

## PROPERTY G AND THE 4-GENUS

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ABSTRACT. We say a null-homologous knot  $K$  in a 3-manifold  $Y$  has Property G, if the Thurston norm and fiberedness of the complement of  $K$  is preserved under the zero surgery on  $K$ . In this paper, we will show that, if the smooth 4-genus of  $K \times \{0\}$  (in a certain homology class) in  $(Y \times [0, 1]) \# \widehat{NCP^2}$ , where  $Y$  is a rational homology sphere, is smaller than the Seifert genus of  $K$ , then  $K$  has Property G. When the smooth 4-genus is 0,  $Y$  can be taken to be any closed, oriented 3-manifold.

### 1. INTRODUCTION

A general theme in the study of Dehn surgery is to get the information about the topology of Dehn surgery from the topology of the original knot. In this paper, we will consider the following situation. If  $F$  is a Seifert surface for a null-homologous knot  $K \subset Y$ , then there is a closed surface  $\widehat{F}$  in the zero surgery on  $K$  obtained by capping off  $\partial F$  with a disk. Suppose that we know a certain property of  $F$  as a subsurface of  $Y \setminus K$ , can we deduce similar properties for  $\widehat{F}$ ?

Before we explain this problem in more detail, let us establish some notations we will use. If  $Z$  is a submanifold of a manifold  $Y$ , let  $\nu(Z)$  be a closed tubular neighborhood of  $Z$  in  $Y$ , let  $\nu^\circ(Z)$  be the interior of  $\nu(Z)$ , and let  $Y \setminus\!\!\setminus Z = Y \setminus \nu^\circ(Z)$ . Given a null-homologous knot  $K$  in a 3-manifold  $Y$ , and  $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ , let  $Y_{\frac{p}{q}}(K)$  be the manifold obtained by  $\frac{p}{q}$ -surgery on  $K$ .

All manifolds are smooth and oriented unless otherwise stated.

**Definition 1.1.** Suppose that  $K$  is a null-homologous knot in a closed 3-manifold  $Y$ . A compact surface  $F \subset Y$  is a *Seifert-like surface* for  $K$ , if  $\partial F = K$ . The homology class  $[F] \in H_2(Y, K)$  is called a *Seifert homology class*. When  $F$  is connected, we say that  $F$  is a *Seifert surface* for  $K$ . We also view a Seifert-like surface as a properly embedded surface in  $Y \setminus\!\!\setminus K$ .

**Definition 1.2.** Suppose that  $M$  is a compact 3-manifold, a properly embedded surface  $S \subset M$  is *taut* if  $\chi_-(S) = \chi_-([S])$  in  $H_2(M, \partial M)$ , no proper subsurface of  $S$  is null-homologous, and for any nonsphere component  $S_i$  of  $S$ ,  $[S_i]$  is not represented by a sphere. Here  $\chi_-(\cdot)$  is the Thurston norm, see Section 2.4.

**Definition 1.3.** Suppose that  $K$  is a null-homologous knot in a closed 3-manifold  $Y$ ,  $\varphi \in H_2(Y, K)$  is the homology class of a Seifert-like surface. We say  $K$  has *Property G* with respect to  $\varphi$ , if the following conditions hold:

(G1) If  $F$  is a taut Seifert-like surface with  $[F] = \varphi$ , then  $\widehat{F}$  is taut in  $Y_0(K)$ ;

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(G2) if  $Y_0(K)$  fibers over  $S^1$ , such that the homology class of the fiber is the natural extension  $\widehat{\varphi}$  of  $\varphi$ , then  $K$  is a fibered knot, and the homology class of the fiber is  $\varphi$ .

If the first (or second) condition holds, then we say that  $K$  has *Property* (G1) (or (G2)) with respect to  $\varphi$ . If  $K$  has Property G (or (G1), (G2)) with respect to every Seifert homology class, then we say  $K$  has *Property G* (or (G1), (G2)).

Gabai [5] proved that knots in  $S^3$  have Property G. As he remarked in [5], the proof also works when  $Y$  is reducible,  $Y - K$  is irreducible and  $H_1(Y)$  is torsion-free. It is not hard to see connected sums of knots have Property G. In [2, 13], it is showed that null-homologous knots in L-spaces have Property (G2), and the same argument also implies that such knots have Property (G1). In [16], we proved that if  $Y$  contains a nonseparating sphere and  $Y - K$  is irreducible, then  $K$  has Property G.

*Remark 1.4.* The reason that we want to use Seifert-like surfaces instead of Seifert surfaces in Definition 1.3 is that if  $b_1(Y) > 0$ , not every Seifert homology class is represented by a taut Seifert surface. For example, let  $Y = Y_1 \# Y_2$ ,  $K \subset Y_1$ , and  $Y_2$  is irreducible. If a Seifert homology class  $\varphi \in H_2(Y, K) \cong H_2(Y_1, K) \oplus H_2(Y_2)$  has a nonzero component in  $H_2(Y_2)$ , then any taut surface representing  $\varphi$  has to be disconnected.

