{\frac{n+2}{n-2}} w_\nu\right| &\le C \left(\int_\Omega |\mathcal{R}-1|^{\frac{2n}{n+2}} v_\nu^{\frac{2n}{n-2}}\right)^{\frac{n+2}{2n}} \left(\int_\Omega |w_\nu|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \\
&\le \varepsilon \left(\int_\Omega |w_\nu|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + C(\varepsilon) \left(\int_\Omega |\mathcal{R}-1|^{\frac{2n}{n+2}} v_\nu^{\frac{2n}{n-2}}\right)^{\frac{n+2}{n}}.
\end{aligned}$$

Finally, by Corollaries 4.4 and 4.13, we have

$$\int_\Omega |\nabla w_\nu|^2 - b w_\nu^2 - \frac{n+2}{n-2} U_\nu^{\frac{4}{n-2}} w_\nu^2 \ge \frac{1}{C} \left(\int_\Omega |w_\nu|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

Since $\int_\Omega |w_\nu|^{\frac{2n}{n-2}} \to 0$, we have, by choosing ε being small, that

$$\begin{aligned}
F(v_\nu) - F(U_\nu) &\le C \left(\int |w_\nu|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n} \min\{\frac{n}{n-2}, \frac{3}{2}\}} + C \left(\int |\mathcal{R}-1|^{\frac{2n}{n+2}} v_\nu^{\frac{2n}{n-2}}\right)^{\frac{n+2}{n}} \\
&\quad - \frac{1}{2C} \left(\int |w_\nu|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\
&\le C \left(\int |\mathcal{R}-1|^{\frac{2n}{n+2}} v_\nu^{\frac{2n}{n-2}}\right)^{\frac{n+2}{n}}.
\end{aligned}$$

By Corollary 4.17, the proof is completed. ☐

Then we can show the estimate for all large time.

Corollary 5.2. *There exist real numbers $\gamma \in (0,1)$ and $t_0 > 0$ such that*

$$F(v(t)) - F_\infty \le \left(\int_\Omega |\mathcal{R}-1|^{\frac{2n}{n+2}} v(x,t)^{\frac{2n}{n-2}} \, \mathrm{d}x\right)^{\frac{n+2}{2n}(1+\gamma)}$$

for all $t \ge t_0$.

Proof. Suppose this is not true. Then, there exists a sequence of times $\{t_\nu : \nu \in \mathbb{N}\}$ such that $t_\nu > \nu$ and

$$F(v(t_\nu)) - F_\infty \geq \left(\int_\Omega |\mathcal{R}(t_\nu) - 1|^{\frac{2n}{n+2}} v(x,t_\nu)^{\frac{2n}{n-2}} \,\mathrm{d}x \right)^{\frac{n+2}{2n}(1+\frac{1}{\nu})}$$

for all $\nu \in \mathbb{N}$. By applying Proposition 5.1 to this sequence $\{t_\nu : \nu \in \mathbb{N}\}$, there exists an infinite subset $I \subset \mathbb{N}$, a real number $\alpha \in (0,1)$ and $C > 0$ such that

$$F(v(t_\nu)) - F_\infty \leq C \left(\int_\Omega |\mathcal{R}(t_\nu) - 1|^{\frac{2n}{n+2}} v(x,t_\nu)^{\frac{2n}{n-2}} \,\mathrm{d}x \right)^{\frac{n+2}{2n}(1+\alpha)}$$

for all $\nu \in I$. Thus, we have

$$1 \leq C \left(\int_\Omega |\mathcal{R}(t_\nu) - 1|^{\frac{2n}{n+2}} v(x,t_\nu)^{\frac{2n}{n-2}} \,\mathrm{d}x \right)^{\frac{n+2}{2n}(\alpha-\frac{1}{\nu})}$$

for all $\nu \in I$. However, from Proposition 2.6, we have

$$\lim_{\nu\to\infty} \left(\int_\Omega |\mathcal{R}(t_\nu) - 1|^{\frac{2n}{n+2}} v(x,t_\nu)^{\frac{2n}{n-2}} \,\mathrm{d}x \right)^{\frac{n+2}{2n}(\alpha-\frac{1}{\nu})} = 0.$$

We have reached a contradiction. □

Now we can use a differential inequality of F to obtain a decay estimate.

Proposition 5.3. *There exist $\theta > 0$ and $C > 0$ such that for all $T > 1$, there holds*

$$\int_T^\infty M_2(t)^{1/2} \,\mathrm{d}t \leq CT^{-\theta},$$

where M_2 is defined in (20)*.*

Proof. It follows from Corollary 5.2, Hölder's inequality, and (15) that

$$0 \leq F(v(t)) - F_\infty \leq \left(\int_\Omega |\mathcal{R} - 1|^{\frac{2n}{n+2}} v(x,t)^{\frac{2n}{n-2}} \,\mathrm{d}x \right)^{\frac{n+2}{2n}(1+\gamma)} \leq CM_2(t)^{\frac{1+\gamma}{2}}.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(F(v(t)) - F_\infty) = -\frac{2(n-2)}{n+2} M_2(t) \leq -C(F(v(t)) - F_\infty)^{\frac{2}{1+\gamma}}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}(F(v(t)) - F_\infty)^{\frac{\gamma-1}{\gamma+1}} \geq C\frac{1-\gamma}{1+\gamma} > 0.$$

