TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 10, Pages 1356–1410 (November 3, 2023)
https://doi.org/10.1090/btran/164

HOLOMORPHIC HÖRMANDER-TYPE FUNCTIONAL CALCULUS ON SECTORS AND STRIPS

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ABSTRACT. In this paper, the abstract multiplier theorems for 0-sectorial and 0-strip type operators by Kriegler and Weis [Math. Z. 289 (2018), pp. 405–444] are refined and generalized to arbitrary sectorial and strip-type operators. To this end, holomorphic Hörmander-type functions on sectors and strips are introduced, with even a finer scale of smoothness than the classical polynomial scale. Moreover, we establish alternative descriptions of these spaces involving Schwartz and "holomorphic Schwartz" functions. Finally, the abstract results are combined with a result by Carbonaro and Dragičević [Duke Math. J. 166 (2017), pp. 937–974] to obtain an improvement—with respect to the smoothness condition—of the known Hörmander-type multiplier theorem for general symmetric contraction semigroups.

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1. INTRODUCTION

In 1960, Hörmander made an important contribution to the theory of Fourier multiplier theorems by proving what is now known as the *Mikhlin–Hörmander multiplier theorem*, see [Hoe60, Theorem 2.5]:

Received by the editors July 15, 2022, and, in revised form, March 15, 2023, and May 23, 2023.
2020 *Mathematics Subject Classification.* Primary 47A60, 47D03, 46E35; Secondary 42B15.
Key words and phrases. Functional calculus, Hörmander function, abstract multiplier theorem.

Theorem 1.1 (Hörmander, 1960). *Let $m \in \mathrm{L}^\infty(\mathbb{R}_{>0})$, $N, d \in \mathbb{N}$ with $N > \frac{d}{2}$, and suppose that*

$$\sup_{R>0} \int_{\frac{R}{2}}^{2R} \left|s^k m^{(k)}(s)\right|^2 \frac{\mathrm{d}s}{s} < \infty \tag{1.1}$$

for all $1 \le k \le N$. Then, for every $1 < p < \infty$, the operator

$$\mathcal{S}(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d), \quad f \mapsto \mathcal{F}^{-1}\big(m(|\mathbf{x}|^2)\widehat{f}\big) = m(-\Delta)f$$

extends to a bounded operator on $\mathrm{L}^p(\mathbb{R}^d)$.

Here, $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions on $\mathbb{R}^d$. It has been noted subsequently that Hörmander's theorem can be extended to all functions f satisfying

$$\sup_{t>0} \|\eta \cdot m(t\mathbf{s})\|_{\mathrm{W}^{\beta,2}(\mathbb{R})} < \infty, \tag{1.2}$$

where $\eta \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}_{>0})$ is any non-zero test function with support contained in $\mathbb{R}_{>0} = (0,\infty)$, $\mathrm{W}^{\beta,2}(\mathbb{R})$ is the usual fractional L^2-Sobolev space, and $\beta > \frac{d}{2}$.

Hörmander's theorem is a result about the functional calculus for the operator $A = -\Delta$, where the Laplacian Δ is the generator of the Gauss-Weierstrass semigroup on $\mathbb{R}^d$. Attempts to generalise it have a long history, see e.g. [Heb90], [Med90], [MM90], [Chr91], [CM93], [Mue98], [GCM+99], [GCM+01], [Blu03], [MMS04], [Kri14], [KW17], [KW18], [DKP21]. A classical generalization is towards (negative) generators $-A$ of so-called *symmetric contraction semigroups*. That means, $-A$ generates a C_0-semigroup $(T_t)_{t\ge 0}$ of symmetric contractions on $\mathrm{L}^2(\Omega)$ for some σ-finite measure space Ω, such that each operator T_t extends to a contraction on each L^p-space.[1] As A is positive and self-adjoint on L^2, the spectral theorem provides a functional calculus in which the operator $m(A)$ is (well-defined and) bounded on L^2 whenever $m : \mathbb{R}_{\ge 0} \to \mathbb{C}$ is bounded and measurable. One says that m is an L^p*-spectral multiplier* if $m(A)$ extends to a bounded operator on $\mathrm{L}^p(\Omega)$. For $1 \le p < \infty$ this property is equivalent to an estimate of the form

$$\|m(A)f\|_p \le c\|f\|_p \qquad (f \in \mathrm{L}^2 \cap \mathrm{L}^p).$$

Generalizing (or adapting) Hörmander's theorem to this setting amounts to proving that m is an L^p-spectral multiplier if it satisfies (1.2) for some $\beta > 0$. Clearly, as the situation is more abstract than before, this requires additional assumptions. The following seminal result was obtained by Meda [Med90, Theorem 4].

Theorem 1.2 (Meda, 1990). *Let $-A$ be the (injective) generator of a symmetric contraction semigroup on some σ-finite measure space, and let $1 < p < \infty$. Suppose, in addition, that there are constants c, $\alpha \ge 0$ with*

$$\|A^{-\mathrm{i}s} f\|_p \le c(1+|s|)^\alpha \|f\|_p \qquad (f \in \mathrm{L}^p,\ s \in \mathbb{R}).$$

Then A has a bounded $\mathrm{H\ddot{o}r}^{\beta,2}(\mathbb{R}_{>0})$-calculus on L^p for all $\beta > \alpha + 1$.

Here, $\mathrm{H\ddot{o}r}^{\beta,2}(\mathbb{R}_{>0})$ is our notation for the space of all functions $m : \mathbb{R}_{>0} \to \mathbb{C}$ satisfying (1.2). (Whether this condition holds or not is independent of the chosen test function η, see also Remark 9.2.) Note that here we require A to be injective, to the effect that m need not be defined at 0. This restriction is inessential, cf. Remark 9.7.

[1] See Section 10 for a slightly more detailed account.

Meda's theorem has been generalized in two directions. The first one is by García-Cuerva, Mauceri, Meda, Sjögren, and Torrea [GCM+01, Theorem 2.2] and allows for an additional exponential factor in the assumptions about the growth of the imaginary powers.

Theorem 1.3 (Garcia-Cuerva et al., 2001)**.** *Let $-A$ be the (injective) generator of a symmetric contraction semigroup on some σ-finite measure space, let $1 < p < \infty$, and suppose there are constants c, $\alpha \ge 0$, and an angle $\omega \in (0, \frac{\pi}{2})$ with*

$$\|A^{-\mathrm{i}s} f\|_p \le c(1+|s|)^\alpha \mathrm{e}^{\omega|s|}\|f\|_p \qquad (f \in \mathrm{L}^p,\, s \in \mathbb{R}). \tag{1.3}$$

Then $m(A)$ extends to a bounded operator on L^p for every $m \in \mathrm{Hör}^{\beta,2}(\mathrm{S}_\omega)$, $\beta > \alpha + 1$.

Here, $\mathrm{Hör}^{\beta,2}(\mathrm{S}_\omega)$ is the space of all functions m which are bounded and holomorphic on the sector $\mathrm{S}_\omega := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \omega\}$ such that its boundary values on the rays $\mathrm{e}^{\pm \mathrm{i}\omega} \cdot \mathbb{R}_{>0}$ are contained in the "Hörmander space" $\mathrm{Hör}^{\beta,2}(\mathbb{R}_{>0})$. The sector S_ω as the (essential) domain of definition of m is natural here, as the exponential growth condition on the imaginary powers corresponds to the L^p-spectrum of A being contained in $\overline{\mathrm{S}_\omega}$.

More recently, Carbonaro and Dragičević in [CD17] could show that *each* negative generator A of a symmetric contraction semigroup satisfies the growth estimate (1.3) with $\alpha = \frac{1}{2}$ and $\omega = \omega_p$, where

$$\omega_p := \arcsin|1 - \tfrac{2}{p}|,$$

cf. Theorem 10.1. Combining this result with Theorem 1.3 then led them to Corollary 1.4, which up to now has been the strongest result about the Hörmander calculus for arbitrary generators of symmetric contraction semigroups.

Corollary 1.4 (Carbonaro and Dragičević, 2017)**.** *Let $-A$ be the (injective) generator of a symmetric contraction semigroup on some σ-finite measure space, and let $1 < p < \infty$. Then $m(A)$ extends to a bounded operator on L^p for every $m \in \mathrm{Hör}^{\beta,2}(\mathrm{S}_{\omega_p})$ and $\beta > \frac{3}{2}$.*

A different generalization of Meda's theorem was achieved by Kriegler and Weis in [Kri09, KW17, KW18] building on the holomorphic functional calculus for sectorial and strip-type operators on general Banach spaces. The authors construct, for a special class of 0-sectorial and 0-strip type operators, an abstract Hörmander functional calculus and infer abstract multiplier theorems (= boundedness of this Hörmander calculus) under additional "geometrical" conditions. As an application of their theory, they improved Meda's result as follows. (See [KW18, Theorem 6.1.(2)] for the even more general actual formulation.)

Theorem 1.5 (Kriegler and Weis, 2018)**.** *Let, on some Banach space X, A be a 0-sectorial operator with a bounded $\mathrm{H}^\infty(\mathrm{S}_\theta)$-calculus for some angle $\theta \in (0, \pi)$, and let $\alpha \ge 0$.*

(a) *Suppose that $\frac{1}{2} > \frac{1}{\mathrm{type}(X)} - \frac{1}{\mathrm{cotype}(X)}$ and that the set $\{(1+|s|)^{-\alpha} A^{-\mathrm{i}s} \mid s \in \mathbb{R}\}$ is bounded. Then A has a bounded $\mathrm{Hör}^{\beta,2}(\mathbb{R}_{>0})$-calculus for all $\beta > \alpha + \frac{1}{2}$. If, in addition, X has Pisier's contraction property, then this calculus is R-bounded.*

(b) *Suppose that the set* $\{(1+|s|)^{-\alpha}A^{-\mathrm{i}s} \mid s \in \mathbb{R}\}$ *is semi-R-bounded and X has Pisier's contraction property, then A has an R-bounded* $\mathrm{Hör}^{\beta,2}(\mathbb{R}_{>0})$*-calculus for each* $\beta > \alpha + \frac{1}{2}$.

Here, type(X) and cotype(X) denote the Rademacher type and cotype, respectively, of the Banach space X. For the other notions see Appendix B.

Meda's result is contained in Theorem 1.5. Indeed, it has been proved already by Cowling in [Cow83, Theorem 2] that if $-A$ is the (injective) generator of a symmetric contraction semigroup then A has a bounded H^∞-calculus on some sector. (This is also a consequence of Corollary 1.4.) Moreover, if $X = \mathrm{L}^p(\Omega)$ for $1 < p < \infty$ then $\mathrm{type}(X) = \min\{2, p\}$, $\mathrm{cotype}(X) = \max\{2, p\}$, and X has Pisier's contraction property. Hence, part (a) of Theorem 1.5 can be applied and one obtains an R-bounded $\mathrm{Hör}^{\beta,2}(\mathbb{R}_{>0})$-calculus for each $\beta > \alpha + \frac{1}{2}$ (and not just a bounded calculus for $\beta > \alpha + 1$ as in Meda's theorem).

Main results. In the present paper we generalize the Kriegler–Weis results towards general sectorial (i.e., not necessarily 0-sectorial) operators. Moreover, following a suggestion by Strichartz from [Str71] we refine the "smoothness" conditions in the description of the Hörmander and underlying Sobolev spaces employing more general (not necessarily polynomial) weights, the so-called "strongly admissible" functions (see Section 2).

Theorem 1.6. *Let A be a densely defined operator with dense range and a bounded* $\mathrm{H}^\infty(\mathrm{S}_\theta)$*-calculus for some* $\theta > 0$ *on some Banach space X. Suppose, in addition, that*

$$\|A^{-\mathrm{i}s}\| \le \tilde{v}(s)\mathrm{e}^{\omega|s|} \qquad (s \in \mathbb{R})$$

for some $\omega \ge 0$ *and some measurable function* $\tilde{v} : \mathbb{R} \to \mathbb{R}_{>0}$. *Let, furthermore,* $v : \mathbb{R} \to [1, \infty)$ *be strongly admissible. Then the following assertions hold.*

(a) *A has a bounded* $\mathrm{Hör}^2_v(\mathrm{S}_\omega)$*-calculus whenever* $\frac{\tilde{v}}{v} \in \mathrm{L}^1(\mathbb{R})$.

(b) *If* $r \in [1, 2]$ *is such that* $\frac{1}{r} > \frac{1}{\mathrm{type}(X)} - \frac{1}{\mathrm{cotype}(X)}$ *then A has a bounded* $\mathrm{Hör}^2_v(\mathrm{S}_\omega)$*-calculus whenever* $\frac{\tilde{v}}{v} \in \mathrm{L}^r(\mathbb{R})$. *If, in addition, X has Pisier's contraction property, then this calculus is R-bounded.*

(c) *If* $\{\tilde{v}(s)^{-1}\mathrm{e}^{-\omega|s|}A^{-\mathrm{i}s} \mid s \in \mathbb{R}\}$ *is semi-R-bounded and X has Pisier's contraction property, then A has an R-bounded* $\mathrm{Hör}^2_v(\mathrm{S}_\omega)$*-calculus whenever* $\frac{\tilde{v}}{v} \in \mathrm{L}^2(\mathbb{R})$.

Parts (b) and (c) generalize Theorem 1.5 (take $\omega = 0$, $\tilde{v}(s) = (1 + |s|)^\alpha$, $v(s) = (1 + |s|)^\beta$). Part (a) has no analogue in [KW18] but is somehow the most direct generalization of (the proof of) Theorem 1.3 to general Banach spaces.

If we combine Theorem 1.6 with the result of Carbonaro and Dragičević (Theorem 10.1), we obtain the following improvement of Corollary 1.4.

Corollary 1.7. *Let* $1 < p < \infty$ *and let* $-A_p$ *be the* L^p*-generator of a symmetric contraction semigroup. Then (the injective part of)* A_p *has an R-bounded* $\mathrm{Hör}^2_v(\mathrm{S}_{\omega_p})$*-calculus for each admissible function* $v : \mathbb{R} \to [1, \infty)$ *such that*

$$\frac{(1 + |\mathbf{s}|)^{\frac{1}{2}}}{v} \in \mathrm{L}^2(\mathbb{R}).$$

In particular, A_p *has an R-bounded* $\mathrm{Hör}^{\beta,2}(\mathrm{S}_{\omega_p})$*-calculus for each* $\beta > 1$.

Parts (a) and (b) of Theorem 1.6 (as well as the corresponding results about strip-type operators and parts of the underlying function theory) and Corollary 1.7 have first been established in the second author's Ph.D. thesis [Pa19].

Synopsis. Having sketched the main achievements of the paper, let us give a short synopsis. Roughly the paper is divided into four parts: function theory (Sections 2–5), operator theory (Sections 6–9), applications (Section 10) and appendices (Appendices A and B).

Function theory. In Section 2 we introduce (strongly) admissible functions, following an idea of Strichartz [Str71]. These functions shall then replace (viz. generalize) the polynomial weights in describing L^2-Sobolev spaces.

In Section 3 we introduce certain function spaces on vertical strips St_ω, e.g., the space of holomorphic Schwartz functions $\mathrm{H}^\infty_0(\mathrm{St}_\omega)$, the generalized Fourier algebra $\mathrm{A}_v(\overline{\mathrm{St}_\omega})$ and the generalized Sobolev space $\mathrm{W}^2_v(\mathrm{St}_\omega)$. In order to deal with boundary values we employ the classical Paley–Wiener theorem for the Hardy space $\mathrm{H}^2(\mathrm{St}_\omega)$, which is sufficient for our purposes. The function theory on sectors is reduced to the strip case via the exp / log-correspondence and is part of Section 9. (This allows us to work with Fourier instead of Mellin transforms.)

Section 4 is devoted to the spaces $\mathrm{H\ddot{o}r}^2_v(\mathbb{R})$ of (generalized) Hörmander functions on the real line. The main novelties here are that we replace test functions by general Schwartz functions in the definition and use admissible functions in contrast to just polynomial weights. The latter forbids using the standard complex interpolation technique (like in [CD17, Proof of Lemma 6]), which is why the quite detailed exposition in the preparatory Section 3 is necessary.

The subsequent Section 5 deals with the space $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$ of generalized Hörmander functions on a vertical strip. Theorem 5.3 is the central technical result, and the "Calderón type reproducing formula" (5.2)

$$\varphi f = \int_{\mathbb{R}} (\tau_t \psi^* \cdot \varphi)(\tau_t \psi \cdot f)\,\mathrm{d}t,$$

established in its proof, is the main auxiliary result. This formula needs "holomorphic cut-off functions" φ, ψ, which is the reason why our definition of the holomorphic Hörmander spaces differs from the classical one by virtue of boundary values. (Note that there are no non-trivial holomorphic test functions.) This also motivates in hindsight the generalization to Schwartz functions in the definition of $\mathrm{H\ddot{o}r}^2_v(\mathbb{R})$ in Section 4. At the end of the section, in Theorem 5.6, we confine to Hörmander spaces with respect to strongly admissible functions and show, in particular, that they coincide with what one would get by using the classical approach via boundary values.

Operator theory. In Section 6 we review the theory of holomorphic functional calculus for strip type operators and give conditions when an operator admits a (possibly unbounded) Hörmander calculus (Proposition 6.5). This generalizes the 0-sectorial case treated by Kriegler and Weis (Remark 6.6).

Section 7 is devoted to operators with a bounded Sobolev calculus. Section 8 then contains the principal results of the paper, Theorems 8.3, 8.4 and 8.8, about bounded Hörmander calculus on strips.

In Section 9 we briefly transfer the function theory from strips to sectors and introduce sectorial operators and their functional calculus. Finally, we transfer the chief results from the previous section to sectorial operators, leading to Theorems 9.4–9.6 (which are summarized in Theorem 1.6).

Applications. This short Section 10 specializes the results of the previous sections to generators of C_0-groups on Hilbert spaces, sectorial operators on L^p-spaces, generators of symmetric contraction semigroups (taking up the discussion in this section) and, as a special case, the Ornstein–Uhlenbeck operator.

Appendices. In Appendix A we recall some results from classical analysis related to Young's inequality and important for establishing the Calderón type reproducing formula for holomorphic Hörmander functions. In Appendix B we review some "Abstract Littlewood–Paley" theory, i.e., the Kalton–Weis theory of abstract square function estimates and its connection with bounded H^∞-calculus, and related notions from Banach space geometry (spaces of finite cotype, Pisier's contraction property, R-boundedness).

Notation and terminology. For a Banach space X the dual space is denoted by X' and the dual pairing by $\langle \cdot, \cdot \rangle_{X,X'}$. Also, $\mathcal{L}(X)$ is the Banach algebra of all bounded operators on X, with $\mathrm{I} \in \mathcal{L}(X)$ being the identity operator.

A possibly unbounded linear operator A between two Banach spaces X and Y is identified with its graph in $X \oplus Y$. By $\operatorname{dom} A$, $\operatorname{ran} A$, $\ker A$, we denote the domain, the range, and the null space of A, respectively. If $X = Y$ then $\sigma(A)$ and $\rho(A)$ denote the spectrum and the resolvent set of A, respectively. Whenever $\lambda \in \rho(A)$, we write $R(\lambda, A) := (\lambda - A)^{-1}$ for the resolvent operator of A at λ.

We employ the notation

$$F(x, f, \dots) \lesssim G(x, f, \dots) \qquad (x \in \cdots . f \in \dots, \dots)$$

as an abbreviation for "there is a real number $c \geq 0$ such that $F(x, f, \dots) \leq cG(x, f, \cdots)$ for all $x \in \dots$, $f \in \dots$ and $\dots$." That means, c may depend on other objects but not on $x, f, \dots$.

By $\mathbb{N}$ we denote the positive integers, i.e., $\mathbb{N} = \{1, 2, \dots\}$, and $\mathbb{R}_{>0} := (0, \infty)$. For $\omega > 0$ we use the notation

$$\mathrm{St}_\omega := \big\{ z \in \mathbb{C} \,\big|\, |\operatorname{Im} z| < \omega \big\}$$

for the strip of width 2ω symmetric about the real axis, and $\mathrm{St}_\omega := \mathbb{R}$ for $\omega = 0$. Similarly, for $0 < \omega \leq \pi$

$$\mathrm{S}_\omega := \exp(\mathrm{St}_\omega) = \big\{ z \in \mathbb{C} \setminus \{0\} \,\big|\, |\arg z| < \omega \big\}$$

and $\mathrm{S}_\omega = (0, \infty)$ for $\omega = 0$.

On any set M, $\mathbf{1}$ denotes the function which is constantly equal to 1. Also we sometimes write $\mathbf{z}$ or $\mathbf{s}$ to denote the identity mapping on M. Usually we use $\mathbf{z}$ whenever $M = O$ is a domain in the complex plane, and $\mathbf{s}$ or $\mathbf{t}$ when $M \subseteq \mathbb{R}$ is a real interval.

Whenever f is a function defined on a domain $M \subseteq \mathbb{C}$ and $z \in \mathbb{C}$, we let

$$\tau_z f := f(\cdot - z)$$

with domain $M + z$. Moreover, f^* is the function

$$f^*(z) := \overline{f(\overline{z})} \quad (\overline{z} \in M).$$

If $r \in \mathbb{R}$ is such that $\mathbb{R} + \mathrm{i}r \subseteq M$, then $f_{|r} := f(\mathbf{s} + \mathrm{i}r)$ is a function on $\mathbb{R}$.

For functions f, $g : \mathbb{R} \to \mathbb{C}$,

$$(f * g)(t) := \int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s \qquad (t \in \mathbb{R})$$

is the usual convolution of f and g (when well-defined). The Fourier transform of a function $f \in \mathrm{L}^1(\mathbb{R})$ is

$$(\mathcal{F}f)(t) := \widehat{f}(t) := \int_{\mathbb{R}} f(s)\mathrm{e}^{-\mathrm{i}st}\,\mathrm{d}s \qquad (t \in \mathbb{R})$$

and the inverse Fourier transform of f is

$$(\mathcal{F}^{-1}f)(t) := f^{\vee}(t) := \frac{1}{2\pi}\int_{\mathbb{R}} f(s)\mathrm{e}^{\mathrm{i}st}\,\mathrm{d}s \qquad (t \in \mathbb{R}).$$

We shall freely use standard results of elementary Fourier analysis.

Glossary of function spaces. For a metric space X we write $\mathrm{C}(X), \mathrm{C_b}(X)$ and $\mathrm{UC_b}(X)$ for the space of continuous, bounded and continuous, bounded and uniformly continuous functions on X, respectively. If X is, in addition, locally compact, we let $\mathrm{C_c}(X)$ be the space of continuous functions with compact support and $\mathrm{C}_0(X)$ the space of continuous functions vanishing at infinity.

For a non-empty open set $O \subseteq \mathbb{C}$ we let

$$\begin{aligned}\mathrm{Hol}(O) &:= \bigl\{f : O \to \mathbb{C} \bigm| f \text{ is holomorphic}\bigr\},\\ \mathrm{H}^\infty(O) &:= \bigl\{f \in \mathrm{Hol}(O) \bigm| f \text{ is bounded}\bigr\} \;=\; \mathrm{Hol}(O) \cap \mathrm{C_b}(O),\end{aligned}$$

the latter space being equipped with the supremum norm $\|f\|_{O,\infty} := \sup_{z\in O}|f(z)|$. The space of test functions on $\mathbb{R}$ is $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$, and $\mathcal{S}(\mathbb{R})$ denotes the space of Schwartz functions. For $\omega > 0$ and $1 \le p < \infty$, the Hardy space of order p on St_ω is

$$\mathrm{H}^p(\mathrm{St}_\omega) = \{f \in \mathrm{Hol}(\mathrm{St}_\omega) \mid \|f\|_{\mathrm{H}^p(\mathrm{St}_\omega)} < \infty\},$$

where

$$\|f\|^p_{\mathrm{H}^p(\mathrm{St}_\omega)} := \sup_{|r|<\omega}\int_{\mathbb{R}} |f(s+\mathrm{i}r)|^p\,\mathrm{d}s.$$

The following function spaces are defined in the course of the paper (page number in parentheses):

- $\mathrm{L}^p_{v,\omega}(\mathbb{R})$, $\mathrm{L}^p_v(\mathbb{R})$: weighted L^p-spaces (1366)
- $\mathrm{H}^\infty_0(\mathrm{St}_\omega)$, $\mathrm{H}^\infty_0[\overline{\mathrm{St}}_\omega]$, $\mathrm{H}^\infty_0(\mathrm{S}_\omega)$: holomorphic Schwartz functions (1368 and 1398)
- $\mathrm{A}(\overline{\mathrm{St}}_\omega)$: generalized Fourier algebra (1369)
- $\mathrm{W}^2_v(\mathrm{St}_\omega)$, $\mathrm{W}^2_v(\mathrm{S}_\omega)$ generalized L^2-Sobolev spaces (1370 and 1398)
- $\mathrm{W}^{\alpha,2}(\mathbb{R})$, $\mathrm{W}^{\alpha,2}(\mathrm{S}_\omega)$: classical L^2-Sobolev spaces (1370 and 1398)
- $\mathrm{H}^p(\mathrm{S}_\omega)$: Hardy space of order p on a sector (1397)
- $\mathrm{Hör}^2_v(\mathbb{R};\psi)$, $\mathrm{Hör}^2_v(\mathbb{R})$: generalized Hörmander functions on $\mathbb{R}$ (1373 and 1375)
- $\mathrm{Hör}^2_v(\mathrm{St}_;\psi)$, $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$: generalized Hörmander functions on St_ω (1376 and 1380)
- $\mathrm{Hör}^2_v(\mathrm{S}_\omega)$, $\mathrm{Hör}^{\beta,2}(\mathrm{S}_\omega)$: generalized and classical Hörmander functions on a sector, (1398)

- $\mathcal{E}(\mathrm{St}_\theta)$, $\mathcal{E}[\overline{\mathrm{St}}_\omega]$, $\mathcal{E}(\mathrm{S}_\theta)$, $\mathcal{E}[\overline{\mathrm{S}}^*_\omega]$ elementary functions on strips and sectors, (1382 and 1398).

2. Admissible weight functions

Classical fractional Sobolev spaces are defined via the (inverse) Fourier transform:

$$\mathrm{W}^{\alpha,2}(\mathbb{R}) = \big\{ f \in \mathrm{L}^2(\mathbb{R}) \,\big|\, (1+|\mathbf{s}|)^\alpha f^\vee \in \mathrm{L}^2(\mathbb{R}) \big\}, \tag{2.1}$$

where $\alpha > \frac{1}{2}$, see [Gra, Definition 6.2.2.] for example. The first step to a more general class of Sobolev spaces is to replace the polynomial weights $(1+|\mathbf{s}|)^\alpha$ in (2.1) by more general functions. This goes back to Strichartz [Str71].

Definition 2.1. A measurable function $v : \mathbb{R} \to (0,\infty)$ is called **admissible** if v has the following properties:

(1) $\forall\, s \in \mathbb{R}: \quad v(s) \geq 1$;

(2) $M_v := \sup\limits_{s,t\in\mathbb{R}} \dfrac{v(s+t)}{v(s)+v(t)} < \infty$.

An admissible function v is called **strongly admissible** if, in addition,

(3) $\frac{1}{v} \in \mathrm{L}^2(\mathbb{R})$.

Note that it is immediate from (1) and (2) that

$$v(s+t) \leq 2M_v v(s)v(t) \quad \text{and} \quad v(2t) \leq 2M_v\, v(t) \qquad (s,t \in \mathbb{R}).$$

Let us call a function $v : \mathbb{R} \to \mathbb{R}_{>0}$ **quasi-monotonic** if it is decreasing on $\mathbb{R}_{\leq 0}$ and increasing on $\mathbb{R}_{\geq 0}$. Lemma 2.2 helps to recognize admissible functions.

Lemma 2.2. *Let $v : \mathbb{R} \to (0,\infty)$ be quasi-monotonic with $v(0) \geq 1$. Then v is admissible if and only if*

$$\sup_{t\in\mathbb{R}} \frac{v(2t)}{v(t)} < \infty.$$

Proof. As seen above, each admissible function has the required property. Suppose $c \geq 0$ is such that $v(2t) \leq cv(t)$ for all $t \in \mathbb{R}$, and fix $s,t \in \mathbb{R}$ with $s \leq t$. Then $2s \leq s+t \leq 2t$. Hence, if $s+t \geq 0$, then $v(s+t) \leq v(2t) \leq cv(t) \leq c\,(v(t)+v(s))$ by monotonicity of v on $\mathbb{R}_{\geq 0}$. Likewise, if $s+t < 0$, then $v(s+t) \leq v(2s) \leq cv(s) \leq c(v(s)+v(t))$. ☐

We collect some properties of admissible functions.

Lemma 2.3. *Let v, $\tilde{v} : \mathbb{R} \to [1,\infty)$ be admissible functions. Then the following statements hold:*

(a) *v is locally bounded.*

(b) *v^α is admissible for each $\alpha \in \mathbb{R}_{\geq 0}$.*

(c) *If both v and $\tilde{v}$ are quasi-monotonic, then $v\tilde{v}$ is admissible and quasi-monotonic as well.*

(d) *For all θ, $\tilde{\theta} \in \mathbb{R}_{\geq 0}$ with $\theta + \tilde{\theta} = 1$ the function $\theta v + \tilde{\theta}\tilde{v}$ is admissible with $M_{\theta v + \tilde{\theta}\tilde{v}} \leq \max\{M_v, M_{\tilde{v}}\}$.*

Proof. (a) It suffices to show that v is locally bounded at $t = 0$, i.e., there is $\varepsilon > 0$ such that v is bounded on $(-\varepsilon, \varepsilon)$. Choose $R > 0$ such that

$$A_R := \{s \in \mathbb{R} \,|\, v(s) + v(-s) \le R\}$$

has Lebesgue measure greater than zero. By a theorem of Steinhaus [Str72], there is $\varepsilon > 0$ with

$$(-\varepsilon, \varepsilon) \subseteq A_R - A_R.$$

Now, property (2) of Definition 2.1 yields the claim.

(b) and (d) are immediate. For (c) use Lemma 2.2. □

Let us list some examples.

Example 2.4.