**Theorem 1.5.** *Let  $Y$  be a closed, oriented, connected 3-manifold, and let  $K \subset Y$  be a null-homologous knot. Suppose that  $F$  is a taut Seifert surface for  $K$ . Let  $X = (Y \times [0, 1]) \# N\overline{CP^2}$  for some  $N$ , and let*

$$\iota : (Y, K) \rightarrow (X, K \times \{0\})$$

*be the inclusion map. If there exists a properly embedded smooth connected surface  $G \subset X$  with  $\partial G = K \times \{0\}$ ,  $[G] = \iota_*[F] \in H_2(X, K \times \{0\})$ , and  $g(G) < g(F)$ , then  $K$  has Property G with respect to  $[F]$ .*

The conclusion of Theorem 1.5 is just about Property G with respect to a special homology class  $[F]$ , and  $F$  is indeed a Seifert surface. When  $Y$  is a rational homology sphere, there is only one such homology class, and any taut Seifert-like surface must be connected, so we have the following special case which is worth mentioning.

**Corollary 1.6.** *Let  $Y$  be a closed connected 3-manifold with  $b_1(Y) = 0$ , and let  $K \subset Y$  be a null-homologous knot. If the smooth 4-genus of  $K \times \{0\}$  in  $X = (Y \times [0, 1]) \# N\overline{CP^2}$ , with the homology class of the surface in the image of  $\iota_*$ , is smaller than the Seifert genus of  $K$  for some  $N$ , then  $K$  has Property G.*

If the above 4-genus is 0, then the conclusion of Theorem 1.5 can be strengthened to assert that  $K$  has Property G even when  $b_1(Y) > 0$ .

**Theorem 1.7.** *Let  $Y$  be a closed, oriented 3-manifold, and let  $K \subset Y$  be a knot. If  $K \times \{0\}$  bounds a properly embedded smooth disk  $G$  in  $X = (Y \times [0, 1]) \# N\overline{CP^2}$ , with  $[G]$  in the image of  $\iota_*$ , then  $K$  has Property G.*

Since Property G does not depend on the orientation of the manifolds involved, one can replace the  $X$  in Theorems 1.5 and 1.7 with  $(Y \times [0, 1]) \# NCP^2$ , the conclusions of these two theorems are still true. However, our current methods break down if  $X$  is replaced with  $(Y \times [0, 1]) \# N(\overline{CP^2} \# \overline{CP^2})$ .

**Conjecture 1.8.** *The conclusion of Theorem 1.5 still holds if  $X$  is replaced with  $(Y \times [0, 1]) \# N(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$ .*

If  $K$  bounds an immersed surface in  $Y$ , then the immersed surface can be perturbed to an immersed surface with normal self-intersections in  $Y \times [0, 1]$ . One can then resolve the self-intersections to get an embedded surface in

$$(Y \times [0, 1]) \# N(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$$

for some  $N$  so that the homology class of the embedded surface is in the image of  $\iota_*$ . Thus the condition in Conjecture 1.8 essentially says that the “immersed genus” of  $K$  is smaller than the Seifert genus in the relative homology class. Similar to Theorem 1.7, one can hope that if the “immersed genus” of  $K$  is zero, then  $K$  has Property G.

**Conjecture 1.9.** *Null-homotopic knots have Property G.*

Boileau has conjectured that null-homotopic knots have Property (G2) [10, Problem 1.80].<sup>1</sup> The question whether null-homotopic knots have Property (G1) was asked to the author by David Gabai.

A special case of Conjecture 1.9 is the following assertion: If  $K \subset Y$  is a nontrivial null-homotopic knot,  $F$  is a Seifert surface, then  $[\widehat{F}]$  is not represented by a sphere. This special case was proved by Hom and Lidman [8] in the case when  $Y$  is a prime rational homology sphere. In fact, using a result due to Gabai [4] and Lackenby [11], it is straightforward to prove the case  $b_1(Y) > 0$ .

**Proposition 1.10.** *Let  $Y$  be a closed, oriented, connected 3-manifold with  $b_1(Y) > 0$ , and  $K \subset Y$  be a nontrivial null-homotopic knot such that  $Y - K$  is irreducible, then  $Y_0(K)$  does not have an  $S^1 \times S^2$  connected summand.*

*Remark 1.11.* Hom and Lidman [8] also proved that if  $b_1(Y) = 0$  and  $K \subset Y$  is a null-homologous knot, then  $Y_0(K)$  is not homeomorphic to  $Y \# S^1 \times S^2$ . Using the same argument as in their paper combined with Theorem 3.3, we can remove the  $b_1(Y) = 0$  condition in their paper. In a later paper [18], the author removed the  $b_1(Y) > 0$  condition in Proposition 1.10.