It follows that

$$F(v(t)) - F_\infty \leq Ct^{-\frac{1+\gamma}{1-\gamma}}$$

for sufficiently large t. Then we have

$$\begin{aligned}
\left(\int_T^{2T} M_2(s)^{1/2} \,\mathrm{d}s \right)^2 &\leq T \int_T^{2T} M_2(s) \,\mathrm{d}s \\
&\leq \frac{n+2}{n-2} T(F(v(T)) - F(v(2T))) \\
&\leq \frac{n+2}{n-2} T(F(v(T)) - F_\infty) \\
&\leq CT^{-\frac{2\gamma}{1-\gamma}},
\end{aligned}$$

where we used the monotonicity of F. It follows that

(65)
$$\int_T^\infty M_2(t)^{1/2}\,\mathrm{d}t=\sum_{k=0}^\infty\int_{2^kT}^{2^{k+1}T}M_2(t)^{1/2}\,\mathrm{d}t\le CT^{-\frac{\gamma}{1-\gamma}}\sum_{k=1}^\infty 2^{-\frac{\gamma}{1-\gamma}k}\le CT^{-\frac{\gamma}{1-\gamma}}.$$

This finishes the proof. □

We are ready to show the uniform boundedness, and uniform higher order estimates.

Proposition 5.4. *For any $\varepsilon>0$, there exists $T_0>0$ such that*

$$\|v(\cdot,t)-v(\cdot,T_0)\|_{L^{\frac{2n}{n-2}}(\Omega)}<\varepsilon\quad\text{for all } t>T_0. \tag{66}$$

Consequently, there exists $C>0$ depending only n,b,Ω and u_0 such that

$$v(x,t)\le C\quad\text{ in }\ \Omega\text{ for all } t>1. \tag{67}$$

Proof. For $b>a>1$, using the pointwise estimate

$$|v(x,b)-v(x,a)|^{\frac{n}{n-2}}\le|v(x,b)^{\frac{n}{n-2}}-v(x,a)^{\frac{n}{n-2}}|,$$

we have

$$\begin{aligned}\left(\int_\Omega|v(x,b)-v(x,a)|^{\frac{2n}{n-2}}\,\mathrm{d}x\right)^{1/2}&\le\left(\int_\Omega|v(x,b)^{\frac{n}{n-2}}-v(x,a)^{\frac{n}{n-2}}|^2\,\mathrm{d}x\right)^{1/2}\\&\le\left(\int_\Omega\Big(\int_a^b|\partial_t(v(x,t)^{\frac{n}{n-2}})|\,\mathrm{d}t\Big)^2\,\mathrm{d}x\right)^{1/2}\\&\le\int_a^b\left(\int_\Omega|\partial_t(v(x,t)^{\frac{n}{n-2}})|^2\,\mathrm{d}x\right)^{1/2}\,\mathrm{d}t\\&\le C\int_a^\infty M_2(t)^{1/2}\,\mathrm{d}t\\&\le Ca^{-\theta},\end{aligned}\tag{68}$$

where we used Minkowski's integral inequality in the third inequality, and Proposition 5.3 in the last inequality. Hence, for any $\varepsilon>0$, there exists $T_0>0$ such that (66) holds.

To show the L^∞ bound in (67), we need to use the following Brézis-Kato [11] estimate (see also Lemma B.3 in Appendix B of Struwe [40]): there exists $\delta>0$ depending only on n and Ω such that if $v\in H_0^1(\Omega)$ is a weak solution of

$$-\Delta v=c_1v+c_2v\quad\text{in }\Omega,$$

where $\|c_1\|_{L^{\frac{n}{2}}(\Omega)}\le\delta$ and $c_2\in L^p(\Omega)$ for some $p>\frac{n}{2}$, then there exist $C>0$ and $q>\frac{2n}{n-2}$ depending only on n,δ,p,Ω and $\|c_2\|_{L^p(\Omega)}$ such that

$$\|v\|_{L^q(\Omega)}\le C\|v\|_{L^2(\Omega)}.$$

Let $T_0>0$ be the one in (66) with some $\varepsilon<\delta/2$ that $(2\varepsilon)^{\frac{4}{n-2}}\le\delta/2$. By Proposition 2.6, there exists $T_1>0$ such that $M(t)_{\frac{n}{2}}<\delta/2$ for all $t>T_1$. Let $T_2=\max(T_0,T_1)$,