(1) (Polynomial weights) Let $v := 1 + |\mathbf{s}|$. Then v is admissible with $M_v = 1$. Consequently, by Lemma 2.3(b), $v_\alpha := v^\alpha$ is admissible for each $\alpha \ge 0$, and strongly admissible for $\alpha > \frac{1}{2}$. Note that v_α is even quasi-monotonic.

(2) (Mixed weights) Let $v_{\log} := \ln(\mathrm{e} + |\mathbf{s}|)$. Then $v_{\log}$ is quasi-monotonic. For each $s \in \mathbb{R}$

$$\frac{v_{\log}(2s)}{v_{\log}(s)} = \frac{\ln\left(\frac{\mathrm{e}}{2} + |s|\right) + \ln 2}{\ln(\mathrm{e} + |s|)} \le 1 + \frac{\ln 2}{\ln(\mathrm{e} + |s|)} \le 1 + \ln 2.$$

Hence, by Lemma 2.2, $v_{\log}$ is admissible. By Lemma 2.3,

$$v_{\alpha,\beta} := (1 + |\mathbf{s}|)^\alpha v_{\log}^\beta = (1 + |\mathbf{s}|)^\alpha \big(\ln(\mathrm{e} + |\mathbf{s}|)\big)^\beta$$

is a quasi-monotonic admissible function for each α, $\beta > 0$. Whenever either $\alpha > \frac{1}{2}$ and $\beta \ge 0$, or $\alpha = \frac{1}{2}$ and $\beta > 1$, the function $v_{\alpha,\beta}$ is strongly admissible.

(3) (Counterexample) The function

$$v := \exp\left(\sqrt{|\mathbf{s}|}\right)$$

is quasi-monotonic. However,

$$\frac{v(2s)}{v(s)} = \exp\left((\sqrt{2} - 1)\sqrt{|s|}\right) \xrightarrow{s \to \pm\infty} \infty.$$

Hence, v is not admissible.

The counterexample from above becomes clear also from Lemma 2.5.

Lemma 2.5. *Every admissible function grows at most polynomially. More precisely: Let $v : \mathbb{R} \to [0, \infty)$ be a function which is locally bounded at zero, and suppose that there is a constant $c > 0$ with*

$$v(2s) \le c v(s) \qquad (s \in \mathbb{R}).$$

Then there is $\alpha \ge 0$ with

$$\sup_{s \in \mathbb{R}} \frac{v(s)}{(1 + |s|)^\alpha} < \infty.$$

Proof. By passing, if necessary, to the function $\max\{v(s), v(-s)\}$ we may suppose that v is even. The hypotheses imply that v is bounded on $[0, 2]$. Choose $\alpha > 0$ large enough so that $c \leq 2^\alpha$. Then

$$\frac{v\big(2^n 2^r\big)}{\big(2^n 2^r\big)^\alpha} \leq c^n \frac{v\big(2^r\big)}{2^{\alpha n} 2^{\alpha r}} = \Big(\frac{c}{2^\alpha}\Big)^n \frac{v\big(2^r\big)}{2^{\alpha r}} \leq \sup_{1 \leq s \leq 2} v(s)$$

for each $0 \leq r \leq 1$ and $n \in \mathbb{N}_0$. As each $s \geq 1$ has a representation $s = 2^{n+r}$ for some $n \in \mathbb{N}_0$ and $r \in [0, 1]$, we obtain

$$\sup_{s \geq 1} \frac{v(s)}{s^\alpha} \leq \sup_{1 \leq s \leq 2} v(s).$$

Consequently,

$$\sup_{s \in \mathbb{R}} \frac{v(s)}{(1 + |s|)^\alpha} = \sup_{s \geq 0} \frac{v(s)}{(1 + s)^\alpha} \leq \sup_{0 \leq s \leq 2} v(s) < \infty,$$

which yields the claim. □

Two $\mathbb{R}_{>0}$-valued functions $v, \tilde{v}$ are called **equivalent** if there is $c > 0$ such that

$$\frac{1}{c} \tilde{v} \leq v \leq c \tilde{v}.$$

Evidently, if $v, \tilde{v}$ are equivalent, $\tilde{v} \geq 1$ and v is (strongly) admissible, then so is $\tilde{v}$.

Lemma 2.6. *Let $v : \mathbb{R} \to [1, \infty)$ be an admissible function. Then there is an admissible function $\tilde{v} \in \mathrm{C}^\infty(\mathbb{R})$ equivalent to v. If v is even then $\tilde{v}$ can be chosen to be even as well.*

Proof. Let $\eta \in \mathrm{C}_c^\infty(\mathbb{R})$ be a positive, even function with

$$\int_{\mathbb{R}} \eta(r) \, \mathrm{d}r = 1 \tag{2.2}$$

and such that $\eta|_{[0,\infty)}$ is decreasing. Then $\eta(2t) \leq \eta(t)$ for all $t \in \mathbb{R}$. Set $\tilde{v} := v * \eta$. Then $1 \leq \tilde{v} \in \mathrm{C}^\infty(\mathbb{R})$, and one has, for all $s, t \in \mathbb{R}$,

$$\begin{aligned}\tilde{v}(s + t) = \int_{\mathbb{R}} v(s + t - r)\eta(r) \, \mathrm{d}r &\leq M_v \bigg(\int_{\mathbb{R}} v(s - \tfrac{r}{2})\eta(r) \, \mathrm{d}r + \int_{\mathbb{R}} v(t - \tfrac{r}{2})\eta(r) \, \mathrm{d}r \bigg) \\ &= 2M_v \bigg(\int_{\mathbb{R}} v(s - r)\eta(2r) \, \mathrm{d}r + \int_{\mathbb{R}} v(t - r)\eta(2r) \, \mathrm{d}r \bigg) \leq 2M_v \big(\tilde{v}(s) + \tilde{v}(t)\big).\end{aligned}$$

Hence, $\tilde{v}$ is an admissible function. As v is locally bounded, choose $C > 0$ such that $v(t) \leq C$ for $t \in \operatorname{supp}(\eta)$. Then, by (2.2),

$$v(s) = \int_{\mathbb{R}} v(s)\eta(r) \, \mathrm{d}r \leq 2M_v \int_{\mathbb{R}} v(r)v(s - r)\eta(r) \, \mathrm{d}r \leq 2M_v C \, \tilde{v}(s),$$

and, as η is even,

$$\begin{aligned}\tilde{v}(s) = \int_{\mathbb{R}} v(s - r)\eta(r) \, \mathrm{d}r &= \int_{\mathbb{R}} v(s + r)\eta(r) \, \mathrm{d}r \\ &\leq 2M_v \int_{\mathbb{R}} v(r)\eta(r) \, \mathrm{d}r \, v(s) \leq 2M_v C \, v(s).\end{aligned}$$

If v is even then so is $\tilde{v}$, hence the claim is proved. □

Remark 2.7. We shall see later that equivalent admissible functions lead to the same function spaces with equivalent norms (see, e.g., Propositions 2.8, 3.3. 5.1). Thanks to Lemma 2.6, any appearing admissible function v can always be assumed to be smooth.

E.g., instead of the non-smooth polynomial function $v_\alpha(t) = (1 + |t|)^\alpha$, $\alpha \geq 0$, we may work with the equivalent smooth function $(1 + t^2)^{\frac{\alpha}{2}}$.

Let v be admissible, $\omega \in \mathbb{R}_{\geq 0}$ and $p \in [1, \infty]$. Then we may form the space

$$\mathrm{L}^p_{v,\omega}(\mathbb{R}) := \{f \in \mathrm{L}^p(\mathbb{R}) \mid \mathrm{e}^{\omega|\mathbf{s}|} v f \in \mathrm{L}^p(\mathbb{R})\},$$

where $\mathrm{L}^p_v(\mathbb{R}) := \mathrm{L}^p_{v,0}(\mathbb{R})$, for simplicity. Note that $\mathrm{e}^{\omega|\mathbf{s}|}$ and $\cosh(\omega\mathbf{s})$ are equivalent, and hence to describe the space $\mathrm{L}^p_{v,\omega}(\mathbb{R})$ we may replace the former by the latter whenever it is convenient, cf. part (b) below.

Let us collect some more or less well-known or obvious facts.

Proposition 2.8. *Let v be admissible, $\omega \in \mathbb{R}_{\geq 0}$ and $p \geq 1$. Then the following assertions hold:*

(a) *$\mathrm{L}^p_{v,\omega}(\mathbb{R})$ is a Banach space with respect to the norm $\|f\|_{\mathrm{L}^p_{v,\omega}} = \|v\mathrm{e}^{\omega|\mathbf{s}|} f\|_{\mathrm{L}^p}$, continuously embedded into $\mathrm{L}^p(\mathbb{R})$ and isomorphic to $\mathrm{L}^p(\mathbb{R})$ via the mapping $f \mapsto v\mathrm{e}^{\omega|\mathbf{s}|} f$.*

(b) *If $\tilde{v}$ is admissible with $\tilde{v} \lesssim v$, $\mathrm{L}^p_{v,\omega}(\mathbb{R}) \subseteq \mathrm{L}^p_{\tilde{v},\omega}(\mathbb{R})$, continuously. In particular, if $\tilde{v}$ and v are equivalent, then $\mathrm{L}^p_{v,\omega}(\mathbb{R}) = \mathrm{L}^p_{\tilde{v},\omega}(\mathbb{R})$ with equivalent norms.*

(c) *If $0 \leq \tilde{\omega} < \omega$ and $\tilde{v}$ is admissible, then $\mathrm{L}^p_{v,\omega}(\mathbb{R}) \subseteq \mathrm{L}^1_{\tilde{v},\tilde{\omega}}(\mathbb{R}) \cap \mathrm{L}^p_{\tilde{v},\tilde{\omega}}(\mathbb{R})$, continuously.*

(d) *If $p < \infty$ then $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ is dense in $\mathrm{L}^p_{v,\omega}(\mathbb{R})$.*

(e) *If $p < \infty$ then translation $(\tau_t)_{t\in\mathbb{R}}$ is a strongly continuous group on $\mathrm{L}^p_{v,\omega}(\mathbb{R})$ with $\|\tau_t\| \leq 2M_v \mathrm{e}^{\omega|t|} v(t)$ for $t \in \mathbb{R}$.*

(f) *If $f \in \mathrm{L}^1_{v,\omega}(\mathbb{R})$ and $g \in \mathrm{L}^p_{v,\omega}(\mathbb{R})$ then $f * g \in \mathrm{L}^p_{v,\omega}(\mathbb{R})$ with*

$$\|f * g\|_{\mathrm{L}^p_{v,\omega}(\mathbb{R})} \leq 2M_v \|f\|_{\mathrm{L}^1_{v,\omega}(\mathbb{R})} \|g\|_{\mathrm{L}^p_{v,\omega}(\mathbb{R})}.$$

If $f \in \mathrm{L}^{p'}_{v,\omega}(\mathbb{R})$ and $g \in \mathrm{L}^p_{v,\omega}(\mathbb{R})$ then $f * g \in \mathrm{C}(\mathbb{R}) \cap \mathrm{L}^\infty_{v,\omega}(\mathbb{R})$ with

$$\|f * g\|_{\mathrm{L}^\infty_{v,\omega}(\mathbb{R})} \leq 2M_v \|f\|_{\mathrm{L}^{p'}_{v,\omega}(\mathbb{R})} \|g\|_{\mathrm{L}^p_{v,\omega}(\mathbb{R})}.$$

(g) *If v is strongly admissible, then $\mathrm{L}^2_{v,\omega}(\mathbb{R})$ is (after scaling the norm) a Banach algebra with respect to convolution, continuously embedded into $\mathrm{L}^1_{\mathbf{1},\omega}(\mathbb{R})$.*

(h) *Let $\tilde{v}$ be admissible and quasi-monotonic with $v \lesssim \tilde{v}$, and let $\eta \in \mathrm{L}^1_{\tilde{v},\omega}(\mathbb{R})$. Define $\eta_n(\mathbf{s}) := n\eta(n\mathbf{s})$ for $n \in \mathbb{N}$. Then $\sup_{n\in\mathbb{N}} \|\eta_n\|_{\mathrm{L}^1_{v,\omega}(\mathbb{R})} < \infty$ and for each $f \in \mathrm{L}^p_{v,\omega}(\mathbb{R})$ one has $(\mathrm{e}^{\mathrm{i}t\mathbf{s}}\eta_n) * f \overset{n\to\infty}{\longrightarrow} \widehat{\eta}(0) \cdot f$ in $\mathrm{L}^p_{v,\omega}(\mathbb{R})$ uniformly in t from compact subsets of $\mathbb{R}$.*

Proof. (a) and (b) are straightforward. (c) follows from the computation

$$\tilde{v}\mathrm{e}^{\tilde{\omega}|\mathbf{s}|} f = \frac{\tilde{v}}{v} \mathrm{e}^{-(\omega-\tilde{\omega})|\mathbf{s}|}\, v\mathrm{e}^{\omega|\mathbf{s}|} f$$

and the fact that $\tilde{v}$ is dominated by some v_α.

(d) follows from (a) and (b) and the fact that we may replace v by a smooth equivalent and $\mathrm{e}^{\omega|\mathbf{s}|}$ by the (smooth) function $\cosh(\omega\mathbf{s})$.

(e) For $f \in \mathrm{L}^p_{v,\omega}(\mathbb{R})$ one has

$$\|\tau_t f\|_{\mathrm{L}^p_{v,\omega}} = \|v\mathrm{e}^{\omega \mathbf{s}}\tau_t f\|_{\mathrm{L}^p} = \|v(\mathbf{s}+t)\mathrm{e}^{\omega|\mathbf{s}+t|}f\|_{\mathrm{L}^p} \le 2M_v v(t)\mathrm{e}^{\omega|t|}\|f\|_{\mathrm{L}^p_{v,\omega}}$$

as $v(x+t) \le 2M_v v(x)v(t)$ for all $x, t \in \mathbb{R}$. Hence, each translation operator is bounded and the translation group is locally bounded. The strong continuity now follows from a density argument employing (c).

(f) Define $F := v\mathrm{e}^{\omega|\mathbf{s}|}|f|$ and $G := v\mathrm{e}^{\omega|\mathbf{s}|}|g|$. Then, for $s \in \mathbb{R}$,

$$\begin{aligned} v(s)\mathrm{e}^{\omega|s|}|(f*g)(s)| &\le v(s)\int_{\mathbb{R}} \mathrm{e}^{\omega|s-t|}|f(s-t)|\,\mathrm{e}^{\omega|t|}|g(t)|\,\mathrm{d}t \\ &\le 2M_v \int_{\mathbb{R}} v(s-t)\mathrm{e}^{\omega|s-t|}|f(s-t)|\,v(t)\mathrm{e}^{\omega|t|}|g(t)|\,\mathrm{d}t = 2M_v(F*G)(s). \end{aligned}$$

In the first case, $F \in \mathrm{L}^1(\mathbb{R})$ and $G \in \mathrm{L}^p(\mathbb{R})$, hence $F*G \in \mathrm{L}^p(\mathbb{R})$ again. This yields $f*g \in \mathrm{L}^p_{v,\omega}(\mathbb{R})$ and the claimed norm estimate. The same argument applies in the second case, where $F \in \mathrm{L}^{p'}(\mathbb{R})$ and $G \in \mathrm{L}^p(\mathbb{R})$. Note that here $f*g \in \mathrm{UC_b}(\mathbb{R})$ as $f \in \mathrm{L}^{p'}(\mathbb{R})$ and $g \in \mathrm{L}^p(\mathbb{R})$.

(g) Let $f, g \in \mathrm{L}^2_{v,\omega}(\mathbb{R})$. Since $\frac{1}{v} \in \mathrm{L}^2(\mathbb{R})$, one has

$$\mathrm{e}^{\omega|\mathbf{s}|}f = \frac{1}{v}(v\mathrm{e}^{\omega|\mathbf{s}|}f) \in \mathrm{L}^2(\mathbb{R})\cdot\mathrm{L}^2(\mathbb{R}) \subseteq \mathrm{L}^1(\mathbb{R})$$

with (hence)

$$\|f\|_{\mathrm{L}^1_{\mathbf{1},\omega}(\mathbb{R})} \le \|\tfrac{1}{v}\|_{\mathrm{L}^2(\mathbb{R})}\|f\|_{\mathrm{L}^2_{v,\omega}(\mathbb{R})}.$$

This proves the continuous embedding $\mathrm{L}^2_{v,\omega}(\mathbb{R}) \subseteq \mathrm{L}^1_{\mathbf{1},\omega}(\mathbb{R})$.

As $f, g \in \mathrm{L}^2(\mathbb{R})$, the convolution $f*g$ is defined pointwise and a function in $\mathrm{C}_0(\mathbb{R})$. Set $F := \mathrm{e}^{\omega|\mathbf{s}|}|f|$ and $G := \mathrm{e}^{\omega|\mathbf{s}|}|g|$. Then

$$\begin{aligned} v(s)\mathrm{e}^{\omega|s|}|(f*g)(s)| &\le v(s)\int_{\mathbb{R}} \mathrm{e}^{\omega|s-t|}|f(s-t)|\,\mathrm{e}^{\omega|t|}|g(t)|\,\mathrm{d}t \\ &\le M_v \int_{\mathbb{R}} \big(v(s-t)+v(t)\big)F(s-t)\,G(t)\,\mathrm{d}t \\ &= M_v(vF*G)(s) + M_v(F*vG)(s). \end{aligned}$$

By the first part, $F, G \in \mathrm{L}^1(\mathbb{R})$. It follows that

$$\begin{aligned} \|f*g\|_{\mathrm{L}^2_{v,\omega}(\mathbb{R})} &\le M_v\big(\|vF*G\|_{\mathrm{L}^2(\mathbb{R})} + \|F*vG\|_{\mathrm{L}^2(\mathbb{R})}\big) \\ &\le M_v\big(\|vF\|_{\mathrm{L}^2(\mathbb{R})}\,\|G\|_{\mathrm{L}^1(\mathbb{R})} + \|F\|_{\mathrm{L}^1(\mathbb{R})}\,\|vG\|_{\mathrm{L}^2(\mathbb{R})}\big) \\ &= M_v\big(\|f\|_{\mathrm{L}^2_{v,\omega}(\mathbb{R})}\,\|g\|_{\mathrm{L}^1_{\mathbf{1},\omega}(\mathbb{R})} + \|f\|_{\mathrm{L}^1_{\mathbf{1},\omega}(\mathbb{R})}\,\|g\|_{\mathrm{L}^2_{v,\omega}(\mathbb{R})}\big) \\ &\le \big(2M_v\|\tfrac{1}{v}\|_{\mathrm{L}^2(\mathbb{R})}\big)\,\|f\|_{\mathrm{L}^2_{v,\omega}(\mathbb{R})}\,\|g\|_{\mathrm{L}^2_{v,\omega}(\mathbb{R})}. \end{aligned}$$

(h) Observe that

$$v(s)\mathrm{e}^{\omega|s|}\eta_n(s) = \frac{v(s)}{\tilde{v}(s)}\frac{\tilde{v}(s)}{\tilde{v}(ns)}\frac{\mathrm{e}^{\omega|s|}}{\mathrm{e}^{n\omega|s|}}\cdot n\cdot(\tilde{v}\mathrm{e}^{\omega|\cdot|}\eta)(ns) \qquad (s \in \mathbb{R}).$$

This implies $\|\eta_n\|_{\mathrm{L}^1_{v,\omega}(\mathbb{R})} \le \|v/\tilde{v}\|_\infty\|\eta\|_{\mathrm{L}^1_{\tilde{v},\omega}(\mathbb{R})}$ for each $n \in \mathbb{N}$.

For the second assertion fix $f \in \mathrm{L}^p_{v,\omega}(\mathbb{R})$. It suffices to show that

$$(\mathrm{e}^{\mathrm{i}t\mathbf{s}}\eta_n)*f - \widehat{\eta}(\tfrac{-t}{n})f = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ts}\eta_n(s)(\tau_s f - f)\,\mathrm{d}s$$

converges to 0 in $\mathrm{L}^p_{v,\omega}(\mathbb{R})$ as $n \to \infty$ uniformly in $t \in \mathbb{R}$. Taking norms leads to the estimate

$$\int_{\mathbb{R}} |\eta_n(s)| \|\tau_s f - f\|_{\mathrm{L}^p_{v,\omega}(\mathbb{R})} \, \mathrm{d}s \lesssim \sup_{|s| \le \delta} \|\tau_s f - f\|_{\mathrm{L}^p_{v,\omega}(\mathbb{R})} + \int_{|s| \ge n\delta} |\eta(s)| \|\tau_{\frac{s}{n}} f - f\|_{\mathrm{L}^p_{v,\omega}(\mathbb{R})}$$

for each $\delta > 0$. For fixed $\delta > 0$ the second summand converges to 0 as $n \to \infty$. This follows from

$$\|\tau_{s/n} f - f\|_{\mathrm{L}^p_{v,\omega}(\mathbb{R})} \lesssim \mathrm{e}^{\omega|\frac{s}{n}|} v(\tfrac{s}{n}) \lesssim \mathrm{e}^{\omega|\frac{s}{n}|} \tilde{v}(\tfrac{s}{n}) \le \mathrm{e}^{\omega|s|} \tilde{v}(s) \qquad (s \in \mathbb{R}, n \in \mathbb{N})$$

and the fact that $\eta \tilde{v} \mathrm{e}^{\omega|\mathbf{s}|} \in \mathrm{L}^1(\mathbb{R})$. By (e), the first summand converges to 0 as $\delta \to 0$. This proves the claim. □

Remark 2.9.

(1) The second part of Assertion (f) can be strengthened in case $1 < p < \infty$. Indeed, one then has (in the proof) $F * G \in \mathrm{C}_0(\mathbb{R})$, and hence $v\mathrm{e}^{\omega|\mathbf{s}|}(f * g) \in \mathrm{C}_0(\mathbb{R})$ whenever v is continuous.

(2) Assertion (g) is essentially due to Strichartz. It is the only point where the subadditivity of v plays a role. For the convolution results only the submultiplicativity is needed.

3. Generalized Sobolev functions on strips

For this section, we fix an admissible function $v : \mathbb{R} \to \mathbb{R}_{\ge 1}$ and a number $\omega \in \mathbb{R}_{\ge 0}$. The **horizontal strip** of **height** $\omega \ge 0$ is

$$\mathrm{St}_\omega := \begin{cases} \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \omega\} & \text{if } \omega > 0, \\ \qquad \mathbb{R} & \text{if } \omega = 0. \end{cases}$$

Suppose, for the time being, $\omega > 0$. Then $\mathrm{Hol}(\mathrm{St}_\omega)$ is the space of all holomorphic functions on St_ω. If convenient, we identify a holomorphic function on some strip with its restriction to a smaller strip, so that we have

$$0 < \omega' < \omega \quad \Rightarrow \quad \mathrm{Hol}(\mathrm{St}_\omega) \subseteq \mathrm{Hol}(\mathrm{St}_{\omega'}) \subseteq \mathrm{C}^\infty(\mathbb{R}).$$

For $f \in \mathrm{Hol}(\mathrm{St}_\omega)$ and $|y| < \omega$ we denote by

$$f_{|y} : \mathbb{R} \to \mathbb{C}, \qquad f_{|y}(x) := f(x + \mathrm{i}y)$$

the restriction of f to the ordinate y. The space of bounded holomorphic functions on St_ω is

$$\mathrm{H}^\infty(\mathrm{St}_\omega) := \{f \in \mathrm{Hol}(\mathrm{St}_\omega) \mid \|f\|_{\infty,\mathrm{St}_\omega} := \sup_{z \in \mathrm{St}_\omega} |f(z)| < \infty\}.$$

Besides this space, we also shall consider the space

$$\mathrm{H}^\infty_0(\mathrm{St}_\omega) := \{f \in \mathrm{Hol}(\mathrm{St}_\omega) \mid \forall \alpha > 0 : |f(z)| \lesssim \frac{1}{(1 + |\operatorname{Re} z|)^\alpha}\}$$

of **holomorphic Schwartz functions** and, for $\omega \ge 0$,

$$\mathrm{H}^\infty_0[\overline{\mathrm{St}}_\omega] := \bigcup_{\theta > \omega} \mathrm{H}^\infty_0(\mathrm{St}_\theta).$$

(Recall that $\overline{\mathrm{St}}_0 = \overline{\mathbb{R}} = \mathbb{R}$, so that $\mathrm{H}^\infty_0[\mathbb{R}] = \mathrm{H}^\infty_0[\overline{\mathrm{St}}_0]$.) Note that by a classical theorem of Paley and Wiener [Rud, Theorem 7.22], $\widehat{\eta} \in \mathrm{H}^\infty_0(\mathrm{St}_\omega)$ for each test function $\eta \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ and $\omega > 0$. Lemma 3.1 tells that the name "holomorphic Schwartz function" is justified.

Lemma 3.1. *Let* $\omega \geq 0$ *and* $f \in \mathrm{H}_0^\infty[\overline{\mathrm{St}}_\omega]$. *Then* $f' \in \mathrm{H}_0^\infty[\overline{\mathrm{St}}_\omega]$ *as well. More precisely, if* $\alpha > 0$ *and* f *is holomorphic with* $|f| \lesssim (1 + |\operatorname{Re} \mathbf{z}|)^{-\alpha}$ *on* St_θ, *for some* $\theta > 0$, *then* $|f'| \lesssim (|1 + \operatorname{Re} \mathbf{z}|)^{-\alpha}$ *on each smaller strip.*

In particular, each function $f \in \mathrm{H}_0^\infty[\mathbb{R}]$ *restricts to a Schwartz function on* $\mathbb{R}$.

Proof. Suppose that $\theta > 0$ and f is holomorphic on St_θ with $|f| \lesssim (1 + |\operatorname{Re} \mathbf{z}|)^{-\alpha}$. Fix $0 < \varepsilon < \theta$ and consider $z \in \mathrm{St}_{\theta-\varepsilon}$. Then

$$|f'(z)| \leq \frac{1}{2\pi} \int_{|w-z|=\varepsilon} \frac{|f(w)|}{|w-z|^2} \, |\mathrm{d}w| \lesssim \sup_{|t| \leq \varepsilon} (1 + |\operatorname{Re} z - t|)^{-\alpha} \leq \frac{(1 + |\varepsilon|)^\alpha}{(1 + |\operatorname{Re} z|)^\alpha}$$

as claimed. □

We now return to the more general setting with $\omega \geq 0$, where $\omega = 0$ is also allowed. In the remainder of this section we shall be concerned with two spaces of functions on St_ω, $\mathrm{A}_v(\mathrm{St}_\omega)$ and $\mathrm{W}_v^2(\mathrm{St}_\omega)$. The former is a generalization of the classical Fourier algebra on $\mathbb{R}$ and plays an auxiliary role. The latter is a generalization of the classical fractional L^2-Sobolev space and the main object of interest. We start with the former.

The **generalized Fourier algebra** on $\overline{\mathrm{St}}_\omega$ with respect to v is

$$\mathrm{A}_v(\overline{\mathrm{St}}_\omega) := \{\widehat{g} \mid g \in \mathrm{L}^1_{v,\omega}(\mathbb{R})\}$$

endowed with the norm

$$\|\widehat{g}\|_{\mathrm{A}_v(\overline{\mathrm{St}}_\omega)} := \|g\|_{\mathrm{L}^1_{v,\omega}(\mathbb{R})} \qquad (g \in \mathrm{L}^1_{v,\omega}(\mathbb{R})).$$

If $v = \mathbf{1}$, we simply write $\mathrm{A}(\overline{\mathrm{St}}_\omega)$. Note that if, in addition, $\omega = 0$, then $\mathrm{A}(\overline{\mathrm{St}}_0) = \mathrm{A}(\mathbb{R}) = \mathcal{F}(\mathrm{L}^1(\mathbb{R}))$ is the classical Fourier algebra. (This justifies our terminology.)

Recall that if $g \in \mathrm{L}^1_{v,\omega}(\mathbb{R})$ then its Fourier transform $\widehat{g}$ is given by

$$\widehat{g}(z) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}sz} g(s) \, \mathrm{d}s = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}sz} \frac{\mathrm{e}^{-\omega|s|}}{v(s)} \left(v(s)\mathrm{e}^{\omega|s|} g(s)\right) \mathrm{d}s \qquad (z \in \mathbb{R}).$$

If $\omega = 0$, then $\widehat{g} \in \mathrm{C}_0(\mathbb{R})$ (Riemann-Lebesgue). If $\omega > 0$ then the formula above is meaningful for $|\operatorname{Im} z| \leq \omega$ and hence $\widehat{g}$ can be considered to be an element of $\mathrm{Hol}(\mathrm{St}_\omega) \cap \mathrm{C_b}(\overline{\mathrm{St}}_\omega)$. Moreover, one then has

$$(\mathrm{e}^{y\mathbf{s}} g)^\wedge = (\widehat{g})_{|y} \qquad (|y| \leq \omega).$$

Proposition 3.2 collects the relevant properties, some of them being rather obvious.

Proposition 3.2. *Let* $v, \tilde{v} : \mathbb{R} \to \mathbb{R}_{\geq 1}$ *be admissible functions and* $\omega, \tilde{\omega} \geq 0$.

(a) *The space* $\mathrm{A}_v(\overline{\mathrm{St}}_\omega)$ *is a separable Banach space isometrically isomorphic to* $\mathrm{L}^1_{v,\omega}(\mathbb{R})$ *and to* $\mathrm{L}^1(\mathbb{R})$.

(b) *If* $\tilde{v} \lesssim v$, *then* $\mathrm{A}_v(\overline{\mathrm{St}}_\omega) \subseteq \mathrm{A}_{\tilde{v}}(\overline{\mathrm{St}}_\omega)$, *continuously. In particular, if* $v, \tilde{v}$ *are equivalent, then* $\mathrm{A}_v(\overline{\mathrm{St}}_\omega) = \mathrm{A}_{\tilde{v}}(\overline{\mathrm{St}}_\omega)$ *with equivalent norms.*

(c) *Restriction yields a continuous embedding*

$$\mathrm{A}_v(\overline{\mathrm{St}}_\omega) \subseteq \mathrm{A}_{\tilde{v}}(\overline{\mathrm{St}}_{\tilde{\omega}}) \qquad (0 \leq \tilde{\omega} < \omega).$$

(d) *The space* $\mathcal{F}(\mathrm{C}_\mathrm{c}^\infty(\mathbb{R}))$ *is dense in* $\mathrm{A}_v(\overline{\mathrm{St}}_\omega)$.