Using the argument in our paper, it is not hard to prove Property G for null-homologous knots in many 3-manifolds with “simple” Heegaard Floer homology. For example, knots in the Brieskorn sphere  $\Sigma(2, 3, 7)$  have Property G. We will only do this for torus bundles.

**Theorem 1.12.** *Let  $Y$  be a  $T^2$ -bundle over  $S^1$ , then null-homologous knots in  $Y$  have Property G.*

Given what is known about Property G, it is reasonable to expect Conjecture 1.13 to be true.

**Conjecture 1.13.** *If  $Y$  is a closed, oriented, connected, reducible 3-manifold,  $K \subset Y$  is a null-homologous knot with  $Y \setminus K$  being irreducible, then  $K$  has Property G.*

This paper is organized as follows. In Section 2, we recall some basic material about Heegaard Floer homology. In Section 3, we state and prove the general case of the zero surgery formula of Heegaard Floer homology from [28]. In Section 4,

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<sup>1</sup>There are trivial counterexamples to the original (slightly different) question of Boileau. See the remark after [14, Question 1.3].

we use a standard argument to prove a rank inequality relating  $\widehat{HFK}(Y, K)$  and  $HF^+(Y_0(K))$ . Our main results follow from this rank inequality. We will also prove Proposition 1.10. In Section 5, we prove Theorem 1.12. In Appendix A, we sketch a proof of the folklore Theorem 3.4, which has been frequently cited in the literature.

2. PRELIMINARIES ON HEEGAARD FLOER HOMOLOGY

In this section, we will collect some results we need on Heegaard Floer homology.

**2.1. Different versions of Heegaard Floer homology.** Heegaard Floer homology [20], in its most fundamental form, assigns a package of invariants

$$\widehat{HF}, HF^+, HF^-, HF^\infty, HF_{\text{red}}$$

to a closed, connected, oriented 3-manifold  $Y$  equipped with a  $\text{Spin}^c$  structure  $\mathfrak{s} \in \text{Spin}^c(Y)$ . We often use  $HF^\circ$  to denote these invariants, where  $\circ$  is one of  $\widehat{, +, -, \infty, \text{red}}$ .

There is a short exact sequence of chain complexes

$$0 \longrightarrow CF^-(Y, \mathfrak{s}) \xrightarrow{\eta} CF^\infty(Y, \mathfrak{s}) \xrightarrow{\rho} CF^+(Y, \mathfrak{s}) \longrightarrow 0,$$

which induces an exact triangle relating  $HF^+, HF^-, HF^\infty$ . When  $c_1(\mathfrak{s})$  is nontorsion, as shown in [22, Corollary 2.4],  $\rho_* = 0$ . Hence we have a natural short exact sequence

$$(1) \quad 0 \longrightarrow HF^+(Y, \mathfrak{s}) \longrightarrow HF^-(Y, \mathfrak{s}) \xrightarrow{\eta_*} HF^\infty(Y, \mathfrak{s}) \longrightarrow 0.$$

**2.2. The cobordism map.** Suppose that  $W : Y_1 \rightarrow Y_2$  is an oriented cobordism, with  $Y_1, Y_2$  connected. Given  $\mathfrak{S} \in \text{Spin}^c(W)$ , there is a homomorphism

$$F_{W, \mathfrak{S}}^\circ : HF^\circ(Y_1, \mathfrak{S}|Y_1) \rightarrow HF^\circ(Y_2, \mathfrak{S}|Y_2),$$

induced by a chain map

$$f_{W, \mathfrak{S}}^\circ : CF^\circ(Y_1, \mathfrak{S}|Y_1) \rightarrow CF^\circ(Y_2, \mathfrak{S}|Y_2).$$

Suppose that  $W$  is the composition of two cobordisms

$$W_1 : Y_1 \rightarrow Y_3, \quad W_2 : Y_3 \rightarrow Y_2.$$

Given  $\mathfrak{S}_1 \in \text{Spin}^c(W_1), \mathfrak{S}_2 \in \text{Spin}^c(W_2)$  with  $\mathfrak{S}_1|Y_3 = \mathfrak{S}_2|Y_3$ , we have the composition law [26, Theorem 3.4]

$$(2) \quad F_{W_2, \mathfrak{S}_2}^\circ \circ F_{W_1, \mathfrak{S}_1}^\circ = \sum_{\mathfrak{S} \in \text{Spin}^c(W), \mathfrak{S}|W_i = \mathfrak{S}_i} F_{W, \mathfrak{S}}^\circ.$$

The composition law becomes simpler in the following special case.