$$U=\{x\in\Omega:|v(x,t)-v(x,T_2)|>\max_\Omega v(\cdot,T_2)\}$$

and χ_U be the characteristic function of U. Then for $t > T_2$, we have

$$\begin{aligned}\|v^{\frac{4}{n-2}}\chi_U\|_{L^{\frac{n}{2}}(\Omega)} &= (\|v\|_{L^{\frac{2n}{n-2}}(U)})^{\frac{4}{n-2}}\\ &\le (\|v(\cdot,T_2)\|_{L^{\frac{2n}{n-2}}(U)} + \|v(\cdot,t)-v(\cdot,T_2)\|_{L^{\frac{2n}{n-2}}(U)})^{\frac{4}{n-2}}\\ &\le (\max_{\Omega}|v(\cdot,T_2)|\cdot|U|^{\frac{n-2}{2n}} + \|v(\cdot,t)-v(\cdot,T_2)\|_{L^{\frac{2n}{n-2}}(U)})^{\frac{4}{n-2}}\\ &\le (\|v(\cdot,t)-v(\cdot,T_2)\|_{L^{\frac{2n}{n-2}}(U)} + \|v(\cdot,t)-v(\cdot,T_2)\|_{L^{\frac{2n}{n-2}}(U)})^{\frac{4}{n-2}}\\ &\le (2\varepsilon)^{\frac{4}{n-2}}\\ &\le \delta/2,\end{aligned}$$

where we used Chebyshev's inequality in the third inequality, and (68) in the fourth inequality. From the definition of $\mathcal{R}$ in (19), we have

$$-\Delta v = bv + \mathcal{R}v^{\frac{n+2}{n-2}} =: V_1 v + V_2 v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \tag{69}$$

where $V_1 = (\mathcal{R}-1)v^{\frac{4}{n-2}} + v^{\frac{4}{n-2}}\chi_U$ and $V_2 = (1-\chi_U)v^{\frac{4}{n-2}} + b$. Then $V_2 \in L^\infty(\Omega)$ and

$$\|V_1\|_{L^{\frac{n}{2}}(\Omega)} \le M(t)_{\frac{n}{2}} + \|v^{\frac{4}{n-2}}\chi_U\|_{L^{\frac{n}{2}}(\Omega)} \le \delta/2 + \delta/2 = \delta$$

for all $t > T_2$. Then by the Brézis-Kato estimate, there exist $q > \frac{2n}{n-2}$ and $C > 0$ such that

$$\|v\|_{L^q(\Omega)} \le C\|v\|_{L^2(\Omega)} \le C,$$

where we used Hölder's inequality and (15) in the last inequality. Now V_1 belongs to L^p for some $p > \frac{n}{2}$, and then the standard Moser's iteration will lead to (67). $\square$

Theorem 5.5. *There exists $C > 0$ depending only n, b, Ω and u_0 such that*

$$\|v(\cdot,t)\|_{C^{2+\frac{n+2}{n-2}}(\overline{\Omega})} \le C \quad \textit{for all } t > 1. \tag{70}$$

Proof. By using (67), it follows from Proposition 6.2 in [23] (more precisely, its proof) that

$$\frac{1}{C}d(x) \le v(\cdot,t) \le Cd(x) \quad \text{for all } x \in \Omega,\ t > 1,$$

where $d(x) := \operatorname{dist}(x, \partial\Omega)$. Then (70) follows from Theorem 5.1 in [29]. $\square$

Let us conclude this section with the proof of Theorem 1.1.

Proof of Theorem 1.1. It follows from Proposition 5.4, Theorem 5.5 and (15) that there exists a nonzero stationary solution v_∞ of (9) such that

$$\lim_{t\to\infty}\|v(\cdot,t) - v_\infty\|_{C^3(\overline{\Omega})} = 0.$$

From (68), we know that there exist $C > 0$ and $\theta > 0$ such that

$$\|v(\cdot,t) - v_\infty\|_{L^{\frac{2n}{n-2}}(\Omega)} \le Ct^{-\theta} \quad \text{for all } t > 1.$$

Using (70) and Gagliardo interpolation inequalities (see, e.g., (12)–(13) in Blanchet-Bonforte-Dolbeault-Grillo-Vázquez [6]), we have

$$\|v(\cdot,t) - v_\infty\|_{C^1(\overline{\Omega})} \le Ct^{-\theta} \quad \text{for all } t > 1$$

with a possibly different θ. Since $v(\cdot,t)\equiv v_\infty\equiv 0$ on $\partial\Omega$, we have for all $x\in\Omega$ that

$$\left|\frac{v(x,t)-v_\infty(x)}{v_\infty(x)}\right|\le \frac{\|v(\cdot,t)-v_\infty\|_{C^1(\overline{\Omega})}\ d(x)}{d(x)/C}\le C\|v(\cdot,t)-v_\infty\|_{C^1(\overline{\Omega})}\le Ct^{-\theta}$$

for all $t>1$. That is,

$$\left\|\frac{v(\cdot,t)}{v_\infty}-1\right\|_{L^\infty(\Omega)}\le Ct^{-\theta}\quad \text{for all } t\ge 1.$$

Using (70) again, we have

$$\left\|\frac{v(\cdot,t)}{v_\infty}\right\|_{C^{1+\frac{n+2}{n-2}}(\overline{\Omega})}\le C\quad \text{for all } t>1.$$

Then by interpolation inequalities, we have

$$\left\|\frac{v(\cdot,t)}{v_\infty}-1\right\|_{C^2(\overline{\Omega})}\le Ct^{-\theta}\quad \text{for all } t\ge 1. \tag{71}$$

Now let us assume that Ω satisfies the condition (11). Then there exists $C>0$ such that for all $\varphi\in H_0^1$ satisfying

$$-\Delta\varphi-b\varphi-\frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}}\varphi=f\quad \text{in } \Omega,$$

there holds

$$\|\varphi\|_{H_0^1(\Omega)}\le C\|f\|_{L^2(\Omega)}. \tag{72}$$

Here, the C depends only on n,b,Ω and v_∞. This estimate can be proved as follows. First, it follows Theorem 6 in Section 6.2.3 of Evans [25] that there exists $C>0$ such that $\|\varphi\|_{L^2(\Omega)}\le C\|f\|_{L^2(\Omega)}$. Secondly, multiplying φ to its equation and integrating by part, it leads to (72).