(e) *One has* $\mathrm{A}_v(\overline{\mathrm{St}}_\omega) \subseteq \mathrm{C}_0(\overline{\mathrm{St}}_\omega)$, *continuously.*

(f) *With respect to pointwise multiplication,* $\mathrm{A}_v(\overline{\mathrm{St}}_\omega)$ *is a Banach algebra (after a scaling of the norm).*

(g) *The translation group acts strongly continuously and isometrically on* $\mathrm{A}_v(\overline{\mathrm{St}_\omega})$.

Proof. (a)–(d) follow, by construction of $\mathrm{A}_v(\overline{\mathrm{St}_\omega})$, directly from the respective assertions of Proposition 2.8.

(e) By construction and a trivial estimate, $\mathrm{A}_v(\overline{\mathrm{St}_\omega}) \subseteq \ell^\infty(\overline{\mathrm{St}_\omega})$ continuously. By (d) it suffices to prove that $\mathcal{F}(\mathrm{C}_\mathrm{c}^\infty(\mathbb{R})) \subseteq \mathrm{C}_0(\overline{\mathrm{St}_\omega})$. But this is true, since $\mathcal{F}(\mathrm{C}_\mathrm{c}^\infty(\mathbb{R})) \subseteq \mathrm{H}_0^\infty[\overline{\mathrm{St}_\omega}]$ as remarked above.

(f) This follows from Proposition 2.8(f) with $p = 1$.

(g) follows readily from the identity $\tau_t f = (\mathrm{e}^{it\mathbf{s}} g)^\wedge$ for $t \in \mathbb{R}$ and $f = \widehat{g} \in \mathrm{A}_v(\overline{\mathrm{St}_\omega})$, $g \in \mathrm{L}^1_{v,\omega}(\mathbb{R})$. □

As before, $\omega \geq 0$ and v is an admissible function. We now turn to the space of principal interest, which is the **generalized Sobolev space**

$$\mathrm{W}^2_v(\mathrm{St}_\omega) := \{\widehat{g} \mid g \in \mathrm{L}^2_{v,\omega}(\mathbb{R})\},$$

and endow it with the norm

$$\|\widehat{g}\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} := \|g\|_{\mathrm{L}^2_{v,\omega}(\mathbb{R})} \qquad (g \in \mathrm{L}^2_{v,\omega}(\mathbb{R})).$$

(Recall that $\mathrm{St}_0 = \mathbb{R}$, and hence $\mathrm{W}^2_v(\mathbb{R}) = \mathrm{W}^2_v(\mathrm{St}_0)$.) By definition of the Fourier transform, an element $f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$ is—a priori—just a tempered distribution on $\mathbb{R}$. However, since $\mathrm{L}^2_{v,\omega}(\mathbb{R}) \subseteq \mathrm{L}^2(\mathbb{R})$, Plancherel's theorem tells that $f \in \mathrm{L}^2(\mathbb{R})$. Furthermore, for $\omega > 0$ the next result shows that f may be considered to be a holomorphic function on St_ω.

Proposition 3.3. *Let $v, \tilde{v} : \mathbb{R} \to \mathbb{R}_{\geq 1}$ be admissible functions and $\omega, \tilde{\omega} \geq 0$.*

(a) *The space $\mathrm{W}^2_v(\mathrm{St}_\omega)$ is a separable Hilbert space. If $v = \mathbf{1}$ and $\omega = 0$ then $\mathrm{W}^2_v(\mathrm{St}_\omega) = \mathrm{W}^2_{\mathbf{1}}(\mathrm{St}_0) = \mathrm{L}^2(\mathbb{R})$.*

(b) *If $\tilde{v} \lesssim v$ then $\mathrm{W}^2_v(\mathrm{St}_\omega) \subseteq \mathrm{W}^2_{\tilde{v}}(\mathrm{St}_\omega)$ continuously. In particular, if $v, \tilde{v}$ are equivalent, then $\mathrm{W}^2_v(\mathrm{St}_\omega) = \mathrm{W}^2_{\tilde{v}}(\mathrm{St}_\omega)$ with equivalent norms.*

(c) *Restriction yields continuous embeddings*

$$\mathrm{W}^2_v(\mathrm{St}_\omega) \subseteq \mathrm{A}_{\tilde{v}}(\mathrm{St}_{\tilde{\omega}}) \cap \mathrm{W}^2_{\tilde{v}}(\mathrm{St}_{\tilde{\omega}}) \subseteq \mathrm{C}_0(\overline{\mathrm{St}_{\tilde{\omega}}}) \qquad (0 \leq \tilde{\omega} < \omega).$$

In particular, if $\omega > 0$ then $\mathrm{W}^2_v(\mathrm{St}_\omega)$ embeds continuously into $\mathrm{Hol}(\mathrm{St}_\omega)$ endowed with the topology of compact convergence through

$$f(z) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}zs} f^\vee(s)\,\mathrm{d}s \qquad (|\operatorname{Im} z| < \omega) \tag{3.1}$$

for $f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$.

(d) *If $\frac{\tilde{v}}{v} \in \mathrm{L}^2(\mathbb{R})$, then $\mathrm{W}^2_v(\mathrm{St}_\omega) \subseteq \mathrm{A}_{\tilde{v}}(\overline{\mathrm{St}_\omega})$ continuously. In particular, if v is strongly admissible, then*

$$\mathrm{W}^2_v(\mathrm{St}_\omega) \subseteq \mathrm{A}(\overline{\mathrm{St}_\omega}) \subseteq \mathrm{C}_0(\overline{\mathrm{St}_\omega})$$

continuously. Moreover, still with v being strongly admissible, $\mathrm{W}^2_v(\mathrm{St}_\omega)$ is (after a scaling of norm) a Banach algebra with respect to pointwise multiplication.

(e) *For $v = v_\alpha = (1 + |\mathbf{s}|)^\alpha$ and $\omega = 0$ the space $\mathrm{W}^2_{v_\alpha}(\mathrm{St}_\omega)$ coincides with the classical fractional Sobolev space $\mathrm{W}^{\alpha,2}(\mathbb{R})$.*

In particular, for $\alpha = N \in \mathbb{N}$, the space $\mathrm{W}^2_N(\mathbb{R})$ coincides with the classical L^2-Sobolev space of all $\mathrm{L}^2(\mathbb{R})$-functions with (distributional) derivatives up to order N being in L^2.

(f) *Pointwise multiplication is a bounded bilinear mapping*

$$\mathrm{A}_v(\overline{\mathrm{St}}_\omega) \times \mathrm{W}^2_v(\mathrm{St}_\omega) \to \mathrm{W}^2_v(\mathrm{St}_\omega).$$

(g) *The translation group $(\tau_t)_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on $\mathrm{W}^2_v(\mathrm{St}_\omega)$.*

Proof. (a) is clear, (b) follows from Proposition 2.8(b), and (c) follows from Proposition 2.8(c) and Proposition 3.2(e).

(d) If $\frac{\tilde v}{v} \in \mathrm{L}^2(\mathbb{R})$ then

$$\mathrm{L}^2_{v,\omega}(\mathbb{R}) = \frac{\tilde v}{v}\frac{\mathrm{e}^{-\omega|\mathbf{s}|}}{\tilde v}\mathrm{L}^2(\mathbb{R}) \subseteq \frac{\mathrm{e}^{-\omega|\mathbf{s}|}}{\tilde v}\mathrm{L}^1(\mathbb{R}) = \mathrm{L}^1_{\tilde v,\omega}(\mathbb{R}).$$

This proves the first assertion. Suppose that v is strongly admissible. Then one can take $\tilde v := \mathbf{1}$ in the first assertion and obtains the second. The third follows directly from Proposition 2.8(g).

(e) is classical.

(f) follows from Proposition 2.8(f) (with $p = 2$) and (g) follows from the identity $(\tau_t f)^\vee = \mathrm{e}^{\mathrm{i}t\mathbf{s}} f^\vee$ for $f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$ and $t \in \mathbb{R}$. □

Fix $\omega > 0$. One (other) classical theorem of Paley and Wiener [Kat, p.188/198] states that

$$\mathrm{W}^2_{\mathbf{1}}(\mathrm{St}_\omega) = \mathrm{H}^2(\mathrm{St}_\omega),$$

where the latter is the **Hardy space** of all functions $f \in \mathrm{Hol}(\mathrm{St}_\omega)$ such that

$$\|f\|^2_{\mathrm{H}^2} := \sup_{|y|<\omega} \|f_{|y}\|^2_{\mathrm{L}^2(\mathbb{R})} = \sup_{|y|<\omega} \int_{\mathbb{R}} |f(x+\mathrm{i}y)|^2\,\mathrm{d}x < \infty$$

with equivalence of norms

$$\|f\|_{\mathrm{W}^2_{\mathbf{1}}(\mathrm{St}_\omega)} \eqsim \|f\|_{\mathrm{H}^2(\mathrm{St}_\omega)}.$$

From (3.1) it follows readily that

$$f_{|y} = (\mathrm{e}^{y\mathbf{s}} f^\vee)^\wedge \qquad (|y| < \omega),$$

and, in particular, $f^\vee = f^\vee_{|0}$. The two L^2-functions

$$f_{|\pm\omega} := (\mathrm{e}^{\pm\omega\mathbf{s}} f^\vee)^\wedge \in \mathrm{L}^2(\mathbb{R})$$

are the **boundary values** of f on $\partial\mathrm{St}_\omega$. It is easily seen that the mapping

$$[-\omega,\omega] \to \mathrm{L}^2(\mathbb{R}), \quad y \mapsto f_{|y}$$

is continuous. In particular,

$$\lim_{r\nearrow\omega} f_{|r} = f_{|\omega} \quad \text{and} \quad \lim_{r\searrow-\omega} f_{|r} = f_{|_\omega}$$

in $\mathrm{L}^2(\mathbb{R})$ (which justifies the term "boundary value").

Employing these concepts we arrive at the following characterization of elements of $\mathrm{W}^2_v(\mathrm{St}_\omega)$.

Corollary 3.4. *Let $\omega > 0$ and $v : \mathbb{R} \to \mathbb{R}_{\geq 1}$ admissible. Then for a holomorphic function $f \in \mathrm{Hol}(\mathrm{St}_\omega)$ the following statements are equivalent:*

(i) *$f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$;*

(ii) *$f \in \mathrm{H}^2(\mathrm{St}_\omega)$ and $f_{|\pm\omega} \in \mathrm{W}^2_v(\mathbb{R})$;*

(iii) *$f_{|0} \in \mathrm{L}^2(\mathbb{R})$ and $f_{|0}^\vee \in \mathrm{L}^2_{v,\omega}(\mathbb{R})$.*

Moreover, there is equivalence of norms

$$\|f_{|\omega}\|^2_{\mathrm{W}^2_v(\mathbb{R})} + \|f_{|-\omega}\|^2_{\mathrm{W}^2_v(\mathbb{R})} \;\eqsim\; \|f\|^2_{\mathrm{W}^2_v(\mathrm{St}_\omega)}.$$

Proof. (i)⇒(ii): Let $f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$. Since $\mathbf{1} \le v$ we have $f \in \mathrm{W}^2_{\mathbf{1}}(\mathrm{St}_\omega) = \mathrm{H}^2(\mathrm{St}_\omega)$. By definition,

$$v(f_{|\pm\omega})^\vee = v\mathrm{e}^{\pm\omega\mathbf{s}} f^\vee = \frac{\mathrm{e}^{\pm\omega\mathbf{s}}}{\mathrm{e}^{\omega|\mathbf{s}|}}(v\mathrm{e}^{\omega|\mathbf{s}|}f^\vee) \in \mathrm{L}^\infty(\mathbb{R})\cdot\mathrm{L}^2(\mathbb{R}) \subseteq \mathrm{L}^2(\mathbb{R}).$$

This concludes the proof of (ii) and shows $\|f_{|\pm\omega}\|_{\mathrm{W}^2_v(\mathbb{R})} \lesssim \|f\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)}$.

(ii)⇒(iii): Suppose $f \in \mathrm{H}^2(\mathrm{St}_\omega)$ and $f_{|\pm\omega} \in \mathrm{W}^2_v(\mathbb{R})$. By definition of H^2, $f_{|0} \in \mathrm{L}^2(\mathbb{R})$. Let $g := f^\vee = f_{|0}^\vee$. Then

$$v\mathrm{e}^{\pm\omega\mathbf{s}} g = v(f_{|\pm\omega})^\vee \in \mathrm{L}^2(\mathbb{R})$$

and hence

$$v\mathrm{e}^{|\omega|\mathbf{s}} g = v\mathrm{e}^{\omega\mathbf{s}} g\mathbf{1}_{\mathbb{R}_{\ge 0}} + v\mathrm{e}^{-\omega\mathbf{s}} g\mathbf{1}_{\mathbb{R}_{<0}} \in \mathrm{L}^2(\mathbb{R})$$

as well. This proves (iii) and shows $\|f\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \lesssim \|f_{|\omega}\|_{\mathrm{W}^2_v(\mathbb{R})} + \|f_{|-\omega}\|_{\mathrm{W}^2_v(\mathbb{R})}$.

(iii)⇒(i): Suppose that (iii) holds and let $g := f_{|0}^\vee$. Then $g \in \mathrm{L}^2_{v,\omega}(\mathbb{R})$ and hence, by definition, $\widehat{g} \in \mathrm{W}^2_v(\mathrm{St}_\omega)$. By Fourier inversion $\widehat{g}_{|0} = f_{|0}$, and hence $f = \widehat{g}$ on St_ω by the identity theorem for holomorphic functions. □

Observe that $\mathrm{H}^\infty_0(\mathrm{St}_\omega) \subseteq \mathrm{H}^2(\mathrm{St}_\omega)$ and hence, by the Paley–Wiener theorem, $\mathrm{H}^\infty_0(\mathrm{St}_\omega) \subseteq \mathrm{W}^2_{\mathbf{1}}(\mathrm{St}_\omega)$. By Proposition 3.3(c) we therefore obtain

$$\mathrm{H}^\infty_0(\mathrm{St}_\theta) \subseteq \mathrm{W}^2_v(\mathrm{St}_\omega) \cap \mathrm{A}_v(\overline{\mathrm{St}}_\omega) \qquad (0 \le \omega < \theta).$$

We shall need the following approximation results.

Lemma 3.5. *Let v be an admissible function and $\omega \ge 0$.*

(a) *The space $\mathcal{F}(\mathrm{C}^\infty_c(\mathbb{R}))$ is dense in $\mathrm{W}^2_v(\mathrm{St}_\omega)$.*

(b) *The space $\mathcal{S}(\mathbb{R})$ is contained and $\mathrm{C}^\infty_c(\mathbb{R})$ is dense in $\mathrm{W}^2_v(\mathbb{R})$ and $\mathrm{A}_v(\mathbb{R})$.*

(c) *Let $\theta > \omega$. Then the space*

$$\{f \in \mathrm{Hol}(\mathrm{St}_\theta) \mid \exists\, a > 0 : |f| \lesssim \mathrm{e}^{-a|\operatorname{Re}\mathbf{z}|^2}\}$$

is dense in $\mathrm{W}^2_v(\mathrm{St}_\omega)$.

Proof. (a) By Proposition 2.8(d), $\mathrm{C}^\infty_c(\mathbb{R})$ is dense in $\mathrm{L}^2_{v,\omega}(\mathbb{R})$. Hence, $\mathcal{F}(\mathrm{C}^\infty_c(\mathbb{R}))$ is dense in $\mathrm{W}^2_v(\mathrm{St}_\omega)$. (This is analogous to Proposition 3.2(d).)

(b) If $\eta \in \mathcal{S}(\mathbb{R})$ then $\eta^\vee \in \mathcal{S}(\mathbb{R})$ as well, and since v grows at most polynomially, $v\eta^\vee \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$. It follows that $\mathcal{S}(\mathbb{R}) \subseteq \mathrm{A}_v(\mathbb{R}) \cap \mathrm{W}^2_v(\mathbb{R})$.

Next, we have to show that the space

$$E := \{v\eta^\vee \mid \eta \in \mathrm{C}^\infty_c(\mathbb{R})\}$$

is dense in $\mathrm{L}^2(\mathbb{R})$ and in $\mathrm{L}^1(\mathbb{R})$. We treat the L^2-case first, the L^1-case being similar. Take $g \in \mathrm{L}^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} v\eta^\vee g = 0 \quad \text{for all } \eta \in \mathrm{C}^\infty_c(\mathbb{R}).$$

Replacing η by $\eta * \varphi$ for $\eta, \varphi \in \mathrm{C}_c^\infty(\mathbb{R})$ we find

$$\int_{\mathbb{R}} v\eta^\vee \varphi^\vee g = 0 \quad \text{for all } \varphi,\, \eta \in \mathrm{C}_c^\infty(\mathbb{R}).$$

Since $\{\varphi^\vee \mid \varphi \in \mathrm{C}_c^\infty(\mathbb{R})\}$ is dense in $\mathrm{L}^2(\mathbb{R})$, it follows that

$$v\eta^\vee g = 0 \quad \text{for all } \eta \in \mathrm{C}_c^\infty(\mathbb{R}).$$

This implies $g = 0$ and hence (b) is proved for $\mathrm{W}_v^2(\mathbb{R})$. The proof for $\mathrm{A}_v(\mathbb{R})$ is analogous.

(c) Let $\mathrm{G}_\varepsilon = \mathrm{e}^{-\varepsilon \mathbf{z}^2}$, $\varepsilon > 0$, be a scaled Gaussian function and let $f = \widehat{\varphi}$ for some $\varphi \in \mathrm{C}_c^\infty(\mathbb{R})$. Then $f \in \mathrm{H}_0^\infty(\mathrm{St}_\theta) \subseteq \mathrm{H}^2(\mathrm{St}_\theta)$. Hence, $\mathrm{H}_0^\infty(\mathrm{St}_\theta) \ni \mathrm{G}_\varepsilon f \to f$ in $\mathrm{H}^2(\mathrm{St}_\theta)$ and, a fortiori, in $\mathrm{W}_v^2(\mathrm{St}_\omega)$ (Proposition 3.3(c) with $v = \mathbf{1}$). Now (c) follows from (a). □

Also the following result shall be needed later.

Lemma 3.6. *Let $v : \mathbb{R} \to \mathbb{R}_{\geq 1}$ be admissible, $0 \leq \omega \leq \theta$, and $\psi, \varphi \in \mathrm{H}_0^\infty[\overline{\mathrm{St}}_\theta]$. Then*

$$\sup_{z \in \mathrm{St}_{\theta-\omega}} \left(\int_{\mathbb{R}} \|\tau_t \psi \cdot \tau_z \varphi\|_{\mathrm{W}_v^2(\mathrm{St}_\omega)} \, \mathrm{d}t \right) + \sup_{z \in \mathrm{St}_{\theta-\omega}} \left(\int_{\mathbb{R}} \|\tau_t \psi \cdot \tau_z \varphi\|_{\mathrm{A}_v(\overline{\mathrm{St}}_\omega)} \, \mathrm{d}t \right) < \infty.$$

Proof. Fix $\theta' > \theta$ with $\psi, \varphi \in \mathrm{H}_0^\infty(\mathrm{St}_{\theta'})$ and define $\omega' := \omega + (\theta' - \theta) > \omega$. Since $\mathrm{H}^2(\mathrm{St}_{\omega'}) = \mathrm{W}_{\mathbf{1}}^2(\mathrm{St}_{\omega'}) \subseteq \mathrm{W}_v^2(\mathrm{St}_\omega) \cap \mathrm{A}_v(\overline{\mathrm{St}}_\omega)$ continuously, it suffices to prove

$$\sup_{z \in \mathrm{St}_{\theta-\omega}} \left(\int_{\mathbb{R}} \|\tau_t \psi \cdot \tau_z \varphi\|_{\mathrm{H}^2(\mathrm{St}_{\omega'})} \, \mathrm{d}t \right) < \infty.$$

Choose $\alpha, \beta > 1$. Then $|\psi| \lesssim (1 + |\operatorname{Re} \mathbf{z}|)^{-\alpha}$ and $|\varphi| \lesssim (1 + |\operatorname{Re} \mathbf{z}|)^{-(\alpha+\beta)}$ on St_θ. Hence, for $t \in \mathbb{R}$ and $z = x + \mathrm{i}y \in \mathrm{St}_{\theta-\omega}$,

$$\begin{aligned} \|\tau_t \psi \cdot \tau_z \varphi\|_{\mathrm{H}^2(\mathrm{St}_{\omega'})} &= \|\tau_{t-x} \psi \cdot \tau_{\mathrm{i}y} \varphi\|_{\mathrm{H}^2(\mathrm{St}_{\omega'})} \\ &= \sup_{|y'| < \omega} \|\tau_{t-x} \psi_{|y'} \cdot \varphi_{|y'-y}\|_{\mathrm{L}^2(\mathbb{R})} \lesssim (1 + |t - x|)^{-\alpha} \end{aligned}$$

by Lemma A.3 (and its proof). The claim follows. □

4. Generalized Hörmander functions on the real line

In this section we generalize the classical Hörmander spaces on the real line. Unless otherwise specified, v denotes an arbitrary admissible function and $0 \neq \psi \in \mathrm{C}^\infty(\mathbb{R})$.

A distribution f is called a **generalized Hörmander function** with respect to the **localizing function** ψ, if $\tau_t \psi \cdot f \in \mathrm{W}_v^2(\mathbb{R})$ for all $t \in \mathbb{R}$ and

$$\|f\|_{\mathrm{Hör}_v^2(\mathbb{R};\psi)} := \sup_{t \in \mathbb{R}} \|\tau_t \psi \cdot f\|_{\mathrm{W}_v^2(\mathbb{R})} < \infty.$$

In this case, $f \in \mathrm{L}_{\mathbf{loc}}^2(\mathbb{R})$ and hence the use of the word "function" is justified. We define

$$\mathrm{Hör}_v^2(\mathbb{R}; \psi) := \{f \in \mathrm{L}_{\mathbf{loc}}^2(\mathbb{R}) \mid f \text{ is generalized Hörmander w.r.t. } \psi\}$$

and endow it with the norm(!) from above. Let us collect some immediate properties.

Proposition 4.1. *Let $v, \tilde{v}$ be admissible and $0 \neq \psi \in \mathrm{C}^\infty(\mathbb{R})$. Then the following assertions hold.*

(a) $\mathrm{Hör}_v^2(\mathbb{R};\psi)$ *is a Banach space continuously included in* $\mathrm{L}^2_{\mathbf{loc}}(\mathbb{R})$.

(b) *If* $\tilde{v} \lesssim v$, *then* $\mathrm{Hör}_v^2(\mathbb{R};\psi) \subseteq \mathrm{Hör}_{\tilde{v}}^2(\mathbb{R};\psi)$ *continuously. In particular, if* $v, \tilde{v}$ *are equivalent then* $\mathrm{Hör}_v(\mathbb{R};\psi) = \mathrm{Hör}_{\tilde{v}}(\mathbb{R};\psi)$ *with equivalent norms.*

(c) *If* v *is strongly admissible then* $\mathrm{Hör}_v^2(\mathbb{R};\psi) \subseteq \mathrm{C_b}(\mathbb{R})$ *continuously.*

(d) *Suppose* $\theta > 0$ *and* $\psi \in \mathrm{H}^2(\mathrm{St}_\theta)$. *Then restriction to the real line yields a continuous inclusion* $\mathrm{H}^\infty(\mathrm{St}_\theta) \subseteq \mathrm{Hör}_v^2(\mathbb{R};\psi)$.

Proof. (a) and (b) are straightforward.

(c) By Proposition 3.3(d), $\tau_t\psi \cdot f \in \mathrm{W}^2_v(\mathbb{R}) \subseteq \mathrm{A}(\mathbb{R}) \subseteq \mathrm{C}_0(\mathbb{R})$ for each $t \in \mathbb{R}$. As $\psi \neq 0$, it follows that $f \in \mathrm{C}(\mathbb{R})$. Now pick $x_0 \in \mathbb{R}$ with $|\psi(x_0)| > 0$; then for each $x \in \mathbb{R}$

$$f(x) = \frac{1}{\psi(x_0)}(\tau_{x-x_0}\psi \cdot f)(x)$$

and hence $|f(x)| \lesssim \|\tau_{x-x_0}\psi \cdot f\|_\infty \lesssim \|\tau_{x-x_0}\psi \cdot f\|_{\mathrm{W}^2_v(\mathbb{R})} \le \|f\|_{\mathrm{Hör}_v^2(\mathbb{R};\psi)}$.

(d) Let $f \in \mathrm{H}^\infty(\mathrm{St}_\theta)$. Then, by Proposition 3.3(c),

$$\|\tau_t\psi \cdot f\|_{\mathrm{W}^2_v(\mathbb{R})} \lesssim \|\tau_t\psi \cdot f\|_{\mathrm{H}^2(\mathrm{St}_\theta)} \le \|\psi\|_{\mathrm{H}^2(\mathrm{St}_\theta)}\|f\|_{\mathrm{H}^\infty(\mathrm{St}_\theta)}. \qquad \square$$

Our aim is to show that the space $\mathrm{Hör}_v^2(\mathbb{R};\psi)$ and its topology do not depend on ψ when ψ is a Schwartz function. We shall need Lemma 4.2.

Lemma 4.2. *Let* $0 \neq \psi \in \mathrm{C}^\infty(\mathbb{R})$ *and* f *a function on* $\mathbb{R}$.

(a) *Suppose that* f *is* ***locally in*** $\mathrm{W}^2_v(\mathbb{R})$, *i.e.* $\eta f \in \mathrm{W}^2_v(\mathbb{R})$ *for each* $\eta \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$. *Then for each* $\eta \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ *the mapping*

$$\mathbb{R} \to \mathrm{W}^2_v(\mathbb{R}), \qquad t \mapsto \tau_t\eta \cdot f$$

is continuous.

(b) *If* f *is such that* $\tau_t\psi \cdot f \in \mathrm{W}^2_v(\mathbb{R})$ *for each* $t \in \mathbb{R}$ *then* f *is locally in* $\mathrm{W}^2_v(\mathbb{R})$ *and the mapping*

$$\mathbb{R} \to \mathrm{W}^2_v(\mathbb{R}), \qquad t \mapsto \tau_t\psi \cdot f$$

is strongly measurable.

Proof. (a) Fix $a > 0$ and choose a test function φ such that $\varphi \equiv 1$ on $[-a,a] + \mathrm{supp}(\eta)$. Then $\tau_t\eta \cdot f = \tau_t\eta \cdot (\varphi f)$ for $|t| \le a$. Since $\varphi f \in \mathrm{W}^2_v(\mathbb{R})$, it suffices to show that $\tau_\mathbf{t}\eta \in \mathrm{C}(\mathbb{R};\mathrm{A}_v(\mathbb{R}))$. But this follows from Lemma 3.5(b) and Proposition 3.2(g).

(b) For the first assertion, use a smooth partition of unity and the fact that $\mathrm{C}^\infty_\mathrm{c}(\mathbb{R}) \subseteq \mathrm{A}_v(\mathbb{R})$.

Now fix $\varphi \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R})$ with $\varphi(0) = 1$ and let $\varphi_n := \varphi(\mathbf{s}/n)$ for $n \in \mathbb{N}$. As f is locally in $\mathrm{W}^2_v(\mathbb{R})$, by (a) the function $\tau_\mathbf{t}(\psi\varphi_n) \cdot f$ is continuous for each $n \in \mathbb{N}$. Therefore, it suffices to show that $\tau_t(\varphi_n\psi) \cdot f \to \tau_t\psi \cdot f$ in $\mathrm{W}^2_v(\mathbb{R})$ for each $t \in \mathbb{R}$.

Let $\eta := \varphi^\vee$ and $\eta_n := n\eta(n\mathbf{s}) = \varphi_n^\vee$ for $n \in \mathbb{N}$. Then η is a Schwartz function and hence satisfies the assumptions made in Proposition 2.8, part (h) with $\omega = 0$ and $\tilde{v} = v_\alpha = (1+|\mathbf{s}|)^\alpha$ for $\alpha > 0$ large enough. Hence, with $t \in \mathbb{R}$ fixed,

$$(\tau_t(\varphi_n\psi) \cdot f)^\vee = (\mathrm{e}^{\mathbf{i}\mathbf{s}t}\eta_n) * (\tau_t\psi \cdot f)^\vee \to \varphi(0) \cdot (\tau_t\psi \cdot f)^\vee = (\tau_t\psi \cdot f)^\vee$$

in $\mathrm{L}^2_{v,0}(\mathbb{R})$ as $n \to \infty$. Taking Fourier transforms yields $\tau_t(\varphi_n\psi) \cdot f \to \tau_t\psi \cdot f$ in $\mathrm{W}^2_v(\mathbb{R})$, and the proof is complete. $\square$

The classical Hörmander spaces are defined as above (with $v = v_\alpha = (1 + |\mathbf{s}|)^\alpha$) with test functions $\psi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$ as localizers. We shall show that one can, more generally, use Schwartz functions.