**Lemma 2.1.** *Suppose that  $W_1$  is obtained from  $Y_1 \times [0, 1]$  by adding  $b$  one-handles to  $Y_1 \times 1$  and then 1 two-handle, where the attaching curve of the two-handle is a null-homologous knot in  $Y_1 \# b(S^1 \times S^2)$ , and the framing is nonzero. Then for any  $\mathfrak{S} \in \text{Spin}^c(W)$ , we have*

$$F_{W_2, \mathfrak{S}|W_2}^\circ \circ F_{W_1, \mathfrak{S}|W_1}^\circ = F_{W, \mathfrak{S}}^\circ.$$

*Proof.* Consider the exact sequence

$$H^1(W_1) \xrightarrow{\eta^*} H^1(Y_3) \longrightarrow H^2(W_1, Y_3) \xrightarrow{\rho^*} H^2(W_1).$$

By the Poincaré–Lefschetz duality, the map  $\rho^*$  can be identified with

$$H_2(W_1, Y_1)(\cong \mathbb{Z}) \rightarrow H_2(W_1, \partial W_1),$$

which is injective by our condition on  $W_1$ . Hence  $\eta^*$  is surjective.

Now consider the exact sequence

$$H^1(W_1) \oplus H^1(W_2) \xrightarrow{\iota^*} H^1(Y_3) \longrightarrow H^2(W) \xrightarrow{\pi^*} H^2(W_1) \oplus H^2(W_2).$$

As  $\eta^*$  is surjective,  $\iota^*$  is also surjective, so  $\pi^*$  is injective, hence the restriction map

$$\text{Spin}^c(W) \rightarrow \text{Spin}^c(W_1) \times \text{Spin}^c(W_2)$$

is injective. Our conclusion then follows from (2).  $\square$

The following adjunction relation proved in [22] plays an important role in our paper.

**Theorem 2.2.** *For every genus  $g$ , there is an element  $\zeta \in \mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H_1(\Sigma)$  of degree  $2g$  with the following property. Given any smooth, oriented, 4-dimensional cobordism  $W$  from  $Y_1$  to  $Y_2$  (both of which are connected 3-manifolds), any smoothly embedded, connected, oriented submanifold  $\Sigma \subset W$  of genus  $g$ , and any  $\mathfrak{S} \in \text{Spin}^c(W)$  satisfying that*

$$\langle c_1(\mathfrak{S}), [\Sigma] \rangle - [\Sigma] \cdot [\Sigma] = -2g(\Sigma),$$

*we have the relation*

$$F_{W, \mathfrak{S}}^\circ = F_{W, \mathfrak{S} + \epsilon \text{PD}[\Sigma]}^\circ (i_*(\zeta)),$$

*where  $\epsilon$  is the sign of  $\langle c_1(\mathfrak{S}), [\Sigma] \rangle$ , and*

$$i_* : \mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* H_1(\Sigma) \rightarrow \mathbb{Z}[U] \otimes_{\mathbb{Z}} \Lambda^* (H_1(W)/\text{Tors})$$

*is the map induced by the inclusion  $i : \Sigma \rightarrow W$ .*

**2.3. The knot Floer chain complex.** In this subsection, we will briefly recall the construction of the knot Floer chain complex from [23]. (See also [30].) We will then discuss its relationship with large Dehn surgery.

Suppose that  $K$  is a null-homologous knot in  $Y$ . As in [27], any relative  $\text{Spin}^c$  structure  $\xi$  on  $X = Y \setminus K$  is represented by a vector field  $v$  on  $X$  whose restriction to  $\partial X$  is a canonical boundary parallel vector field. Let  $c_1(\xi) \in H^2(Y, K) \cong H^2(X, \partial X)$  be the obstruction to extending the boundary-induced trivialization of  $v^\perp|_{(\partial X)}$  to a trivialization of  $v^\perp$  over  $X$ .

Suppose that

$$(\Sigma, \alpha, \beta, w, z)$$

is a doubly pointed Heegaard diagram for  $(Y, K)$ . Given  $\mathfrak{s} \in \text{Spin}^c(Y)$  and  $\xi \in \underline{\text{Spin}}^c(Y, K)$  such that  $\mathfrak{s}$  extends  $\xi$ , let  $CFK^\infty(Y, K, \xi)$  be the knot Floer chain complex of  $(Y, K, \mathfrak{s})$ . It is generated by  $[\mathbf{x}, i, j]$  satisfying

$$(3) \quad \underline{\mathfrak{s}}_{w, z}(\mathbf{x}) + (i - j)\text{PD}[\mu] = \xi,$$

where  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ . The differential  $\partial$  is given by

$$\partial[\mathbf{x}, i, j] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \mu(\phi)=1} \#\widehat{\mathcal{M}}(\phi)[\mathbf{y}, i - n_w(\phi), j - n_z(\phi)].$$