On one hand, using the equation of v_∞, we have

$$\begin{aligned}
&F(v(t))-F_\infty\\
&=\int_\Omega\Big(|\nabla v(\cdot,t)|^2-bv(\cdot,t)^2-\frac{n-2}{n}v(\cdot,t)^{\frac{2n}{n-2}}\Big)\,\mathrm{d}x\\
&\quad-\int_\Omega\Big(|\nabla v_\infty|^2-bv_\infty^2-\frac{n-2}{n}v_\infty^{\frac{2n}{n-2}}\Big)\,\mathrm{d}x\\
&\quad-2\int_\Omega\Big(\nabla v_\infty\nabla(v(\cdot,t)-v_\infty)-bv_\infty(v(\cdot,t)-v_\infty)-v_\infty^{\frac{n+2}{n-2}}(v(\cdot,t)-v_\infty)\Big)\,\mathrm{d}x\\
&=\int_\Omega|\nabla(v(\cdot,t)-v_\infty)|^2-b(v(\cdot,t)-v_\infty)^2\,\mathrm{d}x\\
&\quad-\frac{n-2}{n}\int_\Omega\Big(v(\cdot,t)^{\frac{2n}{n-2}}-v_\infty^{\frac{2n}{n-2}}-\frac{2n}{n-2}(v(\cdot,t)-v_\infty)\Big)\,\mathrm{d}x\\
&\le C\|v(\cdot,t)-v_\infty\|_{H_0^1(\Omega)}^2.
\end{aligned}$$

On the other hand, using the equation of v_∞, we have

$$\begin{aligned}
&\|\Delta v(\cdot,t)+bv(\cdot,t)+v(\cdot,t)^{\frac{n+2}{n-2}}\|_{L^2(\Omega)}\\
&=\|\Delta(v(\cdot,t)-v_\infty)+b(v(\cdot,t)-v_\infty)+v(\cdot,t)^{\frac{n+2}{n-2}}-v_\infty^{\frac{n+2}{n-2}}\|_{L^2(\Omega)}\\
&\geq\left\|\Delta(v(\cdot,t)-v_\infty)+b(v(\cdot,t)-v_\infty)+\frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}}(v(\cdot,t)-v_\infty)\right\|_{L^2(\Omega)}\\
&\quad-\left\|v(\cdot,t)^{\frac{n+2}{n-2}}-v_\infty^{\frac{n+2}{n-2}}-\frac{n+2}{n-2}v_\infty^{\frac{4}{n-2}}(v(\cdot,t)-v_\infty)\right\|_{L^2(\Omega)}\\
&\geq\frac{1}{C}\|v(\cdot,t)-v_\infty\|_{H_0^1(\Omega)}-C\left\|v_\infty^{\max(0,\frac{4}{n-2}-1)}|v(\cdot,t)-v_\infty|^{\min(2,\frac{n+2}{n-2})}\right\|_{L^2(\Omega)}\\
&\geq\frac{1}{C}\|v(\cdot,t)-v_\infty\|_{H_0^1(\Omega)}-o(1)\|v(\cdot,t)-v_\infty\|_{H_0^1(\Omega)}\quad\text{for large } t\\
&\geq\frac{1}{C}\|v(\cdot,t)-v_\infty\|_{H_0^1(\Omega)}\quad\text{for large } t,
\end{aligned}$$

where we used (72) in the second inequality. The constant C depends only on n, b, Ω and u_0. Combining these two inequalities together, we have

$$F(v(t))-F_\infty\leq\|\Delta v(\cdot,t)+bv(\cdot,t)+v(\cdot,t)^{\frac{n+2}{n-2}}\|_{L^2(\Omega)}^2=CM_2(t).$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(F(v(t))-F_\infty)=-\frac{2(n-2)}{n+2}M_2(t)\leq-C(F(v(t))-F_\infty),$$

and thus,

$$F(v(t))-F_\infty\leq Ce^{-\gamma t}$$

for some $C>0,\gamma>0$. Hence, the proof of (65) will give

$$\int_T^\infty M_2(t)^{1/2}\,\mathrm{d}t\leq Ce^{-\gamma t}.$$

From (68), we obtain that

$$\|v(\cdot,t)-v_\infty\|_{L^{\frac{2n}{n-2}}(\Omega)}\leq Ce^{-\gamma t}.$$

Then the proof of (71) gives

$$\left\|\frac{v(\cdot,t)}{v_\infty}-1\right\|_{C^2(\overline{\Omega})}\leq Ce^{-\gamma t}\quad\text{for all } t\geq 1. \tag{73}$$

This finishes the proof of Theorem 1.1. □

6. Subcritical case

In this last section, we consider the Sobolev subcritical case (5), and prove Theorem 1.2. The proof is similar to that of Theorem 1.1.