Theorem 4.3. *Let $\eta, \psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. If $f \in \mathrm{Hör}_v^2(\mathbb{R}; \eta)$, then $f \in \mathrm{Hör}_v^2(\mathbb{R}; \psi)$ with*

$$\|f\|_{\mathrm{Hör}_v^2(\mathbb{R};\psi)} \lesssim \|f\|_{\mathrm{Hör}_v^2(\mathbb{R};\eta)}.$$

Proof. As $\eta \neq 0$ there is no loss of generality to suppose that $\int_\mathbb{R} |\eta|^2 = 1$. We claim that

$$\psi f = \int_\mathbb{R} (\tau_t \overline{\eta} \cdot \psi)\,(\tau_t \eta \cdot f)\,\mathrm{d}t \tag{4.1}$$

as a Bochner integral in $\mathrm{W}_v^2(\mathbb{R})$. Note that the function $\tau_\mathbf{t} \overline{\eta} \cdot \psi$ is continuous $\mathbb{R} \to \mathrm{A}_v(\mathbb{R})$ and, by Lemma 4.2, the function $\tau_\mathbf{t} \eta \cdot f$ is strongly measurable $\mathbb{R} \to \mathrm{W}_v^2(\mathbb{R})$. Hence, the function $(\tau_\mathbf{t} \overline{\eta} \cdot \psi)\,(\tau_\mathbf{t} \eta \cdot f)$ is strongly measurable $\mathbb{R} \to \mathrm{W}_v^2(\mathbb{R})$. Moreover,

$$\int_\mathbb{R} \|(\tau_t \overline{\eta} \cdot \psi)\,(\tau_t \eta \cdot f)\|_{\mathrm{W}_v^2(\mathbb{R})}\,\mathrm{d}t \lesssim \int_\mathbb{R} \|\tau_t \overline{\eta} \cdot \psi\|_{\mathrm{A}_v(\mathbb{R})}\,\mathrm{d}t\, \|f\|_{\mathrm{Hör}_v^2(\mathbb{R};\eta)}.$$

In order to see that

$$\int_\mathbb{R} \|\tau_t \overline{\eta} \cdot \psi\|_{\mathrm{A}_v(\mathbb{R})}\,\mathrm{d}t < \infty,$$

we choose $N \in \mathbb{N}$ so large that $v/v_N \in \mathrm{L}^2(\mathbb{R})$. Then $\mathrm{W}^{N,2}(\mathbb{R}) = \mathrm{W}_{v_N}^2(\mathbb{R}) \subseteq \mathrm{A}_v(\mathbb{R})$ continuously. Since the norm in $\mathrm{W}^{N,2}(\mathbb{R})$ effectively looks at the L^2-norms of all derivatives up to order N, it suffices to have

$$\int_\mathbb{R} \|\tau_t \overline{\eta}^{(j)} \cdot \psi^{(N-j)}\|_{\mathrm{L}^2(\mathbb{R})}\,\mathrm{d}t < \infty \tag{4.2}$$

for all $0 \le j \le N$. As $\overline{\eta}^{(j)}$ and $\psi^{(N-j)}$ are both Schwartz functions, (4.2) follows from Lemma A.3.

Now that we know that the integral (4.1) exists in $\mathrm{W}_v^2(\mathbb{R})$ in the Bochner sense, it remains to determine its value. To this aim, let $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$ be arbitrary. Then

$$\varphi \int_\mathbb{R} (\tau_t \overline{\eta} \cdot \psi)\,(\tau_t \eta \cdot f)\,\mathrm{d}t = \int_\mathbb{R} \tau_t |\eta|^2 \cdot (\varphi\psi f) = \Big(\int_\mathbb{R} |\eta|^2\Big)\,\varphi\psi f = \varphi\psi f$$

in $\mathrm{L}^2(\mathbb{R})$ by Lemma A.2, since $\varphi\psi f \in \mathrm{L}^2(\mathbb{R})$. This implies (4.1).

Finally, take $s \in \mathbb{R}$ and replace ψ above by $\tau_s \psi$. Then we find

$$\|\tau_s \psi \cdot f\|_{\mathrm{W}_v^2(\mathbb{R})} \le \int_\mathbb{R} \|\tau_t \overline{\eta} \cdot \tau_s \psi\|_{\mathrm{A}_v(\mathbb{R})}\,\mathrm{d}t \|f\|_{\mathrm{Hör}_v^2(\mathbb{R};\eta)} = \int_\mathbb{R} \|\tau_t \overline{\eta} \cdot \psi\|_{\mathrm{A}_v(\mathbb{R})}\,\mathrm{d}t \|f\|_{\mathrm{Hör}_v^2(\mathbb{R};\eta)}$$

since $\|\tau_t \overline{\eta} \cdot \tau_s \psi\|_{\mathrm{A}_v(\mathbb{R})} = \|\tau_{t-s} \overline{\eta} \cdot \psi\|_{\mathrm{A}_v(\mathbb{R})}$ for all $t, s \in \mathbb{R}$. This concludes the proof. □

With the last result at hand we are in the position to define the **generalized Hörmander space** on $\mathbb{R}$ associated with the admissible function v as

$$\mathrm{Hör}_v^2(\mathbb{R}) := \mathrm{Hör}_v^2(\mathbb{R}; \psi),$$

with ψ being *any* non-zero Schwartz function.

5. Generalized Hörmander functions on strips

We now want to pass to holomorphic functions on strips. Again, v denotes an arbitrary admissible function and $\omega \geq 0$ is an arbitrary non-negative real number.

Let $0 \neq \psi \in \mathrm{H}_0^\infty[\overline{\mathrm{St}}_\omega]$. A function f on St_ω is called **generalized Hörmander** with respect to ψ, if $\tau_t\psi \cdot f \in \mathrm{W}_v^2(\mathrm{St}_\omega)$ for all $t \in \mathbb{R}$ and

$$\|f\|_{\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)} := \sup_{t\in\mathbb{R}} \|\tau_t\psi \cdot f\|_{\mathrm{W}_v^2(\mathrm{St}_\omega)} < \infty.$$

We let

$$\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi) := \{f \mid f \text{ is generalized Hörmander w.r.t. } \psi\}.$$

Note that if $\omega = 0$ then this definition is the same as the one in the previous section. More generally, restriction to $\mathbb{R}$ induces an embedding $\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi) \subseteq \mathrm{H\ddot{o}r}_v^2(\mathbb{R})$ (since it also induces an embedding $\mathrm{W}_v^2(\mathrm{St}_\omega) \subseteq \mathrm{W}_v^2(\mathbb{R})$ by Proposition 3.3(c)).

Note that each non-zero $\psi \in \mathrm{H}_0^\infty[\mathbb{R}]$ restricts to a non-zero Schwartz function on $\mathbb{R}$ (Lemma 3.1), hence $\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_0;\psi) = \mathrm{H\ddot{o}r}_v^2(\mathbb{R})$, as defined at the end of Section 4.

Proposition 5.1. *Let $v, \tilde{v}$ be admissible, $\omega > 0$ and $0 \neq \psi \in \mathrm{H}_0^\infty[\overline{\mathrm{St}}_\omega]$. Then the following assertions hold:*

(a) $\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)$ *is a Banach space with respect to the norm* $\|\cdot\|_{\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)}$ *with* $\mathrm{W}_v^2(\mathrm{St}_\omega) \subseteq \mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)$*, continuously.*

(b) *Restriction yields continuous embeddings*

$$\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi) \subseteq \mathrm{H}^\infty(\mathrm{St}_\theta) \quad \text{if } 0 < \theta < \omega \quad \text{and}$$
$$\mathrm{H}^\infty(\mathrm{St}_\theta) \subseteq \mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi) \quad \text{if } \theta > \omega.$$

In particular, $\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)$ *embeds continuously into* $\mathrm{Hol}(\mathrm{St}_\omega)$.

(c) *If $\tilde{v} \lesssim v$ then* $\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi) \subseteq \mathrm{H\ddot{o}r}_{\tilde{v}}^2(\mathrm{St}_\omega;\psi)$ *continuously. In particular, if v and $\tilde{v}$ are equivalent, then* $\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi) = \mathrm{H\ddot{o}r}_{\tilde{v}}^2(\mathrm{St}_\omega;\psi)$ *with equivalent norms.*

(d) *Pointwise multiplication is a bounded bilinear mapping*

$$\mathrm{A}_v(\overline{\mathrm{St}}_\omega) \times \mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi) \to \mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi).$$

Proof. (a) and (b) It is clear that $\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)$ is a linear space and $\|\cdot\|_{\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)}$ is a seminorm on it. The continuous inclusion $\mathrm{W}_v^2(\mathrm{St}_\omega) \subseteq \mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)$ follows since $\psi \in \mathrm{A}_v(\overline{\mathrm{St}}_\omega)$ whereon translation is a strongly continuous and isometric group.

Now let $f \in \mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)$. Then for each $t \in \mathbb{R}$

$$f = \frac{\tau_t\psi \cdot f}{\tau_t\psi} \quad \text{on} \quad [\,\tau_t\psi \neq 0\,].$$

As the zeroes of ψ form a discrete set, we can vary t and find $f \in \mathrm{Hol}(\mathrm{St}_\omega)$. Moreover, it follows that $\|\cdot\|_{\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)}$ is a norm.

Next, fix $0 < \theta < \omega$. Then $\mathrm{W}_v^2(\mathrm{St}_\omega) \subseteq \mathrm{H}^\infty(\mathrm{St}_\theta)$ continuously by Proposition 3.3(c). It follows that for each $t \in \mathbb{R}$ and $\delta > 0$

$$|f| = \frac{|\tau_t\psi \cdot f|}{|\tau_t\psi|} \leq \frac{1}{\delta}\|\tau_t\psi \cdot f\|_{\mathrm{H}^\infty(\mathrm{St}_\theta)} \lesssim \frac{1}{\delta}\|f\|_{\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)}$$

on the set $[\,|\tau_t\psi| > \delta\,] \cap \mathrm{St}_\theta$. Hence, by choosing $\delta > 0$ sufficiently small and varying $t \in \mathbb{R}$ we find

$$\|f\|_{\mathrm{H}^\infty(\mathrm{St}_\theta)} \lesssim \|f\|_{\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega;\psi)}.$$

This proves the first assertion of (b).

To show completeness, note that the mapping

$$\Phi : \mathrm{Hör}^2_v(\mathrm{St}_\omega;\psi) \to \ell^\infty(\mathbb{R};\mathrm{W}^2_v(\mathrm{St}_\omega)), \qquad \Phi f := (\tau_t\psi\cdot f)_{t\in\mathbb{R}}$$

is an isometric embedding. Hence, it suffices to show that $\operatorname{ran}\Phi$ is closed. To this end, let $(f_n)_n$ be a sequence in $\mathrm{Hör}^2_v(\mathrm{St}_\omega;\psi)$ and $g := (g_t)_t \in \ell^\infty(\mathbb{R};\mathrm{W}^2_v(\mathrm{St}_\omega))$ with $\Phi f_n \to g$. Then $(f_n)_n$ is Cauchy in $\mathrm{Hör}^2_v(\mathrm{St}_\omega;\psi)$ and hence there is $f \in \mathrm{Hol}(\mathrm{St}_\omega)$ such that $f_n \to f$ uniformly on each St_θ, $0<\theta<\omega$. It follows that $g_t = \tau_t\psi f$ for each $t\in\mathbb{R}$ and hence $f \in \mathrm{Hör}^2_v(\mathrm{St}_\omega;\psi)$ with $\Phi f = g$. This concludes the proof of (a).

It remains to show the second assertion in (b). Fix $\theta > \omega$ and $f \in \mathrm{H}^\infty(\mathrm{St}_\theta)$. Without loss of generality we may suppose that $\psi \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)$. Then $\tau_t\psi\cdot f \in \mathrm{H}^2(\mathrm{St}_\theta)$ with

$$\|\tau_t\psi\cdot f\|_{\mathrm{H}^2(\mathrm{St}_\theta)} \le \|f\|_{\mathrm{H}^\infty(\mathrm{St}_\theta)}\|\tau_t\psi\|_{\mathrm{H}^2(\mathrm{St}_\theta)} = \|f\|_{\mathrm{H}^\infty(\mathrm{St}_\theta)}\|\psi\|_{\mathrm{H}^2(\mathrm{St}_\theta)}.$$

As $\mathrm{H}^2(\mathrm{St}_\theta) = \mathrm{W}^1_{\mathbf{1}}(\mathrm{St}_\theta) \subseteq \mathrm{W}^2_v(\mathrm{St}_\omega)$ continuously, the assertion follows.

(c) is obvious and (d) follows from Proposition 3.3(f). □

Next, we aim at showing that the generalized Hörmander space $\mathrm{Hör}^2_v(\mathrm{St}_\omega;\psi)$ does not depend on the particular choice of $0 \ne \psi \in \mathrm{H}^\infty_0[\overline{\mathrm{St}_\omega}]$. For $\omega = 0$ this has been done in the previous section (Theorem 4.3), as $\mathrm{H}^\infty_0[\mathbb{R}]$-functions restrict to Schwartz functions on $\mathbb{R}$ (Lemma 3.1).

For $\omega > 0$ the matter is a little more delicate since we do not have the means of a (finite) partition of unity at hand, which played an important role in the proof of Lemma 4.2.

Recall that for $\omega \ge 0$, $\theta > 0$, and $\psi : \mathrm{St}_{\theta+\omega} \to \mathbb{C}$ we write

$$(\tau_z\psi)(w) \;=\; \psi(w-z) \quad (z\in\mathrm{St}_\theta,\ w\in\mathrm{St}_\omega).$$

Still, unless explicitly noted otherwise, v is an arbitrary admissible function.

Lemma 5.2. *Let $\theta > 0$, $\omega \ge 0$, $0 \ne \psi \in \mathrm{H}^\infty_0(\mathrm{St}_{\theta+\omega})$, and $f \in \mathrm{Hör}^2_v(\mathrm{St}_\omega;\psi)$. Then the following statements hold:*

(a) *The mapping*

$$\mathbb{R} \to \mathrm{W}^2_v(\mathrm{St}_\omega), \quad t \mapsto \tau_t\psi\cdot f$$

is weakly continuous.

(b) *If $\tau_z\psi\cdot f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$ for all $z \in \mathrm{St}_{\theta-\omega}$, and*

$$\sup_{z\in\mathrm{St}_\theta} \|\tau_z\psi\cdot f\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \;<\; \infty,$$

then the mapping

$$\mathrm{St}_\theta \to \mathrm{W}^2_v(\mathrm{St}_\omega), \quad z \mapsto \tau_z\psi\cdot f$$

is holomorphic.

Proof. For (a) let $\rho := 0$ and for (b), let $\rho := \theta$. Define

$$H : \mathrm{St}_\rho \to \mathrm{L}^2(\mathbb{R}), \qquad H(z) := \tilde{v}\cosh(\omega\mathbf{s})(\tau_z\psi\cdot f)^\vee,$$

where $\tilde{v}$ is equivalent to v and smooth (Lemma 2.6). By the assumptions made in (a) and (b), H is bounded.

Next, fix $\eta \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$ and define $\varphi := \tilde{v}\cosh(\omega\mathbf{s})\eta \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$. Then

$$I(z) := \int_\mathbb{R} H(z)\eta = \int_\mathbb{R} \tilde{v}\cosh(\omega\mathbf{s})(\tau_z\psi \cdot f)^\vee\, \eta = \int_\mathbb{R} (\tau_z\psi \cdot f)^\vee\, \varphi = \int_\mathbb{R} \tau_z\psi \cdot f \cdot \varphi^\vee$$

for each $z \in \mathrm{St}_\rho$. As $f_{|0} \in \mathrm{Hör}_v^2(\mathbb{R})$ and $\varphi^\vee \in \mathcal{S}(\mathbb{R})$, $f\varphi^\vee \in \mathrm{W}_v^2(\mathbb{R}) \subseteq \mathrm{L}^2(\mathbb{R})$. Hence, $I : \mathrm{St}_\rho \to \mathbb{C}$ is continuous.

Since $\mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$ is dense in $\mathrm{L}^2(\mathbb{R})$, this already completes the proof of (a). In case (b) one employs Morera's theorem in combination with a Fubini argument to see that I is holomorphic. □

Lemma 5.2 is preliminary, as the next result entails much stronger statements.

Theorem 5.3. *Let $\omega \geq 0$ and $0 \neq \psi \in \mathrm{H}_0^\infty[\overline{\mathrm{St}}_\omega]$. Furthermore, let $\theta > 0$ and $\varphi \in \mathrm{H}_0^\infty(\mathrm{St}_{\theta+\omega})$. Then the following assertions hold:*

(a) *For each $f \in \mathrm{Hör}_v^2(\mathrm{St}_\omega;\psi)$*

$$\tau_\mathbf{z}\varphi \cdot f \in \mathrm{H}^\infty(\mathrm{St}_\theta; \mathrm{W}_v^2(\mathrm{St}_\omega))$$

and

$$\sup_{z\in\mathrm{St}_\theta} \|\tau_z\varphi \cdot f\|_{\mathrm{W}_v^2(\mathrm{St}_\omega)} \lesssim \|f\|_{\mathrm{Hör}_v^2(\mathrm{St}_\omega;\psi)}$$

(with the constant not depending on f).

(b) *For each $f \in \mathrm{Hör}_v^2(\mathrm{St}_\omega;\psi)$ and $z \in \mathrm{St}_\theta$ one has $(\tau_z\varphi \cdot f)^\vee \in \mathrm{C}(\mathbb{R})$ with*

$$\sup_{z\in\mathrm{St}_\theta, s\in\mathbb{R}} v(s)\mathrm{e}^{\omega|s|}|(\tau_z\varphi \cdot f)^\vee(s)| \lesssim \|f\|_{\mathrm{Hör}_v^2(\mathrm{St}_\omega;\psi)}$$

(with the constant not depending on f).

(c) *If $\omega > 0$ one has the representation formula*

$$\varphi(0) \cdot f(z) = \int_\mathbb{R} (\tau_z\varphi \cdot f)^\vee(s)\mathrm{e}^{-isz}\,\mathrm{d}s \qquad (z \in \mathrm{St}_\theta \cap \mathrm{St}_\omega) \tag{5.1}$$

whenever $f \in \mathrm{Hör}_v^2(\mathrm{St}_\omega;\psi)$.

Proof. Define $\psi^* := \overline{\psi(\overline{\mathbf{z}})} \in \mathrm{H}_0^\infty[\overline{\mathrm{St}}_\omega]$ and fix $f \in \mathrm{Hör}_v^2(\mathrm{St}_\omega;\psi)$ and $z \in \mathrm{St}_\theta$. Then $\tau_z\varphi \in \mathrm{H}_0^\infty[\mathrm{St}_\omega]$.

(a) By Lemma 5.2, the function $F_z := \tau_\mathbf{t}(\psi\psi^*) \cdot (\tau_z\varphi \cdot f) = (\tau_\mathbf{t}\psi^* \cdot \tau_z\varphi) \cdot (\tau_\mathbf{t}\psi \cdot f)$ is bounded and weakly continuous $\mathbb{R} \to \mathrm{W}_v^2(\mathrm{St}_\omega)$. Moreover,

$$\sup_{z\in\mathrm{St}_\theta} \int_\mathbb{R} \|\tau_t(\psi\psi^*) \cdot (\tau_z\varphi \cdot f)\|_{\mathrm{W}_v^2(\mathrm{St}_\omega)}\,\mathrm{d}t$$
$$\lesssim \sup_{z\in\mathrm{St}_\theta} \int_\mathbb{R} \|\tau_t\psi^* \cdot \tau_z\varphi\|_{\mathrm{A}_v(\overline{\mathrm{St}}_\omega)}\,\mathrm{d}t\ \|f\|_{\mathrm{Hör}_v^2(\mathrm{St}_\omega;\psi)} < \infty$$

by Lemma 3.6. Since $\mathrm{W}_v^2(\mathrm{St}_\omega)$ is reflexive, the integral

$$g_z := \int_\mathbb{R} \tau_t(\psi^*\psi) \cdot \tau_z\varphi \cdot f\,\mathrm{d}t$$

exists in $\mathrm{W}_v^2(\mathrm{St}_\omega)$ in the weak sense (as defined, e.g., in [Rud, Thm. 3.26]). Actually, one can say more here: since $\mathrm{W}_v^2(\mathrm{St}_\omega)$ is separable, Pettis' measurability theorem yields that F_z is strongly measurable $\mathbb{R} \to \mathrm{W}_v^2(\mathrm{St}_\omega)$ and hence, by the estimate above, $F_z \in \mathrm{L}^1(\mathbb{R}; \mathrm{W}_v^2(\mathrm{St}_\omega))$.

We claim that $g_z = \|\psi\|^2_{\mathrm{L}^2(\mathbb{R})} \cdot \tau_z\varphi \cdot f$, i.e.,

$$\|\psi\|^2_{\mathrm{L}^2(\mathbb{R})} \cdot \tau_z\varphi \cdot f = \int_{\mathbb{R}} \tau_t(\psi^*\psi) \cdot \tau_z\varphi \cdot f \,\mathrm{d}t.$$

For $\omega = 0$, this has already been established in the proof of Theorem 4.3. In the case $\omega > 0$ evaluations at points $x \in \mathbb{R}$ are bounded linear functionals on $\mathrm{W}^2_v(\mathrm{St}_\omega)$ (by Proposition 3.3(b)). Inserting $x \in \mathbb{R}$ yields

$$g_z(x) = \int_{\mathbb{R}} |\psi(x-t)|^2 \,\mathrm{d}t \; \varphi(x-z) f(x) = \|\psi\|^2_{\mathrm{L}^2(\mathbb{R})} \cdot (\tau_z\varphi \cdot f)(x).$$

Since both g_z and $\tau_z\varphi \cdot f$ are holomorphic functions on St_ω, our claim is proved.

Now, since $\psi \neq 0$ also $\|\psi\|^2_{\mathrm{L}^2(\mathbb{R})} \neq 0$. Therefore, $\tau_z\varphi \cdot f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$ with

$$\|\tau_z\varphi \cdot f\|_{\mathrm{W}^2(\mathrm{St}_\omega)} \lesssim \|f\|_{\mathrm{H\ddot{o}r}^2(\mathrm{St}_\omega;\psi)},$$

where the hidden constant depends on φ, ψ, v, ω, but neither on f nor on $z \in \mathrm{St}_\theta$.

Finally, apply (b) of Lemma 5.2 to find that $\tau_{\mathbf{z}}\varphi \cdot f$ is holomorphic $\mathrm{St}_\theta \to \mathrm{W}^2_v(\mathrm{St}_\omega)$. This concludes the proof of (a).

(b) We again employ the representation formula

$$\varphi f = \int_{\mathbb{R}} (\tau_t\psi^* \cdot \varphi) \cdot (\tau_t\psi \cdot f) \,\mathrm{d}t. \tag{5.2}$$

By (a) applied with $\varphi = \psi$ we know that $\tau_{\mathbf{t}}\psi \cdot f \in \mathrm{C_b}(\mathbb{R}; \mathrm{W}^2_v(\mathrm{St}_\omega))$, and hence

$$(\tau_{\mathbf{t}}\psi \cdot f)^\vee \in \mathrm{C_b}(\mathbb{R}; \mathrm{L}^2_{v,\omega}(\mathbb{R})).$$

On the other hand, by Lemma 3.6,

$$\tau_{\mathbf{t}}\psi^* \cdot \varphi \in \mathrm{L}^1(\mathbb{R}; \mathrm{A}_v(\overline{\mathrm{St}}_\omega) \cap \mathrm{W}^2_v(\mathrm{St}_\omega))$$

and hence

$$(\tau_{\mathbf{t}}\psi^* \cdot \varphi)^\vee \in \mathrm{L}^1(\mathbb{R}; \mathrm{L}^1_{v,\omega}(\mathbb{R}) \cap \mathrm{L}^2_{v,\omega}(\mathbb{R})).$$

Taking the convolution, we obtain (by Proposition 2.8(f))

$$(\tau_{\mathbf{t}}\psi^* \cdot \varphi)^\vee * (\tau_{\mathbf{t}}\psi \cdot f)^\vee \in \mathrm{L}^1(\mathbb{R}; \mathrm{L}^2_{v,\omega}(\mathbb{R}) \cap \mathrm{L}^\infty_{v,\omega}(\mathbb{R}) \cap \mathrm{C}(\mathbb{R}))$$

and hence

$$h := \int_{\mathbb{R}} (\tau_t\psi^* \cdot \varphi)^\vee * (\tau_t\psi \cdot f)^\vee \,\mathrm{d}t \in \mathrm{L}^2_{v,\omega}(\mathbb{R}) \cap \mathrm{L}^\infty_{v,\omega}(\mathbb{R}) \cap \mathrm{C}(\mathbb{R}).$$

Applying the Fourier transform (which is bounded $\mathrm{L}^2_{v,\omega}(\mathbb{R}) \to \mathrm{W}^2_v(\mathrm{St}_\omega)$ and transforms the convolution into the pointwise product) yields

$$\widehat{h} = \int_{\mathbb{R}} (\tau_t\psi^* \cdot \varphi) \cdot (\tau_t\psi \cdot f) \,\mathrm{d}t = \varphi \cdot f$$

and hence $h = (\varphi \cdot f)^\vee$. Replacing φ by $\tau_z\varphi$ in this argument proves the first assertion of (b). The second follows by tracing through the estimates hidden in this qualitative argument.

(c) For $z \in \mathrm{St}_\omega$ and $w \in \mathrm{St}_\theta$ one has

$$\varphi(z-w) f(z) = (\tau_w\varphi \cdot f)(z) = \int_{\mathbb{R}} (\tau_w\varphi \cdot f)^\vee(s) \mathrm{e}^{-\mathrm{i}sz}.$$

Putting $w = z$ yields (5.1). □

Remark 5.4.

(1) Taking into account Remark 2.9(1), we see that in part (b) of Theorem 5.3 one even has $\tilde{v}\mathrm{e}^{\omega|\mathbf{s}|}(\tau_z\varphi \cdot f)^\vee \in \mathrm{C}_0(\mathbb{R})$ whenever $\tilde{v}$ is continuous and equivalent to v.

(2) In the case $\omega = 0$, assertion (b) from above holds for $\theta = 0$ and an arbitrary Schwartz function φ on $\mathbb{R}$. This follows by making appropriate changes in the given proof.

(3) In general, the representation formula (5.1) need not hold if $\omega = 0$, as in this case $z \in \mathbb{R}$ and $(\tau_z\varphi \cdot f)^\vee$ may only be in $\mathrm{L}^2(\mathbb{R})$. However, if v is strongly admissible then $(\tau_z\varphi f)^\vee \in \mathrm{L}^1(\mathbb{R})$ so that the representation formula (5.1) still holds. More generally, if v is strongly admissible then (5.1) holds for $z \in \mathrm{St}_\theta \cap \overline{\mathrm{St}}_\omega$.

As a consequence of Theorem 5.3 we may now define

$$\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega) := \mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega; \psi),$$

where $0 \neq \psi \in \mathrm{H}^\infty_0[\mathrm{St}_\omega]$ is arbitrary. In order to have a canonical norm, we let

$$\|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} := \|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega;G)} = \sup_{t\in\mathbb{R}} \|\tau_t \mathrm{G} \cdot f\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)},$$

where $\mathrm{G} = \mathrm{e}^{-\mathbf{z}^2}$ is the Gaussian function.

Example 5.5. Let $v : \mathbb{R} \to [1,\infty)$ be admissible and $\omega \geq 0$. Then $\mathrm{e}^{-\mathrm{i}s\mathbf{z}} \in \mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$ for each $s \in \mathbb{R}$, and

$$\|\mathrm{e}^{\mathrm{i}s\mathbf{z}}\|_{\mathrm{H\ddot{o}r}^2_v(\mathbb{R})} \lesssim v(s)\mathrm{e}^{\omega|s|} \qquad (s \in \mathbb{R}).$$

Indeed: fix $0 \neq \psi \in \mathrm{H}^\infty_0[\mathrm{St}_\omega]$ and $s \in \mathbb{R}$. Then

$$\|\psi \mathrm{e}^{-\mathrm{i}s\mathbf{z}}\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} = \left\|v\mathrm{e}^{\omega|\mathbf{r}|}\tau_s\psi^\vee\right\|_2 = \left\|\tau_{-s}v\mathrm{e}^{\omega|\mathbf{r}+s|}\psi^\vee\right\|_2 \lesssim_v v(s)\mathrm{e}^{\omega|s|}\|\psi\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)}.$$

Hence,

$$\sup_{t\in\mathbb{R}} \|\tau_t\psi \cdot \mathrm{e}^{-\mathrm{i}s\mathbf{z}}\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \lesssim_{v,\psi} v(s)\mathrm{e}^{\omega|s|} < \infty.$$

In Theorem 5.6 we collect the properties of Hörmander functions in the strongly admissible case.

Theorem 5.6. *Let $\omega \geq 0$ and $v : \mathbb{R} \to \mathbb{R}_{\geq 1}$ be strongly admissible. Then the following assertions hold:*

(a) *The space $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$ embeds continuously into $\mathrm{UC_b}(\overline{\mathrm{St}}_\omega)$.*

(b) *For $\omega > 0$, each $f \in \mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$ extends continuously to $\overline{\mathrm{St}}_\omega$, and its boundary functions $f_{|\pm\omega}$ are in $\mathrm{H\ddot{o}r}^2_v(\mathbb{R})$. Conversely, if $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$ is such that its L^∞-boundary values $f_{|\pm\omega}$ are contained in $\mathrm{H\ddot{o}r}^2_v(\mathbb{R})$, then $f \in \mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$. Moreover one has equivalence of norms*

$$\|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \simeq \|f_{|\omega}\|_{\mathrm{H\ddot{o}r}^2_v(\mathbb{R})} + \|f_{|-\omega}\|_{\mathrm{H\ddot{o}r}^2_v(\mathbb{R})}.$$

(c) *The space $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$ is (after a scaling of the norm) a Banach algebra with respect to pointwise multiplication.*

(d) *Let $\theta > 0$ and $\varphi \in \mathrm{H}^\infty_0(\mathrm{St}_{\theta+\omega})$. Then for each $f \in \mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$ one has*

$$\varphi(0) \cdot f(z) = \int_{\mathbb{R}} (\tau_z\varphi \cdot f)^\vee(s)\, \mathrm{e}^{-\mathrm{i}sz}\, \mathrm{d}s \qquad (z \in \mathrm{St}_\theta \cap \overline{\mathrm{St}}_\omega).$$

Proof. We fix, once and for all, a function $0 \neq \psi \in \mathrm{H}_0^\infty[\mathrm{St}_{2\omega}]$.