The pair  $(i, j)$  defines a double filtration on  $CFK^\infty(Y, K, \xi)$ . Define the knot Floer homology

$$\widehat{HFK}(Y, K, \xi) = H_*(CFK^\infty(Y, K, \xi)\{i = j = 0\}).$$

Given a homology class  $\varphi \in H_2(Y, K)$  represented by a Seifert surface  $F$  for  $K$ , let  $\widehat{\varphi} \in H_2(Y_0(K))$  be the homology class which extends  $\varphi$ . Given  $\mathfrak{s} \in \text{Spin}^c(Y)$  and  $k \in \mathbb{Z}$ , let  $\xi_k \in \underline{\text{Spin}}^c(Y, K)$  be the extension of  $\mathfrak{s}$  so that

$$(4) \quad \langle c_1(\xi_k), \varphi \rangle = 2k + 1,$$

and let  $\mathfrak{t}_k \in \text{Spin}^c(Y_0(K))$  be the  $\text{Spin}^c$  structure satisfying that  $\mathfrak{s}|(Y \setminus K) = \mathfrak{t}_k| (Y \setminus K)$  and

$$\langle c_1(\mathfrak{t}_k), \widehat{\varphi} \rangle = 2k.$$

Denote

$$C = CFK^\infty(Y, K, \mathfrak{s}, \varphi) := CFK^\infty(Y, K, \xi_0).$$

Note that the  $(i, j)$ -filtration on  $C$  depends on the homology class  $\varphi$  through (3), because the definition of  $\xi_0$  (which is (4)) involves  $\varphi$ . For simplicity, we often suppress  $\varphi$  in this paper. Let

$$\begin{aligned} \widehat{HFK}(Y, K, \mathfrak{s}, \varphi, k) &= \widehat{HFK}(Y, K, \xi_k), \\ \widehat{HFK}(Y, K, \varphi, k) &= \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} \widehat{HFK}(Y, K, \mathfrak{s}, \varphi, k). \end{aligned}$$

*Remark 2.3.* We often abuse the notation by letting

$$C = CFK^\infty(Y, K, \varphi) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} CFK^\infty(Y, K, \mathfrak{s}, \varphi)$$

if a  $\text{Spin}^c$  structure  $\mathfrak{s} \in \text{Spin}^c(Y)$  is not explicitly mentioned in the context.

There are chain complexes

$$A_k^+ = C\{i \geq 0 \text{ or } j \geq k\}, \quad k \in \mathbb{Z}$$

and  $B^+ = C\{i \geq 0\} \cong CF^+(Y, \mathfrak{s})$ . As in [28], there are chain maps

$$v_k^+, h_k^+ : A_k^+ \rightarrow B^+.$$

Here  $v_k^+$  is the vertical projection, and  $h_k^+$  first projects  $A_k^+$  to  $C\{j \geq k\}$ , then maps to  $C\{j \geq 0\}$  via  $U^k$ , finally maps to  $B^+$  via a chain homotopy equivalence  $C\{j \geq 0\} \rightarrow C\{i \geq 0\}$ .

*Remark 2.4.* The chain homotopy equivalence  $C\{j \geq 0\} \rightarrow C\{i \geq 0\}$  is obtained by changing the basepoint from  $z$  to  $w$ . In [28], Ozsváth and Szabó said that the chain homotopy equivalence is canonical up to sign. This assertion is justified by [26, Theorem 2.1] and [6]. (If we use  $\mathbb{Z}/2\mathbb{Z}$  coefficients, which are enough for the applications in our paper, we can also refer to [9].)

There is a commutative diagram

$$\begin{array}{ccc} A_{k+1}^+ & & \\ \uparrow & \searrow^{v_{k+1}^+} & \\ A_k^+ & \xrightarrow{v_k^+} & B^+, \end{array}$$

where the vertical arrow denotes the natural quotient map. Let

$$(v_k^+)_*, (h_k^+)_* : H_*(A_k^+) \rightarrow H_*(B^+)$$

be the induced map on homology, then the previous commutative diagram implies that

$$(5) \quad \text{im}(v_k^+)_* \subset \text{im}(v_{k+1}^+)_*.$$

Similarly,

$$(6) \quad \text{im}(h_k^+)_* \supset \text{im}(h_{k+1}^+)_*.$$

Theorem 2.5 is contained in [23] and [28]. Let  $F$  be a Seifert surface for  $K$  in the homology class  $\varphi$ . Given  $n > 0$  and  $t \in \mathbb{Z}/n\mathbb{Z}$ , let  $W'_n(K) : Y_n(K) \rightarrow Y$  be the natural two-handle cobordism, and let  $\widehat{F}_n \subset W'_n(K)$  be the closed surface obtained by capping off  $\partial F$  with a disk. Let  $\widehat{\varphi}_n = [\widehat{F}_n] \in H_2(W'_n(K))$ . Let  $k$  be an integer satisfying  $k \equiv t \pmod{n}$  and  $|k| \leq \frac{n}{2}$ . Let  $\mathfrak{r}_k, \mathfrak{h}_k \in \text{Spin}^c(W'_n(K))$  satisfy  $\mathfrak{r}_k|_Y = \mathfrak{h}_k|_Y = \mathfrak{s}$  and