Proof of Theorem 1.2. First, we know from Proposition 6.2 in [23] that

$$\frac{1}{C}d(x)\leq v(\cdot,t)\leq Cd(x)\quad\text{for all } x\in\Omega,\ t>1.$$

Secondly, it follows from Theorem 1.1 in [26] and Theorem 5.1 in [29] that there exists a nonzero stationary solution v_∞ of (5) such that

$$\lim_{t\to\infty}\|v(\cdot,t)-v_\infty\|_{C^3(\overline{\Omega})}=0.$$

Let

$$F(v(t)) = \int_\Omega \left(|\nabla v(x,t)|^2 - \frac{2}{p+1} v(x,t)^{p+1} \right) \mathrm{d}x,$$
$$\mathcal{R} = -v^{-p}\Delta v = 1 - \frac{p\partial_t v}{v},$$
$$M_q(t) = \int_\Omega |\mathcal{R}-1|^q v^{p+1} \,\mathrm{d}x,$$

where $q \geq 1$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} F(v(t)) = -2\int_\Omega (\Delta v + v^p)\partial_t v = -2p \int_\Omega |\partial_t v|^2 v^{p-1} = -\frac{2}{p} M_2(t).$$

Hence $F(v(\cdot,t))$ deceases to $F(v_\infty)$ as $t \to \infty$. Furthermore,

$$\mathrm{d}F(v) = -2\Delta v - 2v^p = 2(\mathcal{R}-1)v^p.$$

Hence,

$$\|\mathrm{d}F(v(\cdot,t))\|_{L^2(\Omega)} \leq CM_2(t)^{1/2}.$$

Then it follows from Proposition 6.1 in [26] that there exist $C > 0, T_0 > 0, \gamma > 0$ such that for all $t > T_0$, we have

$$F(v(t)) - F(v_\infty) \leq C\|\mathrm{d}F(v(\cdot,t))\|^{1+\gamma}_{L^2(\Omega)} \leq CM_2(t)^{\frac{1+\gamma}{2}}.$$

Therefore, similar to the proof of Proposition 5.3, there exist $\theta \in (0,1)$ and $C > 0$ such that for all $T > 1$, there holds

$$\int_T^\infty M_2(t)^{1/2} \,\mathrm{d}t \leq CT^{-\theta}.$$

Then it follows from the proof of (68) that

$$\|v(\cdot,t) - v_\infty\|_{L^{p+1}(\Omega)} \leq C\int_t^\infty M_2(s)^{1/2}\,\mathrm{d}s \leq Ct^{-\theta}. \tag{74}$$

Using Theorem 5.1 in [29] (which is the regularity estimate) and interpolation inequalities, we have

$$\|v(\cdot,t) - v_\infty\|_{C^1(\overline{\Omega})} \leq Ct^{-\theta} \quad \text{for all } t > 1.$$

Since $v(\cdot,t) \equiv v_\infty \equiv 0$ on $\partial\Omega$, we have for all $x \in \Omega$ that

$$\left|\frac{v(x,t) - v_\infty(x)}{v_\infty(x)}\right| \leq C\|v(\cdot,t) - v_\infty\|_{C^1(\overline{\Omega})} \leq Ct^{-\theta} \quad \text{for all } t > 1.$$

That is,

$$\left\|\frac{v(\cdot,t)}{v_\infty} - 1\right\|_{L^\infty(\Omega)} \leq Ct^{-\theta} \quad \text{for all } t \geq 1.$$

Using Theorem 5.1 in [29] again, we have

$$\left\|\frac{v(\cdot,t)}{v_\infty}\right\|_{C^{1+p}(\overline{\Omega})} \leq C \quad \text{for all } t > 1.$$

Then by interpolation inequalities, we have

$$\left\|\frac{v(\cdot,t)}{v_\infty} - 1\right\|_{C^2(\overline{\Omega})} \leq Ct^{-\theta} \quad \text{for all } t \geq 1, \tag{75}$$

with a possibly different θ.

Now let us assume that Ω satisfies the condition (6). Similar to (72), there exists $C>0$ such that for all $\varphi \in H_0^1$ satisfying