(a) Let $\theta > 0$ and $\varphi \in \mathrm{H}_0^\infty(\mathrm{St}_{\theta+\omega})$ and $f \in \mathrm{Hör}_v^2(\mathrm{St}_\omega)$. Since v is strongly admissible, for each $w \in \mathrm{St}_\theta$ we have $\mathrm{e}^{\omega|\mathbf{s}|}(\tau_w\varphi \cdot f)^\vee \in \mathrm{L}^1(\mathbb{R})$. Hence,

$$\varphi(z-w)f(z) = (\tau_w\varphi \cdot f)(z) = \int_{\mathbb{R}} (\tau_w\varphi \cdot f)^\vee(s) \cdot \mathrm{e}^{-\mathrm{i}sz}\,\mathrm{d}s$$

for all $z \in \mathrm{St}_\omega$. Observe that the right-hand side has a continuous extension to $\overline{\mathrm{St}}_\omega$. Hence, if we specialize $\varphi = \mathrm{G} = \mathrm{e}^{-\mathbf{z}^2}$ and $w = 0$ we find that

$$f(z) = \mathrm{e}^{z^2} \int_{\mathbb{R}} (\mathrm{G}f)^\vee(s)\mathrm{e}^{-\mathrm{i}sz}\,\mathrm{d}s$$

extends continuously to $\overline{\mathrm{St}}_\omega$. (Compare the proof of Proposition 4.1(c) for the case $\omega = 0$.)

Furthermore, by Theorem 5.3(a), the function $v\mathrm{e}^{\omega|\mathbf{s}|}(\tau_{\mathbf{z}}\mathrm{G} \cdot f)^\vee$ is holomorphic $\mathbb{C} \to \mathrm{L}^2(\mathbb{R})$ and bounded on each strip. In particular, it is uniformly continuous on $\overline{\mathrm{St}}_\omega$. Hence, to prove (a) it suffices to show that the function

$$H : \overline{\mathrm{St}}_\omega \to \mathrm{L}^2(\mathbb{R}), \qquad H(z) = \frac{1}{v}\mathrm{e}^{-\omega|\mathbf{s}|}\mathrm{e}^{-\mathrm{i}\mathbf{s}z}$$

is bounded and uniformly continuous. Since v is strongly admissible and $|H(z)| \leq \frac{1}{v}$, the image of H is indeed a bounded subset of $\mathrm{L}^2(\mathbb{R})$. Moreover,

$$\begin{aligned}\|H(z) - H(w)\|_{\mathrm{L}^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \frac{\mathrm{e}^{-2\omega|s|}}{v(s)^2}|\mathrm{e}^{-\mathrm{i}sz} - \mathrm{e}^{-\mathrm{i}sw}|^2\,\mathrm{d}s \\ &\leq \int_{|s|\geq N} \frac{4}{v(s)^2}\,\mathrm{d}s + \int_{-N}^{N} |1 - \mathrm{e}^{-\mathrm{i}s(w-z)}|^2\,\mathrm{d}s\end{aligned}$$

for all $z, w \in \overline{\mathrm{St}}_\omega$ and $N \in \mathbb{N}$. From this, the uniform continuity follows readily.

(d) Now, coming back to the more general situation we find that

$$\varphi(z-w)f(z) = \int_{\mathbb{R}} (\tau_w\varphi \cdot f)^\vee(s) \cdot \mathrm{e}^{-\mathrm{i}sz}\,\mathrm{d}s$$

holds whenever $z \in \overline{\mathrm{St}}_\omega$ and $z - w \in \mathrm{St}_{\omega+\theta}$. This implies (d).

(b) Fix $\omega > 0$ and $f \in \mathrm{Hör}_v^2(\mathrm{St}_\omega)$. Then, as shown above, f extends continuously to $\overline{\mathrm{St}}_\omega$. Obviously, $(\tau_t\psi \cdot f)_{|\pm\omega} = \tau_t\psi_{|\pm\omega} \cdot f_{|\pm\omega}$ for each $t \in \mathbb{R}$. By Corollary 3.4, (i)⇒(ii), $\psi_{|\pm\omega} \in \mathrm{Hör}_v^2(\mathbb{R})$ with

$$\|f\|_{\mathrm{Hör}_v^2(\mathrm{St}_\omega)} \eqsim \|f_{|\omega}\|_{\mathrm{Hör}_v^2(\mathbb{R})} + \|f_{|-\omega}\|_{\mathrm{Hör}_v^2(\mathbb{R})}.$$

Conversely, suppose that $f \in \mathrm{H}^\infty(\mathrm{St}_\omega)$ and its (distributional or almost everywhere) boundary functions $f_{|\pm\omega} = \lim_{y \nearrow \omega} f_{|\pm y}$ are in $\mathrm{Hör}_v^2(\mathbb{R})$. Then

$$(\tau_t\psi \cdot f)_{|y} = \tau_t\psi_{|y} \cdot f_{|y} \to \tau_t\psi_{|\pm\omega} \cdot f_{|\pm\omega}$$

in $\mathrm{L}^2(\mathbb{R})$ (cf. the discussion of the Paley–Wiener theorem in Section 3). By assumption, the latter functions are $\mathrm{W}_v^2(\mathbb{R})$ uniformly in $t \in \mathbb{R}$. As, obviously, $\tau_t\psi \cdot f \in \mathrm{H}^2(\mathrm{St}_\omega)$, Corollary 3.4, (ii)⇒(i), applies and yields $f \in \mathrm{Hör}_v^2(\mathrm{St}_\omega)$ as desired.

(c) Take $f, g \in \mathrm{Hör}_v^2(\mathrm{St}_\omega)$. Since v is strongly admissible, $\mathrm{W}_v^2(\mathrm{St}_\omega)$ is, after scaling of norm, a Banach algebra with respect to pointwise multiplication (Proposition 3.3(d). It follows that for each $t \in \mathbb{R}$,

$$\tau_t\psi^2 \cdot fg = (\tau_t\psi \cdot f) \cdot (\tau_t\psi \cdot g) \in \mathrm{W}_v^2(\mathrm{St}_\omega)$$

with

$$\|\tau_t\psi^2 \cdot fg\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \lesssim \|\tau_t\psi \cdot f\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \, \|\tau_t\psi \cdot g\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)}.$$

Taking the supremum over $t \in \mathbb{R}$ yields $fg \in \mathrm{Hör}^2_v(\mathrm{St}_\omega)$ with

$$\|fg\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} \lesssim \|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} \, \|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)}.$$

(Note that we employ Theorem 5.3(a) here.) This proves the claim. □

6. Functional calculus

In this section, X always denotes a Banach space. We fix $\omega \geq 0$ and a *strongly* admissible function $v : \mathbb{R} \to [1, \infty)$. Then $\mathrm{W}^2_v(\mathrm{St}_\omega)$ and $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$ are Banach algebras continuously embedded into $\mathrm{H}^\infty(\mathrm{St}_\omega) \cap \mathrm{UC_b}(\overline{\mathrm{St}_\omega})$ (Proposition 3.3(d) and Theorem 5.6).

A closed operator A on X is called **strip type** of **height** ω if $\sigma(A) \subseteq \overline{\mathrm{St}_\omega}$ and

$$\sup_{|\operatorname{Im} \lambda| > \omega'} \|R(\lambda, A)\| < \infty$$

for each $\omega' > \omega$. Such a strip type operator A admits a natural (unbounded) holomorphic functional calculus

$$\Phi : \mathrm{H}^\infty[\overline{\mathrm{St}_\omega}] \to \{\text{closed operators on } X\}$$

satisfying the axioms of an "abstract functional calculus" in the sense of [Haa20]. We briefly recall the construction, cf. also [Haa, Ch. 4] and [Haa18, HH13].

The algebra of **elementary functions** is

$$\mathcal{E}[\overline{\mathrm{St}_\omega}] = \bigcup_{\theta > \omega} \mathcal{E}(\mathrm{St}_\theta),$$

where $\mathcal{E}(\mathrm{St}_\theta) := \mathrm{H}^\infty(\mathrm{St}_\theta) \cap \mathrm{H}^1(\mathrm{St}_\theta)$ for $\theta > 0$. The **elementary calculus**

$$\Phi_0 : \mathcal{E}[\overline{\mathrm{St}_\omega}] \to \mathcal{L}(X)$$

is given by

$$f(A) := \Phi_0(f) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_{\omega'}} f(w) R(w, A) \, \mathrm{d}w \qquad (\omega < \omega' < \theta, \ f \in \mathcal{E}(\mathrm{St}_\theta)).$$

The elementary calculus extends canonically to an (in general unbounded) functional calculus Φ with $\mathrm{H}^\infty[\overline{\mathrm{St}_\omega}] \subseteq \operatorname{dom}(\Phi)$.

Definition 6.1. In the situation above, a set $\mathcal{D} \subseteq \mathcal{E}[\overline{\mathrm{St}_\omega}]$ is called **ample** if

$$\overline{\operatorname{span}} \bigcup_{f \in \mathcal{D}} \operatorname{ran}(f(A)) = X.$$

Later, we shall need the following result.

Lemma 6.2. *Let A be an operator of strip type $\omega \geq 0$ on a Banach space X. Then the following statements hold:*

(a) *For each $0 \neq \psi \in \mathcal{E}[\overline{\mathrm{St}_\omega}]$* $\bigcap_{t \in \mathbb{R}} \ker(\tau_t\psi)(A) = \{0\}$.

(b) $\ker(\mathrm{e}^{-\mathbf{z}^2})(A) = \{0\}$.

(c) *If A is densely defined and $\theta > \omega$ then the sets $\{\varphi^2 \mid \varphi \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)\}$ and $\{\tau_t\psi \mid t \in \mathbb{R}\}$ for given $0 \neq \psi \in \mathcal{E}(\mathrm{St}_\theta)$ are ample.*

Proof. Fix $\theta > \omega$ and $0 \neq \psi \in \mathcal{E}(\mathrm{St}_\theta)$. Without loss of generality we may assume that

$$\|\psi|_{\mathbb{R}}\|_2^2 = \int_{\mathbb{R}} \psi^*(t)\psi(t)\,\mathrm{d}t = \int_{\mathbb{R}} \psi^*(s-t)\psi(s-t)\,\mathrm{d}t = 1 \qquad (s \in \mathbb{R}).$$

Then, by holomorphy,

$$\int_{\mathbb{R}} \psi^*(z-t)\psi(z-t)\,\mathrm{d}t = 1 \tag{6.1}$$

for all $z \in \mathrm{St}_\theta$. Set

$$X_\psi := \overline{\operatorname{span}} \bigcup_{t\in\mathbb{R}} \operatorname{ran}(\tau_t\psi)(A),$$

and let $|\operatorname{Im}\lambda| > \omega$. Then, for a suitable $\delta > 0$,

$$\begin{aligned} R(\lambda, A)^2 &= \frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{St}_\delta} \frac{1}{(\lambda - z)^2} R(z, A)\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{St}_\delta} \left(\int_{\mathbb{R}} \frac{\psi^*(z-t)\psi(z-t)}{(\lambda - z)^2}\,\mathrm{d}t \right) R(z, A)\,\mathrm{d}z \\ &= \int_{\mathbb{R}} \left(\frac{1}{2\pi\mathrm{i}} \int_{\partial \mathrm{St}_\delta} \frac{\psi^*(z-t)\psi(z-t)}{(\lambda - z)^2} R(z, A)\,\mathrm{d}z \right) \mathrm{d}t \\ &= \int_{\mathbb{R}} (\tau_t\psi^*)(A)\ (\tau_t\psi)(A)\ R(\lambda, A)^2\,\mathrm{d}t. \end{aligned}$$

Here we have used identity (6.1) in the second line and Fubini's theorem in the third line. It follows that

$$\bigcap_{t\in\mathbb{R}} \ker(\tau_t\psi)(A) \subseteq \ker R(\lambda, A)^2 = \{0\},$$

and $\operatorname{dom}(A^2) = \operatorname{ran} R(\lambda, A)^2 \subseteq X_\psi$. Hence we obtain (a) and the second part of (c).

Now we specialize $\psi := G := \mathrm{e}^{-\mathbf{z}^2}$. Then, obviously,

$$\{\tau_t G \mid t \in \mathbb{R}\} \subseteq \{\varphi^2 \mid \varphi \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)\}.$$

This yields the second part of (c). For (b) write $\tau_t G = f_t G$ with $f_t = \mathrm{e}^{-t^2}\mathrm{e}^{2t\mathbf{z}}$. Note that $\tau_s G \cdot f_t \in \mathcal{E}[\overline{\mathrm{St}}_\omega]$ for each $s \in \mathbb{R}$. It follows that

$$(\tau_s G)(A)(\tau_t G)(A) = (\tau_s G \cdot \tau_t G)(A) = (\tau_s G \cdot f_t)(A)G(A) \qquad (s, t \in \mathbb{R}).$$

By (a), this implies $\ker G(A) = \{0\}$. □

Returning to our main object of study, the question arises whether the holomorphic calculus can be extended to a functional calculus defined on $\mathrm{Hör}_v^2(\mathrm{St}_\omega)$. Observe that the "algebraic extension" method will not work because the functions in the algebra $\mathrm{H}^\infty[\overline{\mathrm{St}}_\omega]$ are all defined on open strips containing $\overline{\mathrm{St}}_\omega$ and hence a quotient of such functions is necessarily meromorphic on such a larger strip. In contrast, a function in $\mathrm{Hör}_v^2(\mathrm{St}_\omega)$ in general is defined on $\overline{\mathrm{St}}_\omega$ (sharp) and may not have a meromorphic extension to a larger strip.

Therefore, in order to obtain a $\mathrm{Hör}_v^2(\mathrm{St}_\omega)$-calculus for A, one looks for a suitable *topological extension* of the natural holomorphic calculus. To this end, note that restriction induces an embedding

$$\mathcal{E}[\overline{\mathrm{St}}_\omega] \hookrightarrow \bigcup_{\theta>\omega} \mathrm{H}^2(\mathrm{St}_\theta) \subseteq \mathrm{W}^2_v(\mathrm{St}_\omega)$$

and hence we may consider Φ_0 as a non-fully defined operator $\mathrm{W}^2_v(\mathrm{St}_\omega) \supseteq \mathrm{dom}(\Phi_0) \to \mathcal{L}(X)$. Suppose that Φ_0 is closable. Then its closure

$$\Phi_{\mathrm{W}^2_v} := \overline{\Phi_0} : \mathrm{dom}(\overline{\Phi_0}) \to \mathcal{L}(X)$$

is an algebra homomorphism, called the **elementary Sobolev calculus**. This is a "topological extension" in the sense of [Haa20].[2]

As usual, one may extend $\Phi_{\mathrm{W}^2_v}$ by the algebraic method to a still larger algebra of functions on $\overline{\mathrm{St}}_\omega$. By general theory (see e.g. [Haa20, Cor. 9.2]) this constitutes an extension of the natural holomorphic calculus Φ. In effect, there is no danger in keeping the symbol Φ also for this second extension and always write $f(A)$ instead of $\Phi(f)$.

We are now heading for a condition on A guaranteeing that Φ_0 is closable and that $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$ is contained in the domain of the resulting calculus. In this case, we say that A **admits an (unbounded)** $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$**-calculus**.

Definition 6.3. Let A be a strip-type operator of height ω. A function $h \in \mathcal{E}[\overline{\mathrm{St}}_\omega]$ is called a $\mathrm{W}^2_v(\mathrm{St}_\omega)$**-regularizer** for A if there is a constant $c = c(h)$ such that

$$\|(\psi h)(A)\| \le c\|\psi\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \qquad (\psi \in \mathrm{H}^\infty_0[\overline{\mathrm{St}}_\omega]).$$

The operator A is called **almost** $\mathrm{W}^2_v(\mathrm{St}_\omega)$**-regular**, if

$$\bigcap\{\ker h(A) \mid h \text{ is a } \mathrm{W}^2_v(\mathrm{St}_\omega)\text{-regularizer for } A\} = \{0\}.$$

In the terminology of [Haa20], A is almost $\mathrm{W}^2_v(\mathrm{St}_\omega)$-regular iff the set of $\mathrm{W}^2_v(\mathrm{St}_\omega)$-regularizers for A is an *anchor set* within the natural holomorphic calculus.

Proposition 6.4. *Let A be a strip-type operator of height ω. Then, for a function $h \in \mathcal{E}[\mathrm{St}_\omega]$ the following statements are equivalent:*

(i) *h is $\mathrm{W}^2_v(\mathrm{St}_\omega)$-regularizer for A;*

(ii) $\exists\, c \ge 0,\ \theta > \omega\colon\ \forall\, \psi \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)\colon \quad \|(\psi h)(A)\| \le c\|\psi\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)}$;

(iii) $\forall\, x \in X,\ x' \in X'\colon \quad s \mapsto \frac{\mathrm{e}^{-\omega|s|}}{v}\langle(\mathrm{e}^{-\mathrm{i}s\mathbf{z}}h)(A)x, x'\rangle \in \mathrm{L}^2(\mathbb{R})$.

Proof. Clearly (i) implies (ii). For the proof of the remaining implications we abbreviate

$$U^h_s := (\mathrm{e}^{-\mathrm{i}s\mathbf{z}}h)(A) \in \mathcal{L}(X) \qquad (s \in \mathbb{R})$$

and note that

$$(fh)(A) = \int_{\mathbb{R}} f^\vee(s) U^h_s \,\mathrm{d}s \tag{6.2}$$

for all $f \in \mathcal{E}[\mathrm{St}_\omega]$, the integrand being continuous and integrable in $\mathcal{L}(X)$. (Write out $U^h_s = (\mathrm{e}^{-\mathrm{i}s\mathbf{z}}h)(A)$ as a Cauchy-integral over $\partial\mathrm{St}_{\omega'}$, where $\omega' > \omega$ is sufficiently close to ω.)

[2] As Φ_0 is an algebra homomorphism, its graph is a subalgebra of the product algebra $\mathrm{W}^2_v(\mathrm{St}_\omega) \times \mathcal{L}(X)$. Hence, the closure of the graph is also an algebra, and therefore $\mathrm{dom}(\overline{\Phi_0})$ is an algebra and $\overline{\Phi_0}$ is an algebra homomorphism.

Now suppose that (iii) holds. By the closed graph theorem there is a constant $c \geq 0$ with

$$\Big\| \frac{e^{-\omega|\mathbf{s}|}}{v} \langle U_{\mathbf{s}}^h x, x' \rangle \Big\|_2 \leq c\|x\| \, \|x'\| \qquad (x \in X,\, x' \in X').$$

Let $f \in \mathcal{E}[\mathrm{St}_\omega]$, $x \in X$, and $x' \in X'$. Then, by (6.2) and Cauchy–Schwarz

$$\begin{aligned} |\langle (fh)(A)x, x' \rangle| &= \Big| \int_{\mathbb{R}} f^\vee(s) \langle U_s^h x, x' \rangle \, \mathrm{d}s \Big| \\ &= \Big| \int_{\mathbb{R}} \big(v(s) e^{\omega|s|} f^\vee(s) \big) \cdot \Big(\frac{e^{-\omega|s|}}{v(s)} \langle U_s^h x, x' \rangle \Big) \, \mathrm{d}s \Big| \\ &\leq c \|f\|_{\mathrm{W}_v^2(\mathrm{St}_\omega)} \|x\| \, \|x'\|. \end{aligned}$$

This yields (i).

Now suppose that (ii) holds. By Lemma 2.6 we may suppose that $v \in \mathrm{C}^\infty(\mathbb{R})$. Let $\eta \in \mathrm{C}_c^\infty(\mathbb{R})$ and let $x \in X$, and $x' \in X'$. Abbreviate $f = \mathcal{F}\big(\frac{\eta}{\cosh(\omega \mathbf{s}) v}\big) \in \mathcal{F}(\mathrm{C}_c^\infty(\mathbb{R})) \subseteq \mathrm{H}_0^\infty(\mathrm{St}_\theta)$. Then

$$\begin{aligned} \Big| \int_{\mathbb{R}} \eta(s) \frac{\langle U_s^h x, x' \rangle}{\cosh(\omega s) v(s)} \, \mathrm{d}s \Big| &= \Big| \int_{\mathbb{R}} f^\vee(s) \langle U_s^h x, x' \rangle \, \mathrm{d}s \Big| = \big| \langle (fh)(A)x, x' \rangle \big| \\ &\lesssim \|f\|_{\mathrm{W}_v^2(\mathrm{St}_\omega)} \|x\| \, \|x'\| \lesssim \|\eta\|_2 \, \|x\| \, \|x'\|. \end{aligned}$$

As $\mathrm{C}_c^\infty(\mathbb{R})$ is dense in $\mathrm{L}^2(\mathbb{R})$, (iii) follows. □

Next, we show that A admits an unbounded $\mathrm{Hör}_v^2(\mathrm{St}_\omega)$-calculus if A is almost $\mathrm{W}_v^2(\mathrm{St}_\omega)$-regular.

Proposition 6.5. *Let A be a strip type operator of height $\omega \geq 0$ on a Banach space X and let $v : \mathbb{R} \to [1, \infty)$ be strongly admissible. Suppose that A is almost $\mathrm{W}_v^2(\mathrm{St}_\omega)$-regular. Then the following statements hold:*

(a) *A admits a (possibly unbounded) $\mathrm{Hör}_v^2(\mathrm{St}_\omega)$-calculus.*

(b) *For each $f \in \mathrm{W}_v^2(\mathrm{St}_\omega)$ and each $\mathrm{W}_v^2(\mathrm{St}_\omega)$-regularizer $h \in \mathcal{E}[\overline{\mathrm{St}}_\omega]$ one has $(fh)(A) \in \mathcal{L}(X)$ and*

$$\langle (fh)(A)x, x' \rangle = \int_{\mathbb{R}} f^\vee(s) \langle U_s^h x, x' \rangle \, \mathrm{d}s$$

for all $x \in X$ and $x' \in X'$, where $U_s^h = (e^{-is\mathbf{z}} h)(A)$ for $s \in \mathbb{R}$.

Proof. (a), first part: We start with showing that the elementary calculus Φ_0 is $(\mathrm{W}_v^2(\mathrm{St}_\omega), \mathcal{L}(X))$-closable. Let $\psi_n \in \mathcal{E}[\overline{\mathrm{St}}_\omega]$ with $\psi_n \to 0$ in $\mathrm{W}_v^2(\mathrm{St}_\omega)$ and $\psi_n(A) \to T$ in $\mathcal{L}(X)$. We need to show that $T = 0$. Fix a $\mathrm{W}_v^2(\mathrm{St}_\omega)$-regularizer $h \in \mathcal{E}[\overline{\mathrm{St}}_\omega]$. Then

$$\|h(A)\psi_n(T)\| = \|(\psi_n h)(A)\| \lesssim \|\psi_n\|_{\mathrm{W}_v^2(\mathrm{St}_\omega)} \to 0 \qquad (n \to \infty).$$

This implies $h(A)T = 0$. Since $\bigcap_h \ker(h(A)) = \{0\}$ it follows that $T = 0$ as desired.

(b) Fix $f \in \mathrm{W}_v^2(\mathrm{St}_\omega)$ and $h \in \mathcal{E}[\overline{\mathrm{St}}_\omega]$ as before. Choose $\psi_n \in \mathcal{E}[\overline{\mathrm{St}}_\omega]$ with $\psi_n \to f$ in $\mathrm{W}_v^2(\mathrm{St}_\omega)$. Then, $\psi_n h \to fh$ in $\mathrm{W}_v^2(\mathrm{St}_\omega)$. Since h is a $\mathrm{W}_v^2(\mathrm{St}_\omega)$-regularizer, the sequence $((\psi_n h)(A))_n$ is operator norm Cauchy and therefore converges to some operator $T \in \mathcal{L}(X)$. It follows that $f \in \mathrm{dom}(\Phi_{\mathrm{W}_v^2})$ and $f(A) = T$. Furthermore,

$$(\psi_n h)(A) = \int_{\mathbb{R}} \psi_n^\vee(s) U_s^h \, \mathrm{d}s \qquad (n \in \mathbb{N})$$

by (6.2). The claim now follows from Proposition 6.4(iii).

(a), second part: Let $f \in \mathrm{Hör}^2_v(\mathrm{St}_\omega)$. Then $\mathrm{e}^{-\mathbf{z}^2} f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$. Since $(\mathrm{e}^{-\mathbf{z}^2})(A)$ is injective, we may suppose without loss of generality that $f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$. By (b) one has $hf \in \mathrm{dom}(\Phi_{\mathrm{W}^2_v})$ for each $\mathrm{W}^2_v(\mathrm{St}_\omega)$-regularizer h, and since these functions h form an anchor set (by hypothesis), $f(A)$ is defined via the algebraically extended calculus. □

Remark 6.6. The notion of almost $\mathrm{W}^2_v(\mathrm{St}_\omega)$-regularity is coined in order to generalize the setting of Kriegler and Weis in [Kri09], [KW17], and [KW18]. There, 0-strip type operators A and strongly admissible functions $v = (1 + |\mathbf{s}|)^\beta$ with $\beta > \frac{1}{2}$ are considered; in addition it is required that there is $h \in \mathrm{H}^\infty_0[\mathbb{R}]$ such that $h(A)$ is injective, and $\frac{1}{v_\beta} U_{\mathbf{s}} h(A)$ has weakly square integrable orbits. (See also [KW17, (3.9)] for a sectorial formulation.)

We shall see in the following section that if $-\mathrm{i}A$ generates a C_0-group then almost $\mathrm{W}^2_v(\mathrm{St}_\omega)$-regularity, and in fact a stronger property, is automatic. Example 6.7 shows that there are almost $\mathrm{W}^2_v(\mathrm{St}_\omega)$-regular operators A such that $-\mathrm{i}A$ does not generate a group.

Example 6.7. Let $p \in (1, \infty)$, $p \neq 2$, and let $-\Delta$ denote the negative Laplacian on $\mathrm{L}^p(\mathbb{R})$. Then, $-\Delta$ is of strip type 0 (see [Haa, p. 236]) but the operators $\mathrm{e}^{\mathrm{i}s\Delta}$ are unbounded for all $s \neq 0$ [Haa, Proposition 8.3.8]. In particular, $-\mathrm{i}\Delta$ does not generate a C_0-group on $\mathrm{L}^p(\mathbb{R})$. However, by the Mikhlin multiplier theorem [Haa, Theorem E.6.2.b)],

$$\begin{aligned}\big\|\mathrm{e}^{\mathrm{i}t\Delta} R(\lambda, -\Delta)\big\| &\lesssim \max_{k=0,1} \Big\| \mathbf{s}^k \Big(\frac{\mathrm{e}^{\mathrm{i}t\mathbf{s}^2}}{\lambda + \mathbf{s}^2}\Big)^{(k)} \Big\|_{\infty, \mathbb{R}\setminus\{0\}} \\ &\le \Big\| \frac{\mathrm{e}^{\mathrm{i}t\mathbf{s}^2}}{\lambda + \mathbf{s}^2} \Big\|_{\infty, \mathbb{R}\setminus\{0\}} + \Big\| \mathrm{i}t \cdot \frac{2\mathbf{s}^2 \mathrm{e}^{\mathrm{i}t\mathbf{s}^2}}{\lambda + \mathbf{s}^2} - \frac{2\mathbf{s}^2 \mathrm{e}^{\mathrm{i}t\mathbf{s}^2}}{(\lambda + \mathbf{s}^2)^2} \Big\|_{\infty, \mathbb{R}\setminus\{0\}} \lesssim 1 + |t|\end{aligned}$$

for all $t \in \mathbb{R}$ and $|\operatorname{Im} \lambda| > 0$. Hence, $h := (\lambda - \mathbf{z})^{-2}$ is a $\mathrm{W}^2_v(\mathbb{R})$-regularizer whenever $v : \mathbb{R} \to [1, \infty)$ is an admissible function with $\frac{1+|\mathbf{s}|}{v} \in \mathrm{L}^2(\mathbb{R})$. A fortiori, $-\Delta$ is almost $\mathrm{W}^2_v(\mathbb{R})$-regular for such v.

7. Bounded Sobolev calculus

Still, X is a Banach space, $\omega \ge 0$ and $v : \mathbb{R} \to [1, \infty)$ is strongly admissible.

Lemma 7.1. *Let $\lambda \in \mathbb{C}$ with $|\operatorname{Im} \lambda| > \omega$. Then*

$$(\lambda - \mathbf{z})^{-1} \in \mathrm{W}^2_v(\mathrm{St}_\omega).$$

The mapping

$$\mathbb{C} \setminus \overline{\mathrm{St}}_\omega \to \mathrm{W}^2_v(\mathrm{St}_\omega), \qquad \lambda \mapsto r_\lambda := (\lambda - \mathbf{z})^{-1}$$

is holomorphic and for each $\omega' > \omega$ one has

$$\|(\lambda - \mathbf{z})^{-1}\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \lesssim (|\operatorname{Im} \lambda| - \omega')^{-\frac{1}{2}} \qquad (|\operatorname{Im} \lambda| > \omega').$$

Proof. Fix $\omega' > \omega$. Then $\mathrm{H}^2(\mathrm{St}_{\omega'}) \subseteq \mathrm{W}^2_v(\mathrm{St}_\omega)$ continuously. It is elementary to verify

$$\|(\lambda - \mathbf{z})^{-1}\|_{\mathrm{H}^2(\mathrm{St}_{\omega'})} = \sqrt{\pi}(|\operatorname{Im} \lambda| - \omega')^{-\frac{1}{2}}.$$

This yields the first assertion and the claimed norm estimate. To prove holomorphy of the mapping $\lambda \mapsto r_\lambda$ it suffices, by the Arendt–Nikolski theorem, to test this against a point-separating set W of bounded linear functionals on $\mathrm{W}^2_v(\mathrm{St}_\omega)$. Obviously, one can take $W = \{\delta_w \mid w \in \overline{\mathrm{St}_\omega}\}$. □

Definition 7.2. A possibly unbounded operator A on X is said to **have a bounded** $\mathrm{W}^2_v(\mathrm{St}_\omega)$**-calculus** if there exists a bounded algebra homomorphism

$$\Psi : \mathrm{W}^2_v(\mathrm{St}_\omega) \to \mathcal{L}(X)$$

such that $\Psi(r_\lambda) = R(\lambda, A)$ for some/all $\lambda \in \mathbb{C} \setminus \overline{\mathrm{St}_\omega}$.