$$(7) \quad \langle c_1(\mathfrak{r}_k), \widehat{\varphi}_n \rangle = 2k - n \quad \langle c_1(\mathfrak{h}_k), \widehat{\varphi}_n \rangle = 2k + n,$$

and let  $\mathfrak{s}_t = \mathfrak{r}_k|_{Y_n(K)} = \mathfrak{h}_k|_{Y_n(K)}$ .

**Theorem 2.5.** *When  $n \geq 2g(F)$ , the chain complex  $CF^+(Y_n(K), \mathfrak{s}_t)$  is represented by the chain complex  $A_k^+$ , in the sense that there is an isomorphism*

$$\Psi_n^+ : CF^+(Y_n(K), \mathfrak{s}_t) \rightarrow A_k^+.$$

Moreover, the following squares commute:

$$\begin{array}{ccc} CF^+(Y_n(K), \mathfrak{s}_t) & \xrightarrow{f_{W'_n(K), \mathfrak{r}_k}^+} & CF^+(Y, \mathfrak{s}) \\ \downarrow \Psi_n^+ & & \downarrow \Psi_n^+ \\ A_k^+ & \xrightarrow{v_k^+} & B^+ \end{array} \quad \begin{array}{ccc} CF^+(Y_n(K), \mathfrak{s}_t) & \xrightarrow{f_{W'_n(K), \mathfrak{h}_k}^+} & CF^+(Y, \mathfrak{s}) \\ \downarrow \Psi_n^+ & & \downarrow \Psi_n^+ \\ A_k^+ & \xrightarrow{h_k^+} & B^+ \end{array}$$

**2.4. The Thurston norm of  $Y \setminus\setminus K$  and  $Y_0(K)$ .** In this subsection, we will recall a few facts about the Thurston norm [31] and Heegaard Floer homology.

**Definition 2.6.** Let  $S$  be a compact oriented surface with connected components  $S_1, \dots, S_n$ . We define

$$\chi_-(S) = \sum_i \max\{0, -\chi(S_i)\}.$$

Let  $M$  be a compact oriented 3-manifold,  $A$  be a compact codimension-0 submanifold of  $\partial M$ . Let  $h \in H_2(M, A)$ . The *Thurston norm*  $\chi_-(h)$  of  $h$  is defined to be the minimal value of  $\chi_-(S)$ , where  $S$  runs over all the properly embedded surfaces in  $M$  with  $\partial S \subset A$ , and  $[S] = h$ .

The following properties of Heegaard Floer homology are well-known, see [7, 13, 15, 17, 24].

**Theorem 2.7.** *Suppose that  $Y$  is a closed 3-manifold. Let  $G$  be a taut surface in  $Y$ . Then*

$$\chi_-(G) = \max \{ \langle c_1(\mathfrak{s}), [G] \rangle \mid \mathfrak{s} \in \text{Spin}^c(Y), HF^+(Y, \mathfrak{s}) \neq 0 \}.$$

**Theorem 2.8.** *Suppose that  $K$  is a null-homologous knot in a closed 3-manifold  $Y$ . Let  $F$  be a taut Seifert-like surface for  $K$ . Then*

$$\chi_-(F) + 2 = \max \left\{ \langle c_1(\xi), [F] \rangle \mid \xi \in \underline{\text{Spin}}^c(Y \setminus\!\! \setminus K), \widehat{HF}K(Y, K, \xi) \neq 0 \right\}.$$

**Theorem 2.9.** *Suppose that  $K$  is a null-homologous knot in a 3-manifold  $Y$  with a genus  $g$  Seifert surface  $F$ , and  $Y \setminus\!\! \setminus K$  is irreducible. If*

$$\widehat{HF}K(Y, K, [F], g(F)) \cong \mathbb{Z},$$

*then  $K$  is fibered with fiber  $F$ .*

Theorems 2.7–2.9 hold true even when we use field coefficients.

### 3. COMPUTING $HF^+(Y_0(K))$ VIA MAPPING CONE

In this section, we present detailed proofs for the general case of two theorems mentioned in [28, Subsection 4.8]. The original theorems of Ozsváth and Szabó are the (untwisted and twisted) zero surgery formula for knots in integral homology three-spheres. It is not hard to adapt their argument to prove the theorems for null-homologous knots in 3-manifolds with torsion  $\text{Spin}^c$  structures. See [12, Section 2] for a detailed proof of the twisted formula. Our contribution here is to generalize the theorems to the case of nontorsion  $\text{Spin}^c$  structures.