$$-\Delta\varphi - pv_\infty^{p-1}\varphi = f \quad \text{in } \Omega,$$

there holds

$$\|\varphi\|_{H_0^1(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{76}$$

As before, on one hand, using the equation of v_∞, we have

$$F(v(t)) - F_\infty \le C\|v(\cdot,t) - v_\infty\|^2_{H_0^1(\Omega)}.$$

On the other hand, using the equation of v_∞, we have

$$\begin{aligned}
&\|\Delta v(\cdot,t) + v(\cdot,t)^p\|_{L^2(\Omega)}\\
&\ge \left\|\Delta(v(\cdot,t) - v_\infty) + pv_\infty^{p-1}(v(\cdot,t) - v_\infty)\right\|_{L^2(\Omega)}\\
&\quad - \left\|v(\cdot,t)^p - v_\infty^p - pv_\infty^{p-1}(v(\cdot,t) - v_\infty)\right\|_{L^2(\Omega)}\\
&\ge \frac{1}{C}\|v(\cdot,t) - v_\infty\|_{H_0^1(\Omega)} - C\left\|v_\infty^{\max(0,p-2)}|v(\cdot,t) - v_\infty|^{\min(2,p)}\right\|_{L^2(\Omega)}\\
&\ge \frac{1}{C}\|v(\cdot,t) - v_\infty\|_{H_0^1(\Omega)} - o(1)\|v(\cdot,t) - v_\infty\|_{H_0^1(\Omega)} \quad \text{for large } t\\
&\ge \frac{1}{C}\|v(\cdot,t) - v_\infty\|_{H_0^1(\Omega)} \quad \text{for large } t,
\end{aligned}$$

where we used (76) in the second inequality. The constant C depends only on n, p, Ω and u_0. Combining these two inequalities together, we have

$$F(v(t)) - F_\infty \le C\|\Delta v(\cdot,t) + v(\cdot,t)^p\|^2_{L^2(\Omega)} \le CM_2(t).$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(F(v(t)) - F_\infty) = -\frac{1}{p}M_2(t) \le -C(F(v(t)) - F_\infty),$$

and thus,

$$F(v(t)) - F_\infty \le Ce^{-\gamma t}$$

for some $C>0, \gamma>0$. Hence, the proof of (65) will give

$$\int_T^\infty M_2(t)^{1/2}\,\mathrm{d}t \le Ce^{-\gamma t}.$$

From (74), we obtain that

$$\|v(\cdot,t) - v_\infty\|_{L^{p+1}(\Omega)} \le Ce^{-\gamma t}.$$

Then the proof of (75) gives

$$\left\|\frac{v(\cdot,t)}{v_\infty} - 1\right\|_{C^2(\overline{\Omega})} \le Ce^{-\gamma t} \quad \text{for all } t \ge 1. \tag{77}$$

This finishes the proof of Theorem 1.2. ∎

Appendix A. Bubbles interactions

In the end of our proof of Proposition 4.5, we need to calculate and compare the following two quantities:

$$I_1 = \int_{\mathbb{R}^n} \left(\frac{\lambda_{1,\nu}}{1+\lambda_{1,\nu}^2|x-x_{1,\nu}|^2} \right)^{\frac{n+2}{2}} \left(\frac{\lambda_{2,\nu}}{1+\lambda_{2,\nu}^2|x-x_{2,\nu}|^2} \right)^{\frac{n-2}{2}} \mathrm{d}x,$$

$$I_2 = \int_{\mathbb{R}^n} \left\{ \Big(\frac{\lambda_{1,\nu}}{1+\lambda_{1,\nu}^2|x-x_{1,\nu}|^2} \Big) \vee \Big(\frac{\lambda_{2,\nu}}{1+\lambda_{2,\nu}^2|x-x_{2,\nu}|^2} \Big) \right\}^2 \cdot \left\{ \Big(\frac{\lambda_{1,\nu}}{1+\lambda_{1,\nu}^2|x-x_{1,\nu}|^2} \Big) \wedge \Big(\frac{\lambda_{2,\nu}}{1+\lambda_{2,\nu}^2|x-x_{2,\nu}|^2} \Big) \right\}^{n-2} \mathrm{d}x.$$

We want to show that under (32), there holds

$$I_1/\sqrt{I_2} = o(1) \quad \text{as } \nu \to \infty. \tag{78}$$

Recall that $\lambda_{1,\nu} \geq \lambda_{2,\nu}$.

If $x_{1,\nu} = x_{2,\nu}$, then using the change of variables: $y = \lambda_{2,\nu} x$ and $\Lambda = \lambda_{1,\nu}/\lambda_{2,\nu}$, we have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} \left(\frac{\Lambda}{1+\Lambda^2|y|^2} \right)^{\frac{n+2}{2}} \left(\frac{1}{1+|y|^2} \right)^{\frac{n-2}{2}} \mathrm{d}y \\ &= \left(\int_{|y|\leq \Lambda^{-1}} + \int_{\Lambda^{-1}\leq|y|\leq 1} + \int_{|y|\geq 1} \right) \left(\frac{\Lambda}{1+\Lambda^2|y|^2} \right)^{\frac{n+2}{2}} \left(\frac{1}{1+|y|^2} \right)^{\frac{n-2}{2}} \mathrm{d}y \\ &\leq C\Lambda^{\frac{2-n}{2}} \\ &\leq C\lambda_{1,\nu}^{\frac{2-n}{2}} \lambda_{2,\nu}^{\frac{n-2}{2}}, \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^n} \left\{ \Big(\frac{\Lambda}{1+\Lambda^2|y|^2} \Big) \vee \Big(\frac{1}{1+|y|^2} \Big) \right\}^2 \left\{ \Big(\frac{\Lambda}{1+\Lambda^2|y|^2} \Big) \wedge \Big(\frac{1}{1+|y|^2} \Big) \right\}^{n-2} \mathrm{d}y \\ &\geq \int_{|y|\leq 1/\sqrt{\Lambda}} \left(\frac{\Lambda}{1+\Lambda^2|y|^2} \right)^2 \left(\frac{1}{1+|y|^2} \right)^{n-2} \mathrm{d}y \\ &\geq 2^{2-n}\Lambda^{2-n} \int_{|z|\leq\sqrt{\Lambda}} \left(\frac{1}{1+|z|^2} \right)^2 \mathrm{d}y \\ &\geq c\Lambda^{2-n}\log\Lambda \quad (\text{ since } n\geq 4 \text{ and } \Lambda\geq 1) \\ &\geq c\lambda_{1,\nu}^{2-n}\lambda_{2,\nu}^{n-2}\ln(\lambda_{1,\nu}/\lambda_{2,\nu}). \end{aligned}$$