Observe that if Ψ is as in Definition 7.2, then $\lambda \mapsto \Psi(r_\lambda)$ is a pseudo-resolvent and hence there is a linear (possibly multivalued) closed linear operator A on X with $R(\lambda, A) = \Psi(r_\lambda)$. This operator is uniquely determined by each single operator $\Psi(r_\lambda)$ [Haa, Prop. A.2.4].

Proposition 7.3. *For an operator A on a Banach space X the following assertions are equivalent:*

(i) *A has a bounded $\mathrm{W}^2_v(\mathrm{St}_\omega)$-calculus Ψ;*

(ii) *A is of strip type ω and there are $\theta > \omega$ and a constant $c \geq 0$ with*

$$\|\psi(A)\| \leq c\|\psi\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \qquad \text{for all } \psi \in \mathrm{H}^\infty_0(\mathrm{St}_\theta).$$

In this case, A is almost $\mathrm{W}^2_v(\mathrm{St}_\omega)$-regular and Ψ coincides with the topological extension $\Phi_{\mathrm{W}^2_v}$ of the natural holomorphic calculus for A.

Proof. (ii)⇒(i): Suppose (ii) and pick $h \in \mathcal{E}[\overline{\mathrm{St}_\omega}]$. Then

$$\|(\psi h)(A)\| = \|h(A)\|\|\psi(A)\| \leq c\|h(A)\|\|\psi\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \quad (\psi \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)).$$

Hence, by Proposition 6.4, h is a $\mathrm{W}^2_v(\mathrm{St}_\omega)$-regularizer for A.

Next, let $f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$ and let $(\psi_n)_n$ be a sequence in $\mathrm{H}^\infty_0(\mathrm{St}_\theta)$ with $\psi_n \to f$ in $\mathrm{W}^2_v(\mathrm{St}_\omega)$. By (ii), $(\psi_n(A))_n$ is a Cauchy sequence in $\mathcal{L}(X)$ and hence there is $T \in \mathcal{L}(X)$ with $\psi_n(A) \to T$. It follows that $f \in \mathrm{dom}(\Phi_{\mathrm{W}^2_v})$, $f(A) = T$, and $\|f(A)\| \leq c\|f\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)}$. This gives (i) with $\Psi(f) = f(A)$.

(i)⇒(ii): Suppose (i) holds. Then it follows from Lemma 7.1 that A is of strip type ω. Fix $\theta > \omega$ and $f \in \mathcal{E}(\mathrm{St}_\theta) = \mathrm{H}^1(\mathrm{St}_\theta) \cap \mathrm{H}^\infty(\mathrm{St}_\theta)$. Then, with $\omega < \omega' < \theta$

$$f = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_{\omega'}} f(w) r_w \,\mathrm{d}w$$

as an integral in $\mathrm{W}^2_v(\mathrm{St}_\omega)$, again by Lemma 7.1. It follows that

$$\Psi(f) = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_{\omega'}} f(w)\Psi(r_w) \,\mathrm{d}w = \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{St}_{\omega'}} f(w)R(w, A) \,\mathrm{d}w = f(A).$$

This proves (ii) and $\Psi = \Phi_0$ on $\mathcal{E}[\overline{\mathrm{St}_\omega}]$. From this it follows that $\Psi = \Phi_{\mathrm{W}^2_v}$ as claimed. □

Proposition 7.4, a generalization of [KW17, Lemma 3.7], is similar to Proposition 6.4, and yields a characterization of group generators with a bounded Sobolev calculus.

Proposition 7.4. *Let $-\mathrm{i}A$ be the generator of a C_0-group $(U_s)_{s\in\mathbb{R}}$ on a Banach space X, let $\omega \geq 0$ and let $v : \mathbb{R} \to [1, \infty)$ be strongly admissible. Then the following statements are equivalent:*

(i) *The mapping* $\frac{\mathrm{e}^{-\omega|\mathbf{s}|}}{v}U_{\mathbf{s}}$ *has weakly square integrable orbits, i.e., for each* $x \in X$ *and* $x' \in X'$,

$$\frac{\mathrm{e}^{-\omega|\mathbf{s}|}}{v}\langle U_{\mathbf{s}}x, x'\rangle \in \mathrm{L}^2(\mathbb{R}).$$

(ii) *A has a bounded* $\mathrm{W}^2_v(\mathrm{St}_\omega)$*-calculus.*

In that case,

$$\langle f(A)x, x'\rangle = \int_{\mathbb{R}} f^\vee(s)\langle U_s x, x'\rangle \,\mathrm{d}s \qquad (x \in X,\ x' \in X')$$

for all $f \in \mathrm{W}^2_v(\mathrm{St}_\omega)$.

Proof. Suppose that (ii) holds. Then we obtain (i) as in the proof of Proposition 6.4 but with $h = \mathbf{1}$.

Conversely, suppose that (i) holds. By the closed graph theorem we find $c > 0$ with

$$\|v^{-1}\mathrm{e}^{-\omega|\mathbf{s}|}\langle U_{\mathbf{s}}x, x'\rangle\|_{\mathrm{L}^2(\mathbb{R})} \le c\|x\|\|x'\| \qquad (x \in X,\ x' \in X').$$

Next, fix $\theta > \omega$. Then, by Cauchy–Schwarz

$$\int_{\mathbb{R}} \mathrm{e}^{-\theta|s|}|\langle U_s x, x'\rangle| \,\mathrm{d}s \lesssim \|x\|\|x'\| \qquad (x \in X,\ x' \in X').$$

It now follows from [Glu15, Thm. 5.1] that A is of strip type $\le \omega$.

Fix $\theta > \theta(U)$. Then for $\psi \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)$ one has

$$\psi(A) = \int_{\mathbb{R}} \psi^\vee(s) U_s \,\mathrm{d}s.$$

Indeed, the integral converges strongly; and multiplying the identity by $(\lambda - A)^{-2}$ from the left reduces the matter to (6.2) for $h = (\lambda - \mathbf{z})^{-2}$, which is already known to be true. Finally, as in the proof of Proposition 6.4, use Cauchy–Schwarz to estimate

$$|\langle \psi(A)x, x'\rangle| \le \|\psi\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \|v^{-1}\mathrm{e}^{-\omega|\mathbf{s}|}\langle U_s x, x'\rangle\|_{\mathrm{L}^2(\mathbb{R})} \le c\|\psi\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)}\|x\|\|x'\|$$

for $x \in X$, $x' \in X'$. This gives (ii). The additional statement follows by approximation. □

Example 7.5. Let $-\mathrm{i}A$ be the generator of a C_0-group $(U_s)_{s\in\mathbb{R}}$ such that $\|U_{\mathbf{s}}\| \le v_u \mathrm{e}^{\omega|\mathbf{s}|}$ for some measurable function $v_U : \mathbb{R} \to (0, \infty)$. If v is admissible with $v_U/v \in \mathrm{L}^2(\mathbb{R})$, then A is $\mathrm{W}^2_v(\mathrm{St}_\omega)$-regular. This follows from Proposition 7.4 (but can quite easily be seen directly as well).

In order to extend a bounded Sobolev calculus to a bounded Hörmander calculus (which is the subject of the next section), we need the following "convergence lemma".

Theorem 7.6 (Convergence lemma)**.** *Let A be an operator on X with a bounded* $\mathrm{W}^2_v(\mathrm{St}_\omega)$*-calculus, where* $\omega \ge 0$ *and* $v : \mathbb{R} \to [1, \infty)$ *is strongly admissible. Suppose that* $f \in \mathrm{Hör}^2_v(\mathrm{St}_\omega)$ *and that* $(f_n)_n$ *is a sequence in* $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$ *with the following properties:*

(1) *There is an ample set* $\mathcal{D} \subseteq \mathrm{H}^\infty_0[\mathrm{St}_\omega]$ *with*

$$\|\psi(f_n - f)\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \xrightarrow{n\to\infty} 0 \qquad (\psi \in \mathcal{D});$$

(2) $\sup_{n\in\mathbb{N}}\|f_n(A)\| < \infty$.

Then $f(A) \in \mathcal{L}(X)$ and $f_n(A) \to f(A)$ strongly on X.

Proof. Observe that

$$\|(f_n(A) - f(A))\psi(A)\| = \|(\psi(f_n - f))(A)\| \lesssim_A \|\psi(f_n - f)\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \xrightarrow{n\to\infty} 0$$

for all $\psi \in \mathcal{D}$. As $\mathcal{D}$ is ample and the sequence $(f_n(A))_n$ is uniformly bounded, the claim follows. □

In Lemma 7.7 we show that for a given function $f \in \mathrm{Hör}^2_v(\mathrm{St}_\omega)$ one can always find a bounded sequence $(f_n)_n$ in $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$ such that condition (1) of Theorem 7.6 is satisfied. Moreover, this sequence may be chosen in $\mathrm{H}^\infty_0(\mathrm{St}_\theta)$ for any given $\theta > \omega$.

Lemma 7.7. *Let $\theta > \omega \geq 0$, and let $v : \mathbb{R} \to [1, \infty)$ be strongly admissible. Then there is a constant $K \geq 0$ with the following property: For each $f \in \mathrm{Hör}^2_v(\mathrm{St}_\omega)$ there is a sequence $(f_n)_n$ in $\mathrm{H}^\infty_0(\mathrm{St}_\theta)$ with*

$$\sup_{n\in\mathbb{N}} \|f_n\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} \leq K\|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} \tag{7.1}$$

and

$$\|\psi_1\psi_2(f_n - f)\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \xrightarrow{n\to\infty} 0$$

for all $\psi_1, \psi_2 \in \mathrm{H}^\infty_0[\overline{\mathrm{St}_\omega}]$.

Proof. Define $\varphi_n := \mathrm{e}^{-\frac{1}{n}\mathbf{z}^2} \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)$ for $n \in \mathbb{N}$. Then

$$\sup_{n\in\mathbb{N}}\|\varphi_n\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} \lesssim \sup_{n\in\mathbb{N}}\|\varphi_n\|_{\mathrm{H}^\infty(\mathrm{St}_\theta)} < \infty.$$

Furthermore, $\varphi_n \to \mathbf{1}$ uniformly on compacts. Hence, for each $\psi \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)$

$$\|\psi(\mathbf{1} - \varphi_n)\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \lesssim \|\psi(\mathbf{1} - \varphi_n)\|_{\mathrm{H}^2(\mathrm{St}_\theta)} \to 0.$$

Now, fix $f \in \mathrm{Hör}^2_v(\mathrm{St}_\omega)$ and set

$$g_n := \varphi_n f \in \mathrm{W}^2_v(\mathrm{St}_\omega) \qquad (n \in \mathbb{N}).$$

Then

$$\sup_{n\in\mathbb{N}} \|g_n\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} \lesssim_v \big(\sup_{n\in\mathbb{N}} \|\varphi_n\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)}\big)\|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} < \infty,$$

and

$$\|\psi_1\psi_2(g_n - f)\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \lesssim_v \|\psi_1(\phi_n - \mathbf{1})\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \|\psi_2 f\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \xrightarrow{n\to\infty} 0$$

for all $\psi_1, \psi_2 \in \mathrm{H}^\infty_0[\mathrm{St}_\omega]$ by the considerations above. Finally, recall that $\mathrm{H}^\infty_0(\mathrm{St}_\theta)$ is dense in $\mathrm{W}^2_v(\mathrm{St}_\omega)$, which is why one can choose $f_n \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)$ with

$$\|g_n - f_n\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \leq \frac{1}{n}\|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} \qquad (n \in \mathbb{N}).$$

It follows that

$$\begin{aligned}\|f_n\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} &= \sup_{t\in\mathbb{R}} \|\tau_t \mathrm{G}\cdot f_n\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \\ &\le \sup_{t\in\mathbb{R}} \big(\|\tau_t \mathrm{G}\cdot (f_n - g_n)\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} + \|\tau_t \mathrm{G}\cdot g_n\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)}\big) \\ &\lesssim_v \tfrac{1}{n}\|\mathrm{G}\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)}\|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} + \|g_n\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} \\ &\lesssim_v \Big(\|\mathrm{G}\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} + \sup_{k\in\mathbb{N}} \|\varphi_k\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)}\Big)\|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)}.\end{aligned}$$

Therefore, we may set

$$K := \|\mathrm{G}\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} + \sup_{k\in\mathbb{N}} \|\phi_k\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)}$$

(times a constant depending only on v) to establish (7.1). One readily verifies that

$$\|\psi_1\psi_2(f_n - f)\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \xrightarrow{n\to\infty} 0$$

for all $\psi_1, \psi_2 \in \mathrm{H}^\infty_0[\overline{\mathrm{St}_\omega}]$, and this concludes the proof. ☐

8. Bounded Hörmander calculus

As in the previous sections, X always denotes a Banach space, $\omega \in \mathbb{R}_{\ge 0}$ and $v : \mathbb{R} \to [1,\infty)$ is strongly admissible.

Definition 8.1.
(1) An algebra homomorphism $\Phi : \mathcal{A} \to \mathcal{L}(X)$, where $\mathcal{A}$ is some normed algebra, is called γ-**bounded** (R-**bounded**) if the set

$$\{\Phi(a) \colon \|a\|_{\mathcal{A}} \le 1\}$$

is a γ-bounded (R-bounded) subset of $\mathcal{L}(X)$.

(2) An operator A on X is said to have a **bounded** (**γ-bounded, R-bounded**) $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$**-calculus** if there is a bounded (γ-bounded, R-bounded) unital algebra homomorphism

$$\Psi : \mathrm{Hör}^2_v(\mathrm{St}_\omega) \to \mathcal{L}(X)$$

such that $\Psi(r_\lambda) = R(\lambda, A)$ for one/all $\lambda \in \mathbb{C} \setminus \overline{\mathrm{St}_\omega}$.

Cf. Appendix B, in particular Remark B.3, for the notions of γ- and R-boundedness.

Suppose that A has a bounded $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$-calculus. Then, as $\mathrm{W}^2_v(\mathrm{St}_\omega) \subseteq \mathrm{Hör}^2_v(\mathrm{St}_\omega)$ continuously, A also has a bounded Sobolev calculus and hence is of strip type ω. Moreover, the Sobolev calculus extends (topologically) the natural holomorphic calculus (Proposition 7.3). In particular, $\Psi(\mathrm{e}^{-\mathbf{z}^2}) = (\mathrm{e}^{-\mathbf{z}^2})(A)$ is injective (Lemma 6.2). As each $f \in \mathrm{Hör}^2_v(\mathrm{St}_\omega)$ can be written as the quotient $f = (\mathrm{e}^{-\mathbf{z}^2} f)/\mathrm{e}^{-\mathbf{z}^2}$, we conclude that Ψ coincides (on its domain) with the algebraic extension of the $\mathrm{W}^2_v(\mathrm{St}_\omega)$-calculus. In particular, if A has a bounded $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$-calculus, then it is unique.

As a corollary of Theorem 7.6 and Lemma 7.7 we obtain a characterization for when an operator admits a bounded $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$-calculus.

Theorem 8.2. *Let A be a densely defined operator on X and let $\theta > \omega$. Then the following statements are equivalent.*

(i) *A has a bounded (γ-bounded, R-bounded)* $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$*-calculus.*

(ii) *A is of strip type ω and the set*

$$\{\psi(A) \mid \psi \in \mathrm{H}^\infty_0(\mathrm{St}_\theta),\ \|\psi\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \leq 1\} \subseteq \mathcal{L}(X)$$

is bounded (γ-bounded, R-bounded).

Proof. The implication (i)⇒(ii) is clear from the remarks preceding this theorem. For the converse, suppose that (ii) holds. Then, in particular, A has a bounded $\mathrm{W}^2_v(\mathrm{St}_\omega)$-calculus. Choose K as in Lemma 7.7.

To a given $f \in \mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$ with $\|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \leq 1$ we can then find a sequence $(f_n)_n$ in $\mathrm{H}^\infty_0(\mathrm{St}_\theta)$ with $\sup_n \|f_n\|_{\mathrm{H\ddot{o}r}^2_v} \leq K$ and $\|\psi^2(f_n - f)\|_{\mathrm{W}^2_v} \to 0$ for all $\psi \in \mathrm{H}^\infty_0[\overline{\mathrm{St}}_\omega]$. By (ii), $\sup_n \|f_n(A)\| < \infty$. Since the set $\mathcal{D} := \{\psi^2 \mid \psi \in \mathrm{H}^\infty_0[\overline{\mathrm{St}}_\omega]\}$ is ample, Theorem 7.6 is applicable and yields $f(A) \in \mathcal{L}(X)$ and $f_n(A) \to f(A)$ strongly. Hence, the operator $f(A)$ lies in the strong closure of the bounded (γ-bounded, R-bounded) set

$$\{\psi(A) \mid \psi \in \mathrm{H}^\infty_0(\mathrm{St}_\theta),\ \|\psi\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \leq K\}.$$

This yields (i). □

The following is our first result on bounded Hörmander calculus. It extends [GCM+01, Theorem 2.2] to general Banach spaces. The proof is based on the approach in the classical works of Meda [Med90] and Cowling and Meda [CM93], see also [CD17, p. 944 - 946].

Theorem 8.3. *Let $-\mathrm{i}A$ be a densely defined operator with a bounded* $\mathrm{H}^\infty(\mathrm{St}_\theta)$*-calculus for some $\theta > 0$ on a Banach space X. Suppose*

$$\|\mathrm{e}^{-isA}\| \leq \tilde{v}(s)\mathrm{e}^{\omega|s|} \qquad (s \in \mathbb{R})$$

for some $\omega \in \mathbb{R}_{\geq 0}$ and some measurable function $\tilde{v} : \mathbb{R} \to \mathbb{R}_{\geq 0}$. Let $v : \mathbb{R} \to [1, \infty)$ be strongly admissible such that $\tilde{v}/v \in \mathrm{L}^1(\mathbb{R})$. Then A has a bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$*-calculus. Moreover, for each $\varphi \in \mathrm{H}^\infty_0[\overline{\mathrm{St}}_{2\omega}]$ with $\varphi(0) = 1$ one has*

$$f(A) = \int_{\mathbb{R}} F_s(A)\mathrm{e}^{-\mathrm{i}sA}\,\mathrm{d}s \qquad (f \in \mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)),$$

where $F_s = (\tau_{\mathbf{z}}\varphi \cdot f)^\vee(s)$ for $s \in \mathbb{R}$.

Proof. We show first that A is of strip type ω. To this end, let $\operatorname{Re}\lambda \geq \omega' > \omega$. Then, since v is growing at most polynomially,

$$\|\mathrm{e}^{\pm\mathrm{i}sA}\mathrm{e}^{-\lambda s}\| \leq \tilde{v}(s)\mathrm{e}^{\omega s}\mathrm{e}^{-\omega' s} = \frac{\tilde{v}(s)}{v(s)} v(s)\mathrm{e}^{-(\omega'-\omega)s} \lesssim \frac{\tilde{v}(s)}{v(s)} \qquad (s \geq 0).$$

Since $\tilde{v}/v \in \mathrm{L}^1(\mathbb{R})$, it follows that $\lambda \in \rho(\pm\mathrm{i}A)$ and $\sup_{\operatorname{Re}\lambda \geq \omega'} \|R(\lambda, \pm\mathrm{i}A)\| < \infty$. Hence, A is of strip type ω.

Fix, without loss of generality, $\theta > \omega$ such that A has a bounded $\mathrm{H}^\infty(\mathrm{St}_\theta)$-calculus, and fix $\varphi \in \mathrm{H}^\infty_0(\mathrm{St}_{2\theta})$ with $\varphi(0) = 1$. Then by Theorem 5.6 we have the representation formula

$$f(z) = \int_{\mathbb{R}} (\tau_z\varphi f)^\vee(s)\mathrm{e}^{-\mathrm{i}sz}\,\mathrm{d}s = \int_{\mathbb{R}} F_s(z)\mathrm{e}^{-\mathrm{i}sz}\,\mathrm{d}s \qquad (z \in \overline{\mathrm{St}}_\omega)$$

for $f \in \mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$. Note, however, that this formula holds even for $z \in \mathrm{St}_\theta$ whenever $f \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)$. We shall use this fact in the following to establish the formula

$$f(A) = \int_{\mathbb{R}} F_s(A)\mathrm{e}^{-\mathrm{i}sA}\,\mathrm{d}s \tag{8.1}$$

for such f and therefrom the estimate

$$\|f(A)\| \lesssim \|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \qquad (f \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)).$$

From Theorem 8.2 it then follows that A has bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$-calculus. Finally, we shall establish (8.1) for all $f \in \mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$.

Fix $f \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)$ and define (as above)

$$F(z,s) := F_s(z) := (\tau_z\varphi \cdot f)^\vee(s) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(t-z)f(t)\mathrm{e}^{\mathrm{i}ts}\,\mathrm{d}t \qquad (z \in \mathrm{St}_\theta, s \in \mathbb{R}).$$

It is easy to see that F is continuous, $F_s \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)$ (uniformly in $s \in \mathbb{R}$) and

$$s \mapsto F_s(A) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(t-A)f(t)\mathrm{e}^{\mathrm{i}ts}\,\mathrm{d}t$$

is continuous in operator norm. Moreover, by Theorem 5.3(b)

$$\|F_s\|_{\mathrm{H}^\infty(\mathrm{St}_\theta)} \lesssim_\delta \frac{\mathrm{e}^{-\delta|s|}}{v(s)}\|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\delta)} \qquad (s \in \mathbb{R},\ \omega \le \delta \le \theta).$$

Since the $\mathrm{H}^\infty(\mathrm{St}_\theta)$-calculus is bounded,

$$\|F_s(A)\| \lesssim \frac{\mathrm{e}^{-\omega|s|}}{v(s)}\|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \qquad (s \in \mathbb{R}).$$

This shows that the integral

$$\int_{\mathbb{R}} F_s(A)\mathrm{e}^{-\mathrm{i}sA}\,\mathrm{d}s$$

is absolutely convergent. Fix $\lambda \in \mathbb{C} \setminus \overline{\mathrm{St}}_\theta$ and $\delta \in (\omega,\theta)$. Then

$$\begin{aligned}
R(\lambda,A)^2 f(A) &= \frac{1}{2\pi\mathrm{i}}\int_{\partial\mathrm{St}_\delta} \frac{f(z)}{(\lambda-z)^2}R(z,A)\,\mathrm{d}z \\
&= \frac{1}{2\pi\mathrm{i}}\int_{\partial\mathrm{St}_\delta}\Big(\int_{\mathbb{R}}(\tau_z\varphi\cdot f)^\vee(s)\mathrm{e}^{-\mathrm{i}sz}\,\mathrm{d}s\Big)\frac{1}{(\lambda-z)^2}R(z,A)\,\mathrm{d}z \\
&= \frac{1}{2\pi\mathrm{i}}\int_{\partial\mathrm{St}_\delta}\int_{\mathbb{R}}(\tau_z\varphi\cdot f)^\vee(s)\mathrm{e}^{-\mathrm{i}sz}\frac{1}{(\lambda-z)^2}R(z,A)\,\mathrm{d}s\,\mathrm{d}z \\
&= \int_{\mathbb{R}}\Big(\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathrm{St}_\delta}(\tau_z\varphi\cdot f)^\vee(s)\mathrm{e}^{-\mathrm{i}sz}\frac{1}{(\lambda-z)^2}R(z,A)\,\mathrm{d}z\Big)\,\mathrm{d}s \\
&= \int_{\mathbb{R}} F_s(A)\mathrm{e}^{-\mathrm{i}sA}R(\lambda,A)^2\,\mathrm{d}s = R(\lambda,A)^2\Big(\int_{\mathbb{R}}F_s(A)\mathrm{e}^{-\mathrm{i}sA}\,\mathrm{d}s\Big).
\end{aligned}$$

As $R(\lambda,A)^2$ is injective, the identity (8.1) follows. Hence, one can estimate

$$\begin{aligned}
\|f(A)\| &\le \int_{\mathbb{R}}\|F_s(A)\|\,\|\mathrm{e}^{-\mathrm{i}sA}\|\,\mathrm{d}s \lesssim \int_{\mathbb{R}}\frac{\mathrm{e}^{-\omega|s|}}{v(s)}\|f\|_{\mathrm{H\ddot{o}r}^2_v(\overline{\mathrm{St}}_\omega)}\,\|\mathrm{e}^{-\mathrm{i}sA}\|\,\mathrm{d}s \\
&\le \|\tilde{v}/v\|_1\,\|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)}.
\end{aligned}$$

As said before, this by Theorem 8.2 yields that A has a bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$-calculus.

It remains to show that the formula (8.1) holds for all $\varphi \in \mathrm{H}_0^\infty[\overline{\mathrm{St}}_{2\omega}]$ with $\varphi(0) = 1$ and all $f \in \mathrm{Hör}_v^2(\mathrm{St}_\omega)$. Fix $\theta > \omega$ and $\varphi \in \mathrm{H}_0^\infty(\mathrm{St}_{2\theta})$ with $\varphi(0) = 1$. By what we have already shown, A has a bounded $\mathrm{H}^\infty(\mathrm{St}_\theta)$-calculus (simply because $\mathrm{H}^\infty(\mathrm{St}_\theta) \subseteq \mathrm{Hör}_v^2(\mathrm{St}_\omega)$ continuously) and hence the proof from above yields (8.1) for all $f \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)$.

Fix $f \in \mathrm{Hör}_v^2(\mathrm{St}_\omega)$ and pick $K \geq 0$ and a sequence $(f_n)_n$ in $\mathrm{H}_0^\infty(\mathrm{St}_\theta)$ as in Lemma 7.7. Define

$$F_{s,n}(z) := (\tau_z \varphi \cdot f_n)^\vee(s) \qquad (z \in \mathrm{St}_\theta,\ s \in \mathbb{R}).$$

By Theorem 7.6, $f_n(A) \to f(A)$ strongly, and by the classical convergence theorem, $F_{n,s}(A) \to F_s(A)$ strongly. Moreover,

$$\sup_n \|F_{n,s}(A)\| \lesssim \frac{\mathrm{e}^{-\omega|s|}}{v(s)} K \|f\|_{\mathrm{Hör}_v^2(\mathrm{St}_\omega)} \qquad (s \in \mathbb{R}).$$

Hence (dominated convergence!)

$$f_n(A)x = \int_{\mathbb{R}} F_{s,n}(A)\mathrm{e}^{-\mathrm{i}sA} x \,\mathrm{d}s \to \int_{\mathbb{R}} F_s(A)\mathrm{e}^{-\mathrm{i}sA} x \,\mathrm{d}s \qquad (x \in X).$$

This concludes the proof. □

Theorem 8.3 holds for any Banach space and uses the bounded H^∞-calculus in a very coarse way. However, if X has finite cotype, bounded H^∞-calculus implies "square function estimates" and "dual square function estimates", i.e. an abstract Littlewood–Paley theorem. Combining this with γ-boundedness arguments (which guarantee boundedness with respect to the relevant square function norms) then leads to a more refined statement about a bounded Hörmander type functional calculus as follows.

Theorem 8.4. *Let X be a Banach space of finite cotype $q \in [2, \infty)$ and type $p \in [1, 2]$, and let A be a densely defined operator on X with a bounded $\mathrm{H}^\infty(\mathrm{St}_\theta)$-calculus for some $\theta > 0$. Suppose that $\omega \geq 0$ and $\tilde{v} : \mathbb{R} \to (0, \infty)$ is measurable with*

$$\|\mathrm{e}^{-\mathrm{i}sA}\| \leq \tilde{v}(s)\mathrm{e}^{\omega|s|} \qquad (s \in \mathbb{R}).$$

Let $r \in [1, 2]$ with $\frac{1}{r} > \frac{1}{p} - \frac{1}{q}$ and let $v : \mathbb{R} \to [1, \infty)$ be strongly admissible such that $\tilde{v}/v \in \mathrm{L}^r(\mathbb{R})$. Then, A has a bounded $\mathrm{Hör}_v^2(\mathrm{St}_\omega)$-calculus. If, in addition, X has Pisier's contraction property, then this calculus is γ-bounded.

Remark 8.5. Observe that for a Banach space X with type $p \in [1, 2]$ and finite cotype $q < \infty$ one has $\frac{1}{p} - \frac{1}{q} < 1$. Hence Theorem 8.4 is stronger than Theorem 8.3 for such spaces X.

The proof requires several steps. In all what follows $X, p, q, A, \omega, \tilde{v}, v$ are as in Theorem 8.4. As in the beginning of the proof of Theorem 8.3, we see that A is of strip type ω. Fix $\theta > \omega$ such that A has a bounded $\mathrm{H}^\infty(\mathrm{St}_\theta)$-calculus.[3] In order to employ abstract Littlewood–Paley theory, we need the following "partition of unity"-result.

[3]By a theorem of Cowling, Doust, McIntosh and Yagi [Haa, Thm. 5.4.1] or also by Theorem 8.3, θ can be any number $> \omega$. However, this information is inessential here.