**Theorem 3.1.** *Let  $Y, K, C$  be as in Section 2.3. For any  $k \in \mathbb{Z}$ ,  $HF^+(Y_0(K), \mathfrak{t}_k)$  is isomorphic to the homology of the mapping cone of*

$$v_k^+ + h_k^+ : A_k^+ \rightarrow B^+.$$

*Remark 3.2.* Using the same argument as in the proof of Theorem 3.1, one should be able to extend the Dehn surgery formula in [28, 29] to null-homologous knots in any closed oriented 3-manifolds.

Ozsváth and Szabó [21] defined the universal twisted Heegaard Floer chain complex  $\underline{CF}^\circ(Y)$  and the corresponding homology  $\underline{HF}^\circ(Y)$  as modules over the group ring  $\mathbb{Z}[H^1(Y)]$ . Suppose that  $K \subset Y$  is a null-homologous knot. Let  $[\mu]$  be the homology class of a meridian of  $K$  in  $Y_0(K)$ , then the evaluation over  $[\mu]$  defines a ring homomorphism  $\mathbb{Z}[H^1(Y)] \rightarrow \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[T, T^{-1}]$ . Thus we get the twisted Heegaard Floer chain complex  $\underline{CF}^\circ(Y_0(K); \mathbb{Z}[T, T^{-1}])$  as a module over  $\mathbb{Z}[T, T^{-1}]$ . Its homology is denoted  $\underline{HF}^\circ(Y_0(K); \mathbb{Z}[T, T^{-1}])$ .

**Theorem 3.3.** *Let  $Y, K, C$  be as in Section 2.3. For any  $k \in \mathbb{Z}$ , the twisted Heegaard Floer homology  $\underline{HF}^+(Y_0(K), \mathfrak{t}_k; \mathbb{Z}[T, T^{-1}])$  is isomorphic to the homology of the mapping cone of*

$$v_k^+ + Th_k^+ : A_k^+[T, T^{-1}] \rightarrow B^+[T, T^{-1}].$$

Given a chain map  $f : A \rightarrow B$ , let  $MC(f)$  be the mapping cone of  $f$ , namely,  $MC(f) = A \oplus B$ , with the differential

$$\partial = \begin{pmatrix} \partial_A & 0 \\ f & \partial_B \end{pmatrix}.$$

In [21, Section 9], Ozsváth and Szabó constructed a genus  $h$  Heegaard quadruple

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}, w),$$

such that



- $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta} - \{\beta_h\})$  is a Heegaard diagram for  $Y \setminus\!\! \setminus K$ , and  $\beta_h$  represents a meridian of  $K$ ;
- $\gamma_i, \delta_i$  are small Hamiltonian translates of  $\beta_i, i = 1, \dots, h-1$ ;
- $\gamma_h$  represents the canonical longitude of  $K$ , and  $\delta_h$  is isotopic to the juxtaposition of the  $n$ -fold juxtaposition of  $\beta_h$  with  $\gamma_h$ ;
- all the necessary admissibility conditions are satisfied.

The three diagrams

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}), (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}), (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta})$$

represent  $Y, Y_0(K), Y_n(K)$ , respectively.

When  $n$  is sufficiently large,  $\mathfrak{t}_k$  is the only  $\text{Spin}^c$  structure in

$$\left\{ \mathfrak{t}_{k+ln} \in \text{Spin}^c(Y_0(K)) \mid l \in \mathbb{Z} \right\}$$

which is represented by an intersection point in  $\mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ .

We will use the notations from Section 2.3. Let

$$CF^+(Y_0(K), [\mathfrak{t}_k]) = \bigoplus_{l \in \mathbb{Z}} CF^+(Y_0(K), \mathfrak{t}_{k+ln}).$$

Then

$$CF^+(Y_0(K), [\mathfrak{t}_k]) \cong CF^+(Y_0(K), \mathfrak{t}_k)$$

by our choice of  $n$ .

In [21, Section 9], Ozsváth and Szabó proved that the sequence

$$0 \longrightarrow CF^+(Y_0(K), [\mathfrak{t}_k]) \xrightarrow{f_2^+} CF^+(Y_n(K), \mathfrak{s}_t) \xrightarrow{f_3^+} CF^+(Y, \mathfrak{s}) \longrightarrow 0$$

is exact at  $CF^+(Y_0(K), [\mathfrak{t}_k])$  and  $CF^+(Y, \mathfrak{s})$ , and there exists a  $U$ -equivariant null-homotopy

$$H : CF^+(Y_0(K), [\mathfrak{t}_k]) \rightarrow CF^+(Y, \mathfrak{s})$$

for  $f_3^+ \circ f_2^+$ . Here  $f_2^+, f_3^+$  are maps induced by two-handle cobordisms, and

$$H([\mathbf{x}, i]) = \sum_{\mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \Theta_{\gamma, \delta}, \Theta_{\delta, \beta}, \mathbf{z}), \mu(\phi) = -1} \# \mathcal{M}(\phi)[\mathbf{z}, i - n_w(\phi)].$$