Since $\lambda_{1,\nu}/\lambda_{2,\nu} \to \infty$, we have (78).

If $x_{1,\nu} \neq x_{2,\nu}$, then we use the following change of variables:

$$\tilde\lambda_{1,\nu} = \lambda_{1,\nu}|x_{2,\nu}-x_{1,\nu}|,\ \tilde\lambda_{2,\nu} = \lambda_{2,\nu}|x_{2,\nu}-x_{1,\nu}|,\ e_\nu = \frac{x_{2,\nu}-x_{1,\nu}}{|x_{2,\nu}-x_{1,\nu}|}.$$

Then

$$I_1 = \int_{\mathbb{R}^n} \left(\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2} \right)^{\frac{n+2}{2}} \left(\frac{\tilde\lambda_{2,\nu}}{1+\tilde\lambda_{2,\nu}^2|x-e_\nu|^2} \right)^{\frac{n-2}{2}}.$$

By (32), we know that

$$\frac{\tilde\lambda_{1,\nu}}{\tilde\lambda_{2,\nu}}+\frac{\tilde\lambda_{2,\nu}}{\tilde\lambda_{1,\nu}}+\tilde\lambda_{1,\nu}\tilde\lambda_{2,\nu}\to\infty.$$

Recall that $\lambda_{1,\nu}\geq\lambda_{2,\nu}$ for all $\nu=1,2,\dots$. Hence, if $\{\tilde\lambda_{1,\nu}\}$ is bounded, then $\tilde\lambda_{2,\nu}\to 0$ and $\frac{\tilde\lambda_{1,\nu}}{\tilde\lambda_{2,\nu}}\to\infty$.

Case A. $\tilde\lambda_{2,\nu}\geq 1$. Then $\tilde\lambda_{1,\nu}\to\infty$, and thus

$$\begin{aligned} I_1&=\int_{B_{1/4}}\left(\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2}\right)^{\frac{n+2}{2}}\left(\frac{\tilde\lambda_{2,\nu}}{1+\tilde\lambda_{2,\nu}^2|x-e_\nu|^2}\right)^{\frac{n-2}{2}}\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^n\setminus B_{1/4}}\left(\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2}\right)^{\frac{n+2}{2}}\left(\frac{\tilde\lambda_{2,\nu}}{1+\tilde\lambda_{2,\nu}^2|x-e_\nu|^2}\right)^{\frac{n-2}{2}}\,\mathrm{d}x\\ &\leq C\tilde\lambda_{1,\nu}^{\frac{2-n}{2}}\tilde\lambda_{2,\nu}^{\frac{2-n}{2}}. \end{aligned}$$

Case B. Otherwise. Then

$$I_1\leq\tilde\lambda_{2,\nu}^{\frac{n-2}{2}}\int_{\mathbb{R}^n}\left(\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2}\right)^{\frac{n+2}{2}}\,\mathrm{d}x\leq C\tilde\lambda_{1,\nu}^{\frac{2-n}{2}}\tilde\lambda_{2,\nu}^{\frac{n-2}{2}}.$$

For I_2, we have

$$\begin{aligned} I_2=\int_{\mathbb{R}^n}&\left\{\Big(\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2}\Big)\vee\Big(\frac{\tilde\lambda_{2,\nu}}{1+\tilde\lambda_{2,\nu}^2|x-e_\nu|^2}\Big)\right\}^2\\ &\cdot\left\{\Big(\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2}\Big)\wedge\Big(\frac{\tilde\lambda_{2,\nu}}{1+\tilde\lambda_{2,\nu}^2|x-e_\nu|^2}\Big)\right\}^{n-2}\,\mathrm{d}x. \end{aligned}$$

Let $\varepsilon>0$ be sufficiently small. The constant c in the below will be independent of ε.