Lemma 8.6. *Let $\theta > 0$. Then there are $\psi, \psi_1, \psi_2 \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)$ with*

$$\sum_{n\in\mathbb{Z}} \psi_1(z-n)\psi_2(z-n)\psi(z-n) = 1 \qquad (z \in \mathrm{St}_\theta).$$

Proof. Fix $0 < \alpha < \frac{\pi}{2\theta}$ and $c > 0$ such that

$$\eta := \frac{c\mathrm{e}^{\alpha\mathbf{z}}}{(1+\mathrm{e}^{\alpha\mathbf{z}})^2} = \frac{c}{(1+\mathrm{e}^{\alpha\mathbf{z}})(1+\mathrm{e}^{-\alpha\mathbf{z}})} \in \mathrm{H}_0^\infty(\mathrm{St}_\omega)$$

satisfies

$$\int_{\mathbb{R}} \eta(s)\,\mathrm{d}s = 1.$$

Then η maps the strip St_θ into the sector $\mathrm{S}_{\alpha\theta}$. Define

$$\varphi(z) := \int_0^1 \eta(s-z)\,\mathrm{d}s \qquad (z \in \mathrm{St}_\theta).$$

Then, still, $\varphi \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)$ and, by convexity, φ maps into $\overline{\mathrm{S}_{\alpha\theta}}$. Moreover,

$$\sum_{n\in\mathbb{Z}} \varphi(t-n) = \sum_{n\in\mathbb{Z}} \int_0^1 \eta(s-t+n)\,\mathrm{d}s = \int_{\mathbb{R}} \eta(s)\,\mathrm{d}s = 1 \qquad (t \in \mathbb{R}).$$

By uniqueness of holomorphic functions,

$$\sum_{n\in\mathbb{Z}} \varphi(z-n) = 1 \quad \text{for all } z \in \mathrm{St}_\theta.$$

Finally, since η has no zeroes, $\mathrm{Re}\,\eta(z) > 0$ and hence also $\mathrm{Re}\,\varphi(z) > 0$ for each $z \in \mathrm{St}_\theta$. This means that we can take

$$\psi(z) := \psi_1(z) := \psi_2(z) := \big(\varphi(z)\big)^{\frac{1}{3}} \qquad (z \in \mathrm{St}_\theta). \qquad \square$$

Back to the proof of Theorem 8.4, fix $\psi, \psi_1, \psi_2 \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)$ as in Lemma 8.6. By Proposition B.7 we have the norm equivalences

$$\begin{aligned} \|x\| &\eqsim \big\|\big(\psi_1(A-n)x\big)_n\big\|_{\gamma(\mathbb{Z};X)} \\ &\eqsim \big\|\big(\psi(A-n)\psi_1(A-n)x\big)_n\big\|_{\gamma(\mathbb{Z};X)} \qquad (x \in X). \end{aligned}$$

Then

$$\|f(A)x\| \eqsim \big\|\big((\tau_n\psi \cdot f)(A)\psi_1(A-n)x\big)_n\big\|_{\gamma(\mathbb{Z};X)} \qquad (x \in X,\ f \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)).$$

Hence, in order to obtain the desired estimate

$$\|f(A)x\| \lesssim \|f\|_{\mathrm{Hör}_v^2(\mathrm{St}_\omega)}\|x\| \qquad (x \in X,\ f \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)),$$

it is sufficient to establish

$$\|\big((\tau_n\psi \cdot f)(A)\big)_{n\in\mathbb{Z}}\|_{\mathcal{L}(\gamma(\mathbb{Z};X))} \lesssim \|f\|_{\mathrm{Hör}_v^2(\mathrm{St}_\omega)} \qquad (f \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)).$$

These considerations lead to the following intermediate result, a generalization of [KW18, Cor. 5.2]. (Recall from Appendix B that $[\![\mathcal{A}]\!]_\gamma$ denotes the γ-bound of a set of operators $\mathcal{A} \subseteq \mathcal{L}(X)$.)

Proposition 8.7. *Let $\theta > \omega \geq 0$ and let A be a densely defined closed operator with a bounded $\mathrm{H}^\infty(\mathrm{St}_\theta)$ calculus on a Banach space X. Furthermore, let $0 \neq \psi \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)$ as above and $v : \mathbb{R} \to [1,\infty)$ strongly admissible.*

(a) *If X has finite cotype and*

$$[\![(\tau_n\psi \cdot f)(A) \mid n \in \mathbb{Z}]\!]_\gamma \lesssim \|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \qquad (f \in \mathrm{H}^\infty_0(\mathrm{St}_\omega)),$$

then A has a bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$-calculus.

(b) *If X has Pisier's contraction property and*

$$[\![(\tau_n\psi \cdot f)(A) \mid n \in \mathbb{N},\ f \in \mathrm{H}^\infty_0(\mathrm{St}_\theta),\ \|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \le 1]\!]_\gamma < \infty,$$

then A has a γ-bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$-calculus.

Proof. (a) By the considerations above and the identity

$$\|(T_n)_{n\in\mathbb{Z}}\|_{\mathcal{L}(\gamma(\mathbb{Z};X))} \le [\![T_n \mid n \in \mathbb{Z}]\!]_\gamma$$

for any sequence $(T_n)_n \in \mathcal{L}(X)^{\mathbb{Z}}$ (Lemma B.2) we obtain an estimate

$$\|f(A)x\| \lesssim \|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)}\|x\| \qquad (x \in X,\ f \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)).$$

Hence, the claim follows from Theorem 8.2.

(b) Define

$$C := [\![(\tau_n\psi \cdot f)(A) \mid n \in \mathbb{N},\ f \in \mathrm{H}^\infty_0(\mathrm{St}_\theta),\ \|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \le 1]\!]_\gamma < \infty.$$

Pick a finite sequence $(f_k)_k$ in $\mathrm{H}^\infty_0(\mathrm{St}_\theta)$ with $\|f_k\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \le 1$ for each k, and a finite sequence $(x_k)_k$ in X. Then

$$\begin{aligned}
\mathbb{E}\Big\|\sum_k \gamma_k f_k(A)x_k\Big\|_X^2 &\eqsim \mathbb{E}\Big\|\Big(\sum_k \gamma_k(\tau_n\psi \cdot f_k)(A)\psi_1(A-n)x_k\Big)_n\Big\|^2_{\gamma(\mathbb{Z};X)}\\
&= \mathbb{E}\Big\|\sum_k \gamma_k\Big((\tau_n\psi \cdot f_k)(A)\psi_1(A-n)x_k\Big)_n\Big\|^2_{\gamma(\mathbb{Z};X)}\\
&= \Big\|\Big((\tau_n\psi \cdot f_k)(A)\psi_1(A-n)x_k\Big)_{n,k}\Big\|^2_{\gamma(\mathbb{Z};\gamma(\mathbb{Z};X))}\\
&\eqsim \Big\|\Big((\tau_n\psi \cdot f_k)(A)\psi_1(A-n)x_k\Big)_{n,k}\Big\|^2_{\gamma(\mathbb{Z}\times\mathbb{Z};X)}\\
&\le C^2\Big\|\Big(\psi_1(A-n)x_k\Big)_{n,k}\Big\|^2_{\gamma(\mathbb{Z}\times\mathbb{Z};X)} \eqsim C^2\Big\|\sum_k \gamma_k x_k\Big\|^2_X.
\end{aligned}$$

(Note that Pisier's contraction property was employed in passing from $\gamma(\mathbb{Z};\gamma(\mathbb{Z};X))$ to $\gamma(\mathbb{Z}\times\mathbb{Z};X)$ and back.) We conclude that

$$[\![f(A) \mid f \in \mathrm{H}^\infty_0(\mathrm{St}_\theta),\ \|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \le 1]\!]_\gamma \le C < \infty$$

and hence, by Theorem 8.2, that A has γ-bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$-calculus. ☐

We now make the final step in the proof of Theorem 8.4.

Proof of Theorem 8.4. Let, as before, $X, p, q, A, \omega, \tilde{v}, v$ as in the formulation of the theorem, and $\psi, \psi_1, \psi_2 \in \mathrm{H}^\infty_0(\mathrm{St}_\theta)$ as in Lemma 8.6. By Proposition 8.7 it suffices to show

$$[\![(\tau_n\psi \cdot f)(A) \mid n \in \mathbb{N},\ f \in \mathrm{H}^\infty_0(\mathrm{St}_\theta),\ \|f\|_{\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)} \le 1]\!]_\gamma < \infty.$$

To that end, fix $f \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)$, and note that

$$(\tau_n\psi \cdot f)(A)x = \int_{\mathbb{R}} (\tau_n\psi \cdot f)^\vee(s)\mathrm{e}^{-\mathrm{i}sA}x\,\mathrm{d}s = \int_{\mathbb{R}} \Big(v(s)\mathrm{e}^{\omega|s|}(\tau_n\psi \cdot f)^\vee(s)\Big) \cdot \Big(\frac{\mathrm{e}^{-\omega|s|}}{v(s)}\mathrm{e}^{-\mathrm{i}sA}x\Big)\,\mathrm{d}s$$

for each $n \in \mathbb{Z}$ and $x \in X$. By hypothesis,

$$\frac{\mathrm{e}^{-\omega|\mathbf{s}|}}{v}\|\mathrm{e}^{-\mathrm{i}\mathbf{s}A}x\| \lesssim \frac{\tilde{v}}{v}\|x\| \quad \text{and} \quad \frac{\tilde{v}}{v} \in \mathrm{L}^r(\mathbb{R}).$$

Hence, by Theorem B.4 it suffices to establish an estimate of the form

$$\sup_{n\in\mathbb{Z}} \Big(\int_{\mathbb{R}} \big|v(s)\mathrm{e}^{\omega|s|}(\tau_n\psi \cdot f)^\vee(s)\big|^{r'}\,\mathrm{d}s\Big)^{\frac{1}{r'}} \lesssim \|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} \qquad (f \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)),$$

where $\frac{1}{r'} = 1 - \frac{1}{r}$ is the dual exponent. For this, note that on one hand

$$\sup_{n\in\mathbb{Z}} \Big(\int_{\mathbb{R}} \big|v(s)\mathrm{e}^{\omega|s|}(\tau_n\psi \cdot f)^\vee(s)\big|^{2}\,\mathrm{d}s\Big)^{\frac{1}{2}} = \sup_{n\in\mathbb{Z}} \|\tau_n\psi \cdot f\|_{\mathrm{W}^2_v(\mathrm{St}_\omega)} \lesssim \|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)}.$$

On the other hand, by Theorem 5.3(b),

$$\sup_{n\in\mathbb{Z}}\sup_{s\in\mathbb{R}} \big|v(s)\mathrm{e}^{\omega|s|}(\tau_n\psi \cdot f)^\vee(s)\big| \lesssim \|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)}.$$

As $r' \in [2,\infty]$, the desired estimate follows by interpolation. □

One can do with weaker geometric assumptions if one assumes a stronger boundedness condition on the group. Recall from Appendix B the notion of a semi-γ-bounded family of operators.

Theorem 8.8. *Let X be a Banach space with Pisier's contraction property and let A be a densely defined operator on X with a bounded $\mathrm{H}^\infty(\mathrm{St}_\theta)$-calculus for some $\theta > 0$. Suppose that $\omega \ge 0$ and $\tilde{v} : \mathbb{R} \to (0,\infty)$ is measurable such that the operator family*

$$\{\tfrac{\mathrm{e}^{-\omega|s|}}{\tilde{v}(s)}\mathrm{e}^{-\mathrm{i}sA} \mid s \in \mathbb{R}\}$$

is semi-γ-bounded. Let $v : \mathbb{R} \to [1,\infty)$ be strongly admissible such that $\frac{\tilde{v}}{v} \in \mathrm{L}^2(\mathbb{R})$. Then, A has a γ-bounded $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$-calculus.

Proof. We proceed as in the proof of Theorem 8.4 and pick $\theta > \omega$ such that A has bounded $\mathrm{H}^\infty(\mathrm{St}_\theta)$-calculus, as well as functions $\psi, \psi_1, \psi_2 \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)$ as in Lemma 8.6. As before it suffices to show

$$[\![(\tau_n\psi \cdot f)(A) \mid n \in \mathbb{N},\ f \in \mathrm{H}_0^\infty(\mathrm{St}_\theta),\ \|f\|_{\mathrm{Hör}^2_v(\mathrm{St}_\omega)} \le 1]\!]_\gamma < \infty.$$

To that end, we again fix $f \in \mathrm{H}_0^\infty(\mathrm{St}_\theta)$ and consider the formula

$$(\tau_n\psi \cdot f)(A)x = \int_{\mathbb{R}} (\tau_n\psi \cdot f)^\vee(s)\mathrm{e}^{-\mathrm{i}sA}x\,\mathrm{d}s = \int_{\mathbb{R}} \Big(v(s)\mathrm{e}^{\omega|s|}(\tau_n\psi \cdot f)^\vee(s)\Big) \cdot \frac{\tilde{v}(s)}{v(s)} \Big(\frac{\mathrm{e}^{-\omega|s|}}{\tilde{v}(s)}\mathrm{e}^{-\mathrm{i}sA}x\Big)\,\mathrm{d}s$$

for each $n \in \mathbb{Z}$ and $x \in X$. Now an application of Theorem B.6 proves the claim. □

Remark 8.9.

(1) Theorem 8.4 is particularly relevant when $X = \mathrm{L}^p(\Omega)$ for some $1 < p < \infty$ and some measure space Ω. Namely, in this case $\mathrm{type}(X) = \min\{2, p\}$ and $\mathrm{cotype}(X) = \max\{p, 2\}$. This yields

$$\frac{1}{\mathrm{type}(X)} - \frac{1}{\mathrm{cotype} X} = \left|\frac{1}{2} - \frac{1}{p}\right| < \frac{1}{2}.$$

Hence, *each* $r \in [1, 2]$ is a possible choice, and one obtains the largest possible variety of bounded Hörmander calculi. However, as integrability is usually realized via growth conditions, one should think of $\frac{\tilde{v}}{v}$ being bounded. In this situation one has the implication $\frac{\tilde{v}}{v} \in \mathrm{L}^r$, $r \in [1, 2] \Rightarrow \frac{\tilde{v}}{v} \in \mathrm{L}^2$ and, hence, $r = 2$ is the optimal integrability exponent.

(2) Theorems 8.4 and 8.8 extend [KW18, Thm. 10.2] by Kriegler and Weis (for the case $r = 2$) from strip type $\omega = 0$ to general $\omega \geq 0$, and from classical Hörmander spaces (polynomial weights) to the finer scale of admissible functions.

To illustrate the latter, consider the case $\tilde{v} = (1 + |\mathbf{s}|)^\alpha$ and X being an L^p-space as above. Then, restricting to classical Sobolev spaces (=polynomial weights) just allows one to infer a γ-bounded $\mathrm{Hör}^2_{v_\beta}(\mathrm{St}_\omega)$-calculus for $\beta > \alpha + \frac{1}{2}$. For $\omega = 0$, this is Kriegler and Weis' result. However, with our finer regularity scale we are able to infer a γ-bounded $\mathrm{Hör}^2_v(\mathrm{St}_\omega)$ calculus even for the admissible functions

$$v \;=\; (1 + |\mathbf{s}|)^{\alpha + \frac{1}{2}} \big(\ln(\mathrm{e} + |\mathbf{s}|)\big)^\beta \qquad (\beta > 1)$$

or

$$v \;=\; (1 + |\mathbf{s}|)^{\alpha + \frac{1}{2}} \ln(\mathrm{e} + |\mathbf{s}|) \big(\ln\big(\mathrm{e} \ln(\mathrm{e} + |\mathbf{s}|)\big)\big)^\beta \qquad (\beta > 1).$$

(3) However, [KW18, Thm. 10.2] is not completely covered by Theorem 8.4, as we exclusively work with L^2-Sobolev spaces, where Kriegler and Weis also consider Hörmander spaces based on Sobolev spaces $\mathrm{W}^{\beta,p}(\mathbb{R})$ for $p \neq 2$.

9. Sectorial operators

In this section we describe how the obtained (function- and operator-theoretic) results transfer to the sectorial situation. This is the "original" set-up and certainly the one of most interest from the point of view of applications.

We start with the function theory. In all what follows, $\omega \in [0, \pi)$ and $v : \mathbb{R} \to [1, \infty)$ is admissible. Recall that

$$\mathrm{S}_\omega := \begin{cases} \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \omega\} & \text{if } \omega > 0, \\ \mathbb{R}_{>0} & \text{if } \omega = 0 \end{cases}$$

is the open sector of angle 2ω symmetric about the positive real axis. In addition, we let

$$\overline{\mathrm{S}}^*_\omega := \overline{\mathrm{S}_\omega} \setminus \{0\} \qquad (\omega \in [0, \pi]).$$

The function $\mathrm{e}^{\mathbf{z}}$ maps St_ω biholomorphically onto S_ω, with inverse $\log \mathbf{z}$. With the help of this "change of coordinates" we shall transfer concepts from strips to sectors.

For $0 < \omega < \pi$, the **Hardy space** of order $p \in [1, \infty]$ on S_ω is

$$\mathrm{H}^p(\mathrm{S}_\omega) := \{f \in \mathrm{Hol}(\mathrm{S}_\omega) \mid f(\mathrm{e}^{\mathbf{z}}) \in \mathrm{H}^p(\mathrm{St}_\omega)\}$$

endowed with its natural norm. Then the space of **e**lementary functions on S_ω is

$$\mathcal{E}(\mathrm{S}_\omega) = \mathrm{H}^1(\mathrm{S}_\omega) \cap \mathrm{H}^\infty(\mathrm{S}_\omega) = \{f \mid f(\mathrm{e}^{\mathbf{z}}) \in \mathcal{E}(\mathrm{St}_\omega)\}.$$

Similarly,

$$\mathrm{H}_0^\infty(\mathrm{S}_\omega) := \{f \mid f(\mathrm{e}^{\mathbf{z}}) \in \mathrm{H}_0^\infty(\mathrm{St}_\omega)\}.$$

Observe that this space is larger than the space $\mathrm{H}_0^\infty(\mathrm{St}_\omega)$ considered in [Haa, Sec. 2.2]. However, this should not lead to confusion.

The **generalized Sobolev space** on S_ω associated with v is

$$\mathrm{W}_v^2(\mathrm{S}_\omega) := \big\{f : \mathrm{S}_\omega \to \mathbb{C} \mid f(\mathrm{e}^{\mathbf{z}}) \in \mathrm{W}_v^2(\mathrm{St}_\omega)\big\}.$$

It is equipped with the norm

$$\|f\|_{\mathrm{W}_v^2(\mathrm{S}_\omega)} := \|f(\mathrm{e}^{\mathbf{s}})\|_{\mathrm{W}_v^2(\mathrm{St}_\omega)}.$$

Similarly, the **generalized Hörmander space** with respect to v is

$$\mathrm{H\ddot{o}r}_v^2(\mathrm{S}_\omega) := \big\{f : \mathrm{S}_\omega \to \mathbb{C} \,\big|\, f(\mathrm{e}^{\mathbf{z}}) \in \mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega)\big\}.$$

It is equipped with the norm

$$\|f\|_{\mathrm{H\ddot{o}r}_v^2(\mathrm{S}_\omega)} := \|f(\mathrm{e}^{\mathbf{z}})\|_{\mathrm{H\ddot{o}r}_v^2(\mathrm{St}_\omega)}.$$

We abbreviate

$$\mathrm{W}^{\alpha,2}(\mathrm{S}_\omega) := \mathrm{W}_{v_\alpha}^2(\mathrm{S}_\omega), \qquad \text{and} \qquad \mathrm{H\ddot{o}r}^{\alpha,2}(\mathrm{S}_\omega) := \mathrm{H\ddot{o}r}_{v_\alpha}^2(\mathrm{S}_\omega),$$

where $\alpha \geq 0$ and $v_\alpha = (1 + |\mathbf{s}|)^\alpha$. Recall that v_α is strongly admissible if and only if $\alpha > \frac{1}{2}$.

Key properties of the spaces $\mathrm{W}_v^2(\mathrm{S}_\omega)$ and $\mathrm{H\ddot{o}r}_v^2(\mathrm{S}_\omega)$ can easily be deduced from their strip counterparts. Here we just list a few of them for the case that v is strongly admissible.

Proposition 9.1. *Let $0 \leq \omega < \theta < \pi$, and $v : \mathbb{R} \to [1, \infty)$ be strongly admissible. Then the following statements hold:*

(a) $\mathcal{E}(\mathrm{S}_\theta) \subseteq \mathrm{W}_v^2(\mathrm{S}_\omega)$ *and the space*

$$\{f \in \mathrm{Hol}(\mathrm{S}_\theta) \mid \exists\, a > 0 : |f| \lesssim \mathrm{e}^{-a(\ln|\mathbf{z}|)^2}\}$$

is dense in $\mathrm{W}_v^2(\mathrm{S}_\omega)$.

(b) *There are canonical (continuous) embeddings*

$$\mathrm{W}_v^2(\mathrm{S}_\omega) \hookrightarrow \mathrm{C}_0(\overline{\mathrm{S}}_\omega^*), \quad \mathrm{H\ddot{o}r}_v^2(\mathrm{S}_\omega) \hookrightarrow \mathrm{C_b}(\overline{\mathrm{S}}_\omega^*), \quad \text{and} \quad \mathrm{H}^\infty(\mathrm{S}_\theta) \hookrightarrow \mathrm{H\ddot{o}r}_v^2(\mathrm{S}_\omega).$$

(c) *The spaces* $\mathrm{W}_v^2(\mathrm{S}_\omega)$ *and* $\mathrm{H\ddot{o}r}_v^2(\mathrm{S}_\omega)$ *are Banach algebras with respect to pointwise multiplication.*

(d) *In case $\omega > 0$ one has $f \in \mathrm{H\ddot{o}r}_v^2(\mathrm{S}_\omega)$ if and only if $f \in \mathrm{H}^\infty(\mathrm{S}_\omega) \cap \mathrm{C_b}(\overline{\mathrm{S}}_\omega^*)$ and $f(\mathrm{e}^{\pm\mathrm{i}\omega}\mathbf{s}) \in \mathrm{H\ddot{o}r}_v^2(\mathbb{R}_{>0})$.*

Proof. (a) follows from Lemma 3.5. (b) and (c) follow from Proposition 3.3(d) and Theorem 5.6, parts (a) and (c). (d) follows from Theorem 5.6(d). □

Remark 9.2 (Connection to classical Hörmander spaces). Let $\alpha > \frac{1}{2}$ and $\omega > 0$. The classical Hörmander condition of order α for $g \in \mathrm{L}^\infty(\mathbb{R}_+)$ reads

$$\sup_{t>0} \|\eta \cdot g(t\mathbf{s})\|_{\mathrm{W}^{\alpha,2}(\mathbb{R})} < \infty, \tag{9.1}$$

where $0 \neq \eta \in \mathrm{C}_c^\infty(\mathbb{R}_{>0})$ is arbitrary. Since test functions on $\mathbb{R}_{>0}$ and test functions on $\mathbb{R}$ correspond to each other via the exp-log-correspondence, (9.1) is equivalent to $g(\mathrm{e}^{\mathbf{s}}) \in \mathrm{Hör}^2_{v_\alpha}(\mathbb{R})$. (Recall Theorem 4.3 for the independence from the defining function η.) It follows that the Hörmander space $\mathrm{Hör}^{\alpha,2}(\mathrm{S}_0)$ as defined above coincides precisely with the classical Hörmander space $\mathrm{Hör}^{\alpha,2}(\mathbb{R}_+)$.

Consequently, by Proposition 9.1(d), the Hörmander space $\mathrm{Hör}^{\alpha,2}(\mathrm{S}_\omega)$ as defined above coincides precisely with the space $\mathrm{H}^\infty(\mathrm{S}_\omega;\alpha)$ considered in [GCM+01], [MMS04], [Sas05], and [CD17], cf. also (1.2).

Let us now turn to the operator theory. An operator A on a Banach space X is **sectorial** of angle $\omega \in [0,\pi)$ if $\sigma(A) \subseteq \overline{\mathrm{S}_\omega}$ and $\sup\{\|\lambda R(\lambda, A)\| \mid \lambda \in C \setminus \overline{\mathrm{S}_{\omega'}}\} < \infty$ for each $\omega < \omega' < \pi$.

The theory of sectorial operators and their holomorphic functional calculus is presented at many places in the literature, e.g. in [Haa] or [HvNVW, Chap. 10], and we assume the reader to be familiar with the basic facts. In particular, we recall that only for *injective* sectorial operators there is a satisfying notion of a bounded H^∞-calculus on a sector. Given a non-injective sectorial operator A on a Banach space X one either has to pass to $X/\ker(A)$ or to $\overline{\mathrm{ran}}(A)$, cf. [Haa, Prop. 2.2.1.h].

If A is injective and sectorial of angle ω, the operator $\log(A)$ (defined via the functional calculus) is of strip-type ω, and it is densely defined if and only if A is densely defined and has dense range.

The holomorphic functional calculi for A and $\log(A)$ are connected via the **composition rule**

$$f(A) = [f(\mathrm{e}^{\mathbf{z}})](\log(A)), \qquad g(\log(A)) = [g(\log \mathbf{z})](A), \tag{9.2}$$

see [Haa, Cor. 4.2.5]. Analogous to the strip case we can define the notion of a bounded $\mathrm{W}^2_v(\mathrm{S}_\omega)$-calculus and a bounded (γ-bounded) $\mathrm{Hör}^2_v(\mathrm{S}_\omega)$-calculus. The following is straightforward to check.

Proposition 9.3 (Composition rule). *Let A be an injective sectorial operator on a Banach space X, let $\omega \in [0,\pi)$ and $v : \mathbb{R} \to [1,\infty)$ strongly admissible. Then $\log A$ has a bounded $\mathrm{W}^2_v(\mathrm{St}_\omega)$-calculus if and only if A has a bounded $\mathrm{W}^2_v(\mathrm{S}_\omega)$-calculus, and the calculi are related via the composition rule* (9.2)*.*

As in the strip case, a bounded Sobolev calculus implies an (at least) unbounded Hörmander calculus by algebraic regularization. The connection via the composition rule remains valid. Therefore, we can simply transfer theorems about bounded Hörmander calculus on strips to theorems about bounded Hörmander calculus on sectors. The first is the sectorial analogue of Theorem 8.3.

Theorem 9.4. *Let A be a densely defined operator on a Banach space X, with dense range and a bounded $\mathrm{H}^\infty(\mathrm{S}_\theta)$-calculus for some $0 < \theta < \pi$. Suppose*

$$\|A^{-\mathrm{i}s}\| \le \tilde{v}(s)\mathrm{e}^{\omega|s|} \qquad (s \in \mathbb{R})$$

for some $\omega \in [0,\pi)$ and some measurable function $\tilde{v} : \mathbb{R} \to \mathbb{R}_{\ge 0}$. Let $v : \mathbb{R} \to [1,\infty)$ be strongly admissible such that $\tilde{v}/v \in \mathrm{L}^1(\mathbb{R})$. Then A has a bounded $\mathrm{Hör}^2_v(\mathrm{S}_\omega)$-calculus.

Proof. The operator $\log(A)$ satisfies the hypotheses of Theorem 8.3. Hence, $\log(A)$ has a bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$-calculus. Transferring this back to the sector with the composition rule yields that A has a bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{S}_\omega)$-calculus. □

And here is the sectorial analogue of Theorem 8.4.

Theorem 9.5. *Let X be a Banach space of finite cotype $q \in [2,\infty)$ and type $p \in [1,2]$, and let A be a densely defined operator on X with dense range and a bounded $\mathrm{H}^\infty(\mathrm{S}_\theta)$-calculus for some $0 < \theta < \pi$. Suppose that $\omega \in [0,\pi)$ and $\tilde{v} : \mathbb{R} \to (0,\infty)$ is measurable with*

$$\|A^{-\mathrm{i}s}\| \le \tilde{v}(s)\mathrm{e}^{\omega|s|} \qquad (s \in \mathbb{R}).$$

Let $r \in [1,2]$ with $\frac{1}{r} > \frac{1}{p} - \frac{1}{q}$ and let $v : \mathbb{R} \to [1,\infty)$ be strongly admissible such that $\tilde{v}/v \in \mathrm{L}^r(\mathbb{R})$. Then, A has a bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{S}_\omega)$-calculus. If, in addition, X has Pisier's contraction property, then this calculus is γ-bounded.

Proof. Analogous to the proof of Theorem 9.4, now by virtue of Theorem 8.4. □

Finally, here is the analogue of Theorem 8.8.

Theorem 9.6. *Let X be a Banach space with Pisier's contraction property, and let A be a densely defined operator on X with dense range and a bounded $\mathrm{H}^\infty(\mathrm{S}_\theta)$-calculus for some $0 < \theta < \pi$. Suppose that $\omega \in [0,\pi)$ and $\tilde{v} : \mathbb{R} \to (0,\infty)$ is measurable such that the operator family*

$$\left\{\tfrac{\mathrm{e}^{-\omega|s|}}{\tilde{v}(s)} A^{-\mathrm{i}s}\right\}$$

is semi-γ-bounded. Let $v : \mathbb{R} \to [1,\infty)$ be strongly admissible such that $\tilde{v}/v \in \mathrm{L}^2(\mathbb{R})$. Then, A has a γ-bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{S}_\omega)$-calculus.

Theorems 9.5 and 8.8 are (for $r = 2$) a generalization of [KW18, Thm. 6.1(2)] to sectoriality angles $\omega > 0$ and to more general smoothness conditions, cf. also Remark 8.9.

Remark 9.7 (Non-injective operators). Let A be a densely defined sectorial operator of angle ω on a Banach space X, and let $Y := \overline{\mathrm{ran}}(A)$. Then Y is invariant under the resolvent of A and hence the part A_Y of A in Y is sectorial of angle ω, with $R(\lambda, A_Y) = R(\lambda, A)|_Y$ for $\lambda \in \rho(A)$, cf. [Haa, Sec. 2.6.2]. Moreover, A_Y is densely defined and has dense range. (This follows from [Haa, Prop. 2.1.1.c].) We shall call A_Y the **injective part** of A. It is easy to see that $f(A_Y) = f(A)|_Y$ for $f \in \mathcal{E}[\mathrm{S}_\omega]$ [Haa, Prop. 2.3.6].

By abuse of notation, we shall say that A **has a bounded $\mathcal{F}$-calculus** (where $\mathcal{F}$ may be $\mathrm{H}^\infty(\mathrm{S}_\omega)$ or $\mathrm{W}^2_v(\mathrm{S}_\omega)$ or $\mathrm{H\ddot{o}r}^2_v(\mathrm{S}_\omega)$), whenever its injective part A_Y does so. With this terminology, Theorems 9.4—9.6 remain valid even without the assumption that A has dense range.

Note that if X is reflexive then it decomposes as a direct sum

$$X = \overline{\mathrm{ran}}(A) \oplus \ker(A).$$

Clearly $f(A)|_{\ker(A)} = 0$ for each elementary function f. Hence, one may formulate characterizations of a bounded $\mathcal{F}$-calculus for A (i.e., in effect, for A_Y) similar to Proposition 7.3 and Theorem 8.2. We shall not elaborate on this.

An alternative way to deal with a non-injective operator A is to perturb it into $A + \varepsilon$ and then require functional calculus bounds that are independent of $\varepsilon > 0$.

This has, e.g., been done in [GCM+01]. Again, this leads to equivalent concepts, but also here we do not go further into detail.