For any integer  $\delta \geq 0$ , we will consider the subcomplex  $CF^\delta$  of  $CF^+$ , generated by  $[\mathbf{x}, i]$  with  $0 \leq i \leq \delta$ . Similarly, let  $A_k^\delta, B^\delta$  be the corresponding subcomplexes of  $A_k^+, B^+$  which are kernels of  $U^{\delta+1}$ . Let  $f_2^\delta, f_3^\delta, H^\delta$  be the restrictions of  $f_2^+, f_3^+, H$  to the corresponding  $CF^\delta$ , and let  $v_k^\delta, h_k^\delta : A_k^\delta \rightarrow B^\delta$  be the restrictions of  $v_k^+, h_k^+$ .

Define

$$\psi^\delta : CF^\delta(Y_0(K), \mathfrak{t}_k) \rightarrow MC(f_3^\delta), \text{ and } \psi^+ : CF^+(Y_0(K), \mathfrak{t}_k) \rightarrow MC(f_3^+)$$

by

$$\psi^\delta = (f_2^\delta, H^\delta), \text{ and } \psi^+ = (f_2^+, H).$$

**Theorem 3.4.** *The  $U$ -equivariant chain map*

$$\psi^+ : CF^+(Y_0(K), \mathfrak{t}_k) \rightarrow MC(f_3^+)$$

*is a quasi-isomorphism.*

Various versions of the above folklore theorem have been cited in the literature many times, but we are not aware of any complete proof of it. A sketch of a proof of Theorem 3.4 with  $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$  coefficients will be given in Appendix A.

*Proof of Theorem 3.1 when  $c_1(\mathfrak{s})$  is torsion.* In this case, there exists an absolute  $\mathbb{Q}$ -grading on  $CF^+(Y_n(K), \mathfrak{s}_t)$  and  $CF^+(Y, \mathfrak{s})$ . As in [28, Lemma 4.4], there exists an  $N > 0$  such that whenever  $n \geq N$  and  $\mathfrak{S} \in \text{Spin}^c(W'_n(K))$  induces a nontrivial map

$$f_{W'_n(K), \mathfrak{S}}^\delta : CF^\delta(Y_n(K), \mathfrak{s}_t) \rightarrow CF^\delta(Y, \mathfrak{s}),$$

$\mathfrak{S}$  must be  $\mathfrak{r}_k$  or  $\mathfrak{r}_k$ . In particular, by Theorem 2.5, the two chain complexes  $MC(f_3^\delta)$  and  $MC(v_k^\delta + h_k^\delta)$  are equal. It follows from Theorem 3.4 that  $\psi^\delta$  is a quasi-isomorphism.

If  $k = 0$ , there exists a  $\mathbb{Z}$ -grading on  $CF^+(Y_0(K), \mathfrak{t}_0)$ . Moreover, since  $v_0^+$  and  $h_0^+$  have the same grading shift,  $MC(v_0^+ + h_0^+)$  is also  $\mathbb{Z}$ -graded. By [28, Lemma 2.7], the map

$$(\overline{\psi^+})_* : HF^+(Y_0(K), \mathfrak{t}_0) \rightarrow H_*(MC(v_0^+ + h_0^+))$$

is a grading preserving  $\mathbb{Z}[U]$ -isomorphism.

If  $k \neq 0$ ,  $c_1(\mathfrak{t}_k)$  is nontorsion. There exists a  $\delta \geq 0$ , such that the map  $U^{\delta+1}$  on  $HF^+(Y_0(K), \mathfrak{t}_k)$  is zero. So the short exact sequence

$$0 \longrightarrow CF^\delta(Y_0(K), \mathfrak{t}_k) \longrightarrow CF^+(Y_0(K), \mathfrak{t}_k) \xrightarrow{U^{\delta+1}} CF^+(Y_0(K), \mathfrak{t}_k) \longrightarrow 0$$

gives rise to the short exact sequence

$$0 \longrightarrow HF^+(Y_0(K), \mathfrak{t}_k) \longrightarrow HF^\delta(Y_0(K), \mathfrak{t}_k) \longrightarrow HF^+(Y_0(K), \mathfrak{t}_k) \longrightarrow 0.$$

Similarly, we have

$$0 \longrightarrow H_*(MC(v_k^+ + h_k^+)) \longrightarrow H_*(MC(v_k^\delta + h_k^\delta)) \longrightarrow H_*(MC(v_k^+ + h_k^+)) \longrightarrow 