Case A. $\tilde\lambda_{2,\nu}\geq 1$. Then $\tilde\lambda_{1,\nu}\to\infty$. We split it into two cases:

Case A1. $\tilde\lambda_{2,\nu}\geq\varepsilon\tilde\lambda_{1,\nu}$. Then for ν large,

$$\begin{aligned} I_2&\geq\int_{B_{1/2}(e_\nu)}\left\{\frac{\tilde\lambda_{2,\nu}}{1+\tilde\lambda_{2,\nu}^2|x-e_\nu|^2}\right\}^2\cdot\left\{\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2}\right\}^{n-2}\,\mathrm{d}x\\ &\geq c\tilde\lambda_{2,\nu}^{2-n}\tilde\lambda_{1,\nu}^{2-n}\int_{B_{\lambda_{2,\nu}/2}}\left\{\frac{1}{1+|x|^2}\right\}^2\,\mathrm{d}x\\ &\geq c\tilde\lambda_{2,\nu}^{2-n}\tilde\lambda_{1,\nu}^{2-n}(\ln\tilde\lambda_{2,\nu})\quad\text{if } n\geq 4\\ &\geq c\tilde\lambda_{2,\nu}^{2-n}\tilde\lambda_{1,\nu}^{2-n}\ln(\varepsilon\tilde\lambda_{1,\nu})\quad\text{if } n\geq 4. \end{aligned}$$

Case A2. $1 \le \tilde\lambda_{2,\nu} < \varepsilon\tilde\lambda_{1,\nu}$. Then for ν large,

$$\begin{aligned}I_2 &\ge \int_{B_{1/2}\setminus B_{2\sqrt{\varepsilon}}} \left\{\frac{\tilde\lambda_{2,\nu}}{1+\tilde\lambda_{2,\nu}^2|x-e_\nu|^2}\right\}^2 \cdot \left\{\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2}\right\}^{n-2} \mathrm{d}x\\ &\ge c\tilde\lambda_{2,\nu}^{-2}\tilde\lambda_{1,\nu}^{2-n}\int_{B_{1/2}\setminus B_{2\sqrt{\varepsilon}}}|x|^{4-2n}\,\mathrm{d}x\\ &\ge c\tilde\lambda_{2,\nu}^{2-n}\tilde\lambda_{1,\nu}^{2-n}|\ln\varepsilon| \quad \text{if } n\ge 4.\end{aligned}$$

Case B. $\tilde\lambda_{2,\nu} \le 1$. Then $\frac{\tilde\lambda_{1,\nu}}{\tilde\lambda_{2,\nu}} \to \infty$. We split it into two cases.

Case B1. $\tilde\lambda_{1,\nu}\tilde\lambda_{2,\nu} \ge \varepsilon^{-1}$. Then $\tilde\lambda_{1,\nu} \to \infty$. We have

$$\begin{aligned}I_2 &\ge \int_{B_{1/2}\setminus B_{2\sqrt{\varepsilon}}} \left\{\frac{\tilde\lambda_{2,\nu}}{1+\tilde\lambda_{2,\nu}^2|x-e_\nu|^2}\right\}^2 \cdot \left\{\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2}\right\}^{n-2} \mathrm{d}x\\ &\ge c\tilde\lambda_{2,\nu}^{2}\tilde\lambda_{1,\nu}^{2-n}\int_{B_{1/2}\setminus B_{2\sqrt{\varepsilon}}}|x|^{4-2n}\,\mathrm{d}x\\ &\ge c\tilde\lambda_{2,\nu}^{n-2}\tilde\lambda_{1,\nu}^{2-n}|\ln\varepsilon| \quad \text{if } n\ge 4.\end{aligned}$$

Case B2. $\tilde\lambda_{1,\nu}\tilde\lambda_{2,\nu} \le \varepsilon^{-1}$. Then for large ν,

$$\begin{aligned}I_2 &\ge c\int_{B_{\sqrt{1/(2\tilde\lambda_{1,\nu}\tilde\lambda_{2,\nu})}}} \left\{\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2}\right\}^2 \left\{\frac{\tilde\lambda_{2,\nu}}{1+\tilde\lambda_{2,\nu}^2|x-e_\nu|^2}\right\}^{n-2} \mathrm{d}x\\ &\ge c\tilde\lambda_{2,\nu}^{n-2}\int_{B_{\sqrt{1/(2\tilde\lambda_{1,\nu}\tilde\lambda_{2,\nu})}}} \left\{\frac{\tilde\lambda_{1,\nu}}{1+\tilde\lambda_{1,\nu}^2|x|^2}\right\}^2 \mathrm{d}x \quad (\text{since } \tilde\lambda_{2,\nu}^2|x-e_\nu|^2 \le 3)\\ &\ge c\tilde\lambda_{1,\nu}^{2-n}\tilde\lambda_{2,\nu}^{n-2}\ln(\tilde\lambda_{1,\nu}/\tilde\lambda_{2,\nu}) \quad \text{if } n\ge 4.\end{aligned}$$

Therefore, (78) holds.

The following calculus lemma was used.

Lemma A.1. *For $p > 2$, $0 \le \varepsilon \le 1$, we have*

$$(1+\varepsilon)^p \ge 1+\varepsilon^p+p\varepsilon+c_p\varepsilon^2$$

and

$$(1+\varepsilon)^p \ge 1+\varepsilon^p+p\varepsilon^{p-1}+c_p\varepsilon$$

for some $c_p > 0$ independent of ε.

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Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

Email address: `tianlingjin@ust.hk`

School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, MOE, Beijing Normal University, Beijing 100875, People's Republic of China

Email address: `jx@bnu.edu.cn`