10. Applications

10.1. Groups on Hilbert spaces. Let H be a Hilbert space and $-\mathrm{i}A$ the generator of a C_0-group $(U_s)_{s\in\mathbb{R}}$ on H with

$$\|U_s\| \le \tilde{v}(s)\mathrm{e}^{\omega|s|},$$

where $\omega \ge 0$ and $\tilde{v}$ is some polynomially bounded function. By the Boyadzhiev–de Laubenfels theorem [Haa, Thm. 7.2.1] A has a bounded $\mathrm{H}^\infty(\mathrm{St}_\theta)$-calculus for each $\theta > \omega$.

As H has type and cotype 2, Theorem 8.4 implies that A actually has a bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$-calculus whenever v is strongly admissible and $\frac{\tilde{v}}{v} \in \mathrm{L}^r(\mathbb{R})$ for some $r \in [1,2]$. As discussed in Remark 8.9, integrability is usually realized through growth conditions, and one should think of $\frac{\tilde{v}}{v}$ being bounded. In this respect, $r = 2$ represents the optimal exponent.

For example, if $\tilde{v} \equiv c$ is constant, then A has bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{St}_\omega)$-calculus for each strongly admissible function v. In particular, A has a (classical) $\mathrm{H\ddot{o}r}^{\alpha,2}(\mathrm{St}_\omega)$-calculus for each $\alpha > \frac{1}{2}$.

10.2. Sectorial operators on L^p-spaces. Let (Ω,μ) be a measure space, let $1 < p < \infty$ and A a sectorial operator on $X = \mathrm{L}^p(\Omega)$ with a bounded $\mathrm{H}^\infty(\mathrm{S}_\theta)$-calculus for some angle $\theta > 0$. By abuse of notation we write A also for its injective part (see Remark 9.7). Suppose one has an estimate

$$\|A^{-\mathrm{i}s}\| \le \tilde{v}(s)\mathrm{e}^{\omega|s|} \qquad (s \in \mathbb{R}),$$

where $\tilde{v}$ is some positive measurable function and $\omega \ge 0$. As discussed in Remark 8.9, we have

$$\frac{1}{\mathrm{type}(X)} - \frac{1}{\mathrm{cotype}(X)} = \Big|\frac{1}{2} - \frac{1}{p}\Big| < \frac{1}{2}.$$

Hence, $r = 2$ is a possible choice in Theorem 9.5 (and the optimal one if, in addition, $\frac{\tilde{v}}{v}$ is bounded). It follows that A has a γ-bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{S}_\omega)$-calculus whenever $\frac{\tilde{v}}{v} \in \mathrm{L}^2(\mathbb{R})$. In particular, if $\tilde{v} \lesssim (1+|\mathbf{s}|)^\alpha$ for some $\alpha > 0$ is constant, then A has γ-bounded $\mathrm{H\ddot{o}r}^{\beta,2}(\mathrm{S}_\omega)$-calculus for each $\beta > \alpha + \frac{1}{2}$.

10.3. Symmetric contraction semigroups. Let (Ω,μ) denote a σ-finite measure space. We shall abbreviate $\mathrm{L}^1 := \mathrm{L}^1(\Omega,\mu)$ and $\mathrm{L}^\infty := \mathrm{L}^\infty(\Omega,\mu)$. A family of operators

$$T_t : \mathrm{L}^1 \cap \mathrm{L}^\infty \to \mathrm{L}^1 + \mathrm{L}^\infty \qquad (t \ge 0)$$

is called a **symmetric contraction semigroup** if it has the following properties:

(1) $T_t T_s = T_{t+s}$ for all $s,t \ge 0$;

(2) $\|T_t f\|_1 \le \|f\|_1$ and $\|T_t f\|_\infty \le \|f\|_\infty$ for all $f \in \mathrm{L}^1 \cap \mathrm{L}^\infty$ and all $t \ge 0$;

(3) $\|T_t f - f\|_p \xrightarrow{t\searrow 0} 0$ for all $f \in \mathrm{L}^1 \cap \mathrm{L}^\infty$ and $p \in [1,\infty)$;

(4) $\displaystyle\int_\Omega T_t f \cdot \overline{g}\,\mathrm{d}\mu = \int_\Omega f \cdot \overline{T_t g}\,\mathrm{d}\mu$ for each $f,g \in \mathrm{L}^1 \cap \mathrm{L}^\infty$ and $t \ge 0$.

It is an easy consequence of the Riesz-Thorin interpolation theorem that each symmetric contraction semigroup extends to a contractive C_0-semigroup on $\mathrm{L}^p(\Omega,\mu)$ for each $1 \le p < \infty$. We denote the respective generator by $-A_p$. Cowling [Cow83] has shown that A_p has a bounded $\mathrm{H}^\infty(\mathrm{S}_\theta)$-calculus for each angle $\theta > \pi|\frac{1}{p} - \frac{1}{2}|$. In [CM93, Thm. 2.1, Cor. 2.2], Cowling and Meda combined this result with ideas from [Med90] to obtain sharper functional calculus bounds from additional hypotheses about the growth behaviour of the group of imaginary powers $(A_p^{-\mathrm{i}s})_{s\in\mathbb{R}}$. Moreover, a bounded Hörmander-type calculus on S_0 was inferred when the imaginary powers grow at most polynomially [CM93, Cor. 2.3]. As described in Section 1, in [GCM+01] García-Cuerva et al. extended the latter result to the sectorial case, see Theorem 1.3.

For a long time it remained an open problem to determine the optimal angle for the calculus. The angle in Cowling's result from above was obtained by complex interpolation and turned out to be not optimal. Finally, in [CD17], Carbonaro and Dragičević showed that this angle is

$$\omega_p := \arcsin\left|1 - \frac{2}{p}\right|.$$

In fact, they showed more [CD17, Proposition 11]:

Theorem 10.1 (Carbonaro–Dragičević)**.** *Let $-A$ be the generator of a symmetric contraction semigroup over some measure space (Ω,μ). Then for each $1 < p < \infty$ there is a constant $c_p \ge 0$ such that*

$$\|A_p^{-\mathrm{i}s} f\|_p \le c_p(1+|s|)^{\frac{1}{2}} \mathrm{e}^{\omega_p |s|}\|f\|_p \qquad (s \in \mathbb{R},\ f \in \overline{\mathrm{ran}}(A_p)).$$

As said in Section 1, one can pull out of this a bounded $\mathrm{Hör}^{\beta,2}(\mathrm{S}_{\omega_p})$-calculus for A_p for all $\beta > \frac{3}{2}$ with the help of the result by García-Cuerva et al., see [CD17, Theorem 1]. However, with our refined methods, we can say more:

Theorem 10.2. *Let $1 < p < \infty$ and let $-A_p$ be the L^p-generator of a symmetric contraction semigroup. Then (the injective part of) A_p has a γ-bounded $\mathrm{Hör}^2_v(\mathrm{S}_{\omega_p})$-calculus for each admissible function $v : \mathbb{R} \to [1,\infty)$ such that*

$$\frac{(1+|\mathbf{s}|)^{\frac{1}{2}}}{v} \in \mathrm{L}^2(\mathbb{R}).$$

In particular, A_p has a γ-bounded $\mathrm{Hör}^{\beta,2}(\mathrm{S}_{\omega_p})$-calculus for each $\beta > 1$.

Our results also allow for a strengthening of a result (Corollary 4.4) in the recent paper [DKP21] by Domelevo, Kriegler and Petermichl.[4] There, the authors start from a certain symmetric contraction semigroup and then, under certain technical assumptions, infer a bounded $\mathrm{H}^\infty(\mathrm{S}_{\frac{\pi}{2}}; J)$-calculus for the generator A on a *weighted* space $\mathrm{L}^2(\Omega; w\mathrm{d}\mu)$. Here $J > 1$ and $\mathrm{H}^\infty(\mathrm{S}_{\frac{\pi}{2}}; J)$ is the algebra of bounded holomorphic functions on the half-plane $\mathrm{S}_{\frac{\pi}{2}}$ that have traces in the *Besov space* $\mathrm{B}^J_{\infty,1}(\mathbb{R})$ (after pulling the halflines back to $\mathbb{R}$ via the exponential map, of course).

However, in the proof the authors establish exactly the same bilinear estimates as Carbonaro and Dragičević. These imply the bound $\mathrm{e}^{\frac{\pi}{2}|s|} \cdot (1+|s|)^{-\frac{1}{2}}$ on the growth of $\|A^{\mathrm{i}s}\|$ (up to a constant depending in a controlled way on w). Consequently, our results imply the boundedness of a $\mathrm{Hör}^{J',2}(\mathrm{S}_{\frac{\pi}{2}})$-calculus for each $J' > \frac{1}{2}$ (with

[4]This observation is due to Lukas Hagedorn (Kiel).

bound depending on w in the same way as before). And this is stronger than the Domelevo-Kriegler-Petermichl theorem as, for $J > 1 > J' > \frac{1}{2}$,

$$\mathrm{B}^{J}_{\infty,1}(\mathbb{R}) \subseteq \mathrm{H\ddot{o}r}^{1,2}(\mathbb{R}) \subseteq \mathrm{H\ddot{o}r}^{J',2}(\mathbb{R}),$$

with strict inclusions.

Actually, the abovementioned bounds on the imaginary powers yield (by standard semigroup theory) a strictly smaller *exponential* bound on $\|A^{\mathrm{i}s}\|$. Hence, A has a bounded $\mathrm{H}^\infty(\mathrm{S}_\omega)$-calculus even for some angle $\omega < \frac{\pi}{2}$ (see [Haa, Thm. 5.4.1]). The dependence on w of the norm of this calculus, however, is obscure.

10.4. **The Ornstein–Uhlenbeck semigroup.** The Carbonaro–Dragičević result is optimal at least with respect to the angle ω_p. In fact, it had been shown before that a limiting case is provided by the so-called *Ornstein–Uhlenbeck semigroup* on $(\mathbb{R}^d, \gamma_d)$, where γ_d is the standard Gaussian measure on $\mathbb{R}^d$, i.e.,

$$\gamma_d(\mathrm{d}x) \;=\; \frac{1}{(2\pi)^{d/2}} \mathrm{e}^{-\frac{|x|^2}{2}}\,\mathrm{d}x.$$

The L^p-generator is given by $-\mathcal{L}_p$, where

$$\mathcal{L}_p := -\frac{1}{2}\Delta + \mathbf{x}\cdot\nabla,$$

and where Δ is the Laplacian on $\mathbb{R}^d$, $\mathbf{x}$ is the mapping $x \mapsto x$, ∇ is the gradient operator on $\mathbb{R}^d$.

It has been noted in [GCM⁺01, Thm. 2] that $\mathcal{L}_p$ does not have a bounded $\mathrm{H}^\infty(\mathrm{S}_\omega)$-calculus for any $\omega < \omega_p$. Actually, ω_p is the precise sectoriality angle of $\mathcal{L}_p$, see [CFMP05]. Hence, the angle ω_p in Theorem 10.2 is optimal.

Nevertheless, $\mathcal{L}_p$ itself allows for a stronger statement. In [MMS04, Thm. 4.3], Mauceri, Meda and Sjögren proved that for $1 < p < \infty$ there is $c_p \geq 0$ such that

$$\|\mathcal{L}_p^{-\mathrm{i}s} f\|_p \;\leq\; c_p \mathrm{e}^{\omega_p |s|}\|f\|_p \qquad (f \in \overline{\mathrm{ran}}\mathcal{L}_p,\ s \in \mathbb{R}),$$

where by abuse of notation we again write $\mathcal{L}_p$ for the injective part of $\mathcal{L}_p$. Hence, we may apply our findings from above and obtain:

Theorem 10.3. *Let $1 < p < \infty$. Then the Ornstein–Uhlenbeck operator $\mathcal{L}_p$ on $\mathrm{L}^p(\mathbb{R}^d, \gamma_p)$ has a γ-bounded $\mathrm{H\ddot{o}r}^2_v(\mathrm{S}_{\omega_p})$-calculus for each strongly admissible function $v : \mathbb{R} \to [1, \infty)$. In particular, A_p has a γ-bounded $\mathrm{H\ddot{o}r}^{\beta,2}(\mathrm{S}_{\omega_p})$-calculus for each $\beta > \frac{1}{2}$.*

Appendix A. Auxiliary lemmas

The following is related to Fubini's theorem.

Lemma A.1. *Let $f, g \in \mathrm{L}^1(\mathbb{R})$. Then*

(a) $\tau_t f \cdot g \in \mathrm{L}^1(\mathbb{R})$ *for almost all* $t \in \mathbb{R}$;

(b) $\tau_{\mathbf{t}} f \cdot g \in \mathrm{L}^1(\mathbb{R}; \mathrm{L}^1(\mathbb{R}))$;

(c) $\displaystyle\int_{\mathbb{R}} \tau_t f \cdot g \,\mathrm{d}t = \Big(\int_{\mathbb{R}} f(t)\,\mathrm{d}t\Big)\, g$ *as an integral in* $\mathrm{L}^1(\mathbb{R})$.

Proof. First let $f, g \in \mathrm{C_c}(\mathbb{R})$. Then $\tau_t f \cdot g \in \mathrm{L}^1(\mathbb{R})$ for *all* $t \in \mathbb{R}$ and $\tau_{\mathbf{t}} f \cdot g \in \mathrm{C_c}(\mathbb{R}; \mathrm{L}^1(\mathbb{R}) \cap \mathrm{C}_0(\mathbb{R}))$. As point evaluations are continuous on $\mathrm{C}_0(\mathbb{R})$,

$$\int_{\mathbb{R}} \tau_t f \cdot g \,\mathrm{d}t(x) = \int_{\mathbb{R}} f(x-t)g(x)\,\mathrm{d}t = \int_{\mathbb{R}} f(t)\,\mathrm{d}t \cdot g(x) \qquad (x \in \mathbb{R}),$$

and hence (c) holds. It follows that

$$\Big(\int_{\mathbb{R}} |f|\Big)|g| = \int_{\mathbb{R}} \tau_t |f| \cdot |g| \,\mathrm{d}t = \int_{\mathbb{R}} |\tau_t f \cdot g| \,\mathrm{d}t$$

and integrating (which amounts to applying a bounded linear functional on $\mathrm{L}^1(\mathbb{R})$) yields

$$\|f\|_1 \|g\|_1 = \int_{\mathbb{R}} \|\tau_t f \ \cdot g\|_1 \,\mathrm{d}t.$$

Thus we obtain a bounded bilinear mapping

$$\Phi : \mathrm{L}^1(\mathbb{R}) \times \mathrm{L}^1(\mathbb{R}) \to \mathrm{L}^1(\mathbb{R}; \mathrm{L}^1(\mathbb{R})), \qquad \Phi(f,g) := \tau_{\mathbf{t}} f \cdot g \quad \text{if } f, g \in \mathrm{C_c}(\mathbb{R}).$$

Next, let f, g be arbitrary and take $f_n, g_n \in \mathrm{C_c}(\mathbb{R})$ with $f_n \to f$ and $g_n \to g$ in $\mathrm{L}^1(\mathbb{R})$. Without loss of generality $f_n \to f$ and $g_n \to g$ almost everywhere and $\tau_{\mathbf{t}} f_n \cdot g_n = \Phi(f_n, g_n) \to \Phi(f,g)$ almost everywhere. On the other hand, $\tau_t f_n \cdot g_n \to \tau_t f \cdot g$ almost everywhere for *each* $t \in \mathbb{R}$, hence $\Phi(f,g)(t) = \tau_t f \cdot g$ in $\mathrm{L}^1(\mathbb{R})$ for almost all $t \in \mathbb{R}$. This proves (a) and (b). And (c) follows since integration is a bounded linear operator/functional. □

Lemma A.2. *Let $f \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$ and $g \in \mathrm{L}^p(\mathbb{R})$, $1 \le p < \infty$. Then $\tau_{\mathbf{t}} f \cdot g \in \mathrm{C_b}(\mathbb{R}; \mathrm{L}^p(\mathbb{R}))$ and*

$$\int_{\mathbb{R}} \tau_t f \cdot g \,\mathrm{d}t = \Big(\int_{\mathbb{R}} f\Big)\, g$$

as a weak integral.

Proof. Write $\tau_t f \cdot g = \tau_t(f \cdot \tau_{-t} g)$ to see that the function $\tau_{\mathbf{t}} f \cdot g$ is continuous with values in $\mathrm{L}^p(\mathbb{R})$. To see that it is weakly integrable, let $h \in \mathrm{L}^{p'}(\mathbb{R})$ and compute

$$\begin{aligned}\int_{\mathbb{R}} \langle \tau_t f \cdot g, h\rangle \,\mathrm{d}t &= \int_{\mathbb{R}} \langle \tau_t f \cdot (gh), \mathbf{1}\rangle \,\mathrm{d}t \overset{(*)}{=} \langle \int_{\mathbb{R}} \tau_t f \cdot (gh)\,\mathrm{d}t, \mathbf{1}\rangle \\ &= \Big(\int_{\mathbb{R}} f\Big)\, \langle gh, \mathbf{1}\rangle = \Big(\int_{\mathbb{R}} f\Big)\, \langle g, h\rangle,\end{aligned}$$

where we used Lemma A.1 and that $gh \in \mathrm{L}^1(\mathbb{R})$ at $(*)$. □

Lemma A.3. *Let $1 \le p < \infty$ and f, g be measurable such that there are $\alpha > 1$ and $\beta > \frac{1}{p}$ with*

$$|f(s)| \lesssim \frac{1}{(1+|s|)^\alpha}, \quad |g(s)| \lesssim \frac{1}{(1+|s|)^{\alpha+\beta}} \qquad (s \in \mathbb{R}).$$

Then the function $\tau_{\mathbf{t}} f \cdot g$ is in $\mathrm{C_b}(\mathbb{R}; \mathrm{L}^p(\mathbb{R}))$ and

$$\int_{\mathbb{R}} \|\tau_t f \cdot g\|_{\mathrm{L}^p} \,\mathrm{d}t < \infty \quad \text{and} \quad \int_{\mathbb{R}} \tau_t f \cdot g \,\mathrm{d}t = \Big(\int_{\mathbb{R}} f\Big)\, g.$$

Proof. We note that $f \in \mathrm{L}^1 \cap \mathrm{L}^\infty$ and hence Lemma A.2 is applicable. We only need to show the strong integrability. From $1+|x+y| \le (1+|x|)(1+|y|)$ it follows (with $x = t-s$ and $y = s$) that

$$|\tau_t f \cdot g|(s) \lesssim \frac{1}{(1+|t-s|)^\alpha} \frac{1}{(1+|s|)^{(\alpha+\beta)}} \le \frac{1}{(1+|t|)^\alpha} \frac{1}{(1+|s|)^\beta}$$

and hence

$$\|\tau_t f \cdot g\|_p \lesssim \frac{1}{(1+|t|)^\alpha} \qquad (t \in \mathbb{R}).$$

□

Appendix B. Abstract Littlewood–Paley theory

In this appendix we provide some definitions and results from the theory of abstract square function estimates and their connection with functional calculus as promoted by Kalton and Weis in their seminal work [KW16]. A more detailed account can be found in [HH13], an exhaustive treatment in [HvNVW].

In the following, X, Y are Banach spaces and $(\gamma_j)_{j\in J}$ denotes, generically, a family (over a context-depending index set J) of independent $\mathbb{C}$-valued normalized Gaussian random variables. We assume that the reader is familiar with the notion of (Rademacher) type and cotype as developed in [DJT], see also [HvNVW, Chap. 7].

Let $(x_j)_{j\in J}$ be any family in X. It is easy to see (by the symmetry of a standard Gaussian) that the net

$$\Big(\mathbb{E}\Big\|\sum_{j\in F} \gamma_j x_j\Big\|_X^2\Big)_{F\subseteq J \text{ finite}}$$

is increasing. The family $(x_j)_j$ is called **γ-summing** if this net is bounded, i.e. if

$$\|(x_j)_j\|_{\gamma(J;X)} := \sup_F \Big(\mathbb{E}\Big\|\sum_{j\in F} \gamma_n x_j\Big\|_X^2\Big)^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all finite subsets $F \subseteq J$. The space of all γ-summing families $(x_j)_j \in X^J$, endowed with the norm $\|\cdot\|_{\gamma(J;X)}$, is denoted by $\gamma_\infty(J;X)$. The space of **γ-radonifying** families is the closure

$$\gamma(J;X) := \overline{\mathrm{c}_{00}(J;X)}$$

of the set of finite sequences within $\gamma_\infty(J;X)$. By a classical result, $\gamma_\infty(J;X) = \gamma(J;X)$ if X has finite cotype, see [HvNVW, Thm. 6.4.10].

A Banach space X has **(Pisier's) contraction property**[5] if the canonical mapping

$$\mathrm{c}_{00}(\mathbb{Z}^2;X) \to \mathrm{c}_{00}(\mathbb{Z};\gamma(\mathbb{Z};X)), \qquad (x_{ij})_{i,j} \mapsto \big((x_{ij})_i\big)_j$$

extends to an isomorphism

$$\gamma(\mathbb{Z}^2;X) \cong \gamma(\mathbb{Z};\gamma(\mathbb{Z};X)).$$

If X has Pisier's contraction property, then X has finite cotype. For further information, see [HvNVW, Sec. 7.5].

[5]This property is also called "property (α)", but we follow the suggestion in [HvNVW] to change that terminology.

Remarks B.1.
(1) If X has finite cotype, Gaussian and Rademacher random sums are equivalent in the sense that the space $\gamma_\infty(J;X)$ does not change (up to an equivalent norm) if in its definition one employs Rademacher variables instead of Gaussians, see [HvNVW, Cor. 7.2.10].

Similarly, Pisier's contraction property can equivalently be defined by employing Rademacher variables instead of Gaussian variables (because either formulation implies that X has finite cotype).

(2) Each Hilbert space, and hence each finite-dimensional Banach space, has Pisier's contraction property, type 2 and cotype 2.

(3) Let $1 \le p \le \infty$ and $X := \mathrm{L}^p(\Omega', \mu)$ for some measure space (Ω', μ), such that $\dim X = +\infty$. Then, if $1 \le p < \infty$:

(1) X has Pisier's contraction property;
(2) X has type $\min\{2, p\}$, and
(3) X has cotype $\max\{2, p\}$. In particular,
(4) X has finite cotype.

If $p = \infty$, however, X lacks the contraction property, has type 1 and cotype ∞.

A set $\mathcal{T} \subseteq \mathcal{L}(X;Y)$ is called **γ-bounded** if there is a constant $C \ge 0$ such that

$$\|(T_j x_j)_{j\in J}\|_{\gamma(J;Y)} \le C \|(x_j)_j\|_{\gamma(J;X)}$$

for all finite families $(T_j, x_j)_j$ in $\mathcal{T} \times X$. In this case, the least number C with this property is denoted by $[\![\mathcal{T}]\!]_\gamma$ and called the **γ-bound** of $\mathcal{T}$. Clearly, if $\mathcal{T}' \subseteq \mathcal{T}$ then $[\![\mathcal{T}']\!]_\gamma \le [\![\mathcal{T}]\!]_\gamma$. The proof of Lemma B.2 is straightforward.

Lemma B.2. *Let X, Y be Banach space and $\mathcal{T} \subseteq \mathcal{L}(X;Y)$ γ-bounded. Then any (arbitrary) family $(T_j)_{j\in J}$ in $\mathcal{T}$ induces a bounded "diagonal operator"*

$$T := (T_j)_j : \gamma_\infty(J;X) \to \gamma_\infty(J;Y), \qquad (x_j)_j \mapsto (T_j x_j)_j.$$

Moreover, $\|T\|_{\gamma_\infty(J;X)\to\gamma_\infty(J;Y)} \le [\![\mathcal{T}]\!]_\gamma$.

Remark B.3. Exchanging Gaussian variables for Rademachers yields the notion of an **R-bounded** set of operators. However, as long as X, Y have finite cotype, γ-boundedness is equivalent to R-boundedness.

Theorem B.4. *Let X be a Banach space, let (Ω, μ) be a measure space, and let $r \in [1, \infty)$ with*

$$\frac{1}{r} > \frac{1}{\mathrm{type}(X)} - \frac{1}{\mathrm{cotype}(X)}.$$

Furthermore, let $F : \Omega \to \mathcal{L}(X)$ be strongly μ-measurable with

$$\sup_{\|x\|\le 1} \|F(\mathbf{s})x\|_{\mathrm{L}^r(\Omega;X)} < \infty.$$

Consider for each $\varphi \in \mathrm{L}^{r'}(\Omega, \mu)$ the operator

$$I_{F,\varphi} : X \to X, \quad x \mapsto \int_\Omega \varphi(s) \cdot F(s)x \,\mathrm{d}\mu(s).$$

Then, the set $\mathcal{T}_{F,r'} := \{I_{F,\varphi} \mid \|\varphi\|_{r'} \le 1\}$ is γ-bounded with

$$[\![\mathcal{T}_{F,r'}]\!]_\gamma \lesssim \sup_{\|x\|\le 1} \|F(\mathbf{s})x\|_{\mathrm{L}^r(\Omega;X)}.$$

Proof. The hypotheses imply that X has finite cotype and, thus, Rademacher and Gaussian sums are equivalent. Hence, the result is contained in [HvNVW, Theorem 8.5.12]. $\square$

Remark B.5. Theorem B.4 also holds in the case $p = 1$ and $q = \infty$, see [HvNVW, Thm. 8.5.4] and exchange Gaussians for Rademachers in the proof.

A subset $\mathcal{T} \subseteq \mathcal{L}(X;Y)$ is called **semi-γ-bounded** if

$$\sup_{\|x\|\leq 1} [\![\mathbb{C} \ni \lambda \mapsto \lambda T x \mid T \in \mathcal{T}]\!]_\gamma < \infty.$$

If X has finite cotype, semi-γ-boundedness is the same as semi-R-boundedness, a notion that has been introduced and studied in [VW10]. We need the following auxiliary result.

Theorem B.6. *Let X be a Banach space with Pisier's contraction property and let $(T_s)_{s\in\mathbb{R}}$ be a strongly measurable and semi-γ-bounded family. Then*

$$[\![x \mapsto \int_{\mathbb{R}} f(s)g(s)T_s x \,\mathrm{d}s \mid f \in \mathrm{L}^2(\mathbb{R}),\ \|f\|_2 \leq 1]\!]_\gamma \lesssim \|g\|_2 \qquad (g \in \mathrm{L}^2(\mathbb{R})).$$

Proof. Let $H := \mathrm{L}^2(\mathbb{R})$ and let $\gamma(H;X)$ be the set of operators $T : H \to X$ such that $(Te_n)_{n\in\mathbb{N}} \in \gamma(\mathbb{N};X)$ for one/each orthonormal basis $(e_n)_n \in H$. These are the so-called *γ-radonifying operators*, see [HvNVW, Sec. 9.1.b]. Identifying $H' = \mathrm{L}^2(\mathbb{R})'$ with $\mathrm{L}^2(\mathbb{R}) = H$ via the canonical duality we obtain $\gamma(H;\mathbb{C}) = H$. Fix $g \in H$ and $x \in X$ and consider the operator

$$S_{g,x} : H \to X, \qquad f \mapsto \int_{\mathbb{R}} f(s)g(s)T_s x \,\mathrm{d}s.$$

Since the family $(T_s)_{s\in\mathbb{R}}$ is semi-γ-bounded, the so-called *γ-multiplier theorem* [HvNVW, Thm. 9.5.1] implies that $S_{g,x} \in \gamma(H;X)$ and

$$\|T_{g,x}\|_\gamma \lesssim \|g\|_H \|x\| \qquad (x \in X,\ g \in H).$$

Since X has Pisier's contraction property,

$$[\![\gamma(H;X) \ni S \mapsto Sf \in X \mid f \in H, \|f\|_H \leq 1]\!]_\gamma < \infty.$$

Composing this with the bounded operator $x \mapsto T_{g,x}$ yields

$$[\![x \mapsto T_{g,x} f \mid \|f\|_H \leq 1]\!]_\gamma \lesssim \|x \mapsto T_{g,x}\|_{X\to\gamma(H;X)} \lesssim \|g\|_H \qquad (g \in H)$$

as claimed. $\square$

The following is an abstract Littlewood–Paley theorem.

Proposition B.7. *Let A be a densely defined operator on a Banach space X with finite cotype. Suppose that A has a bounded $\mathrm{H}^\infty(\mathrm{St}_\theta)$-calculus for some $\theta > 0$ and $\psi, \eta \in \mathcal{E}[\overline{\mathrm{St}}_\theta]$ are such that*

$$\sum_{n\in\mathbb{Z}} \psi(t-n)\eta(t-n) = 1 \quad \textit{for all } t \in \mathbb{R}.$$

Then one has the norm equivalence

$$\|x\| \eqsim \|(\psi(A-n)x)_{n\in\mathbb{Z}}\|_{\gamma(\mathbb{Z};X)} \qquad (x \in X).$$

Proof. This is similar to [HvNVW, Thms. 10.4.4 and 10.4.8]. Note that

$$\sup_{z \in \mathrm{St}_\theta} \|(\psi(z-n))_{n\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})} < \infty,$$

which in the terminology of [HH13] means that the mapping

$$\mathrm{St}_\theta \to \ell^2(\mathbb{Z}), \quad z \mapsto (\psi(z-n))_n$$

has ℓ^1-frame bounded image. This implies that the associated square function and dual square function are bounded [HH13, Thm. 4.11]. The additional hypothesis implies (by holomorphy)

$$\sum_{n\in\mathbb{Z}} \psi(z-n)\eta(z-n) = 1 \qquad (z \in \mathrm{St}_\theta),$$

and hence the norm equivalence holds by [HH13, Cor. 4.7]. □

Acknowledgments

Florian Pannasch would like to thank Christoph Kriegler and Lutz Weis for inspiring discussions on Hörmander-type functional calculus during the 21st Internet Seminar on "Functional Calculus" (Wuppertal 2018). Markus Haase is grateful to Lukas Hagedorn for giving helpful comments on earlier versions of the paper and for pointing out the relevance of the results in the context of [DKP21]. Finally, both authors thank the anonymous referee for her/his very careful reading of the manuscript to the effect of increased readability and a drastically smaller amount of typos and inaccuracies.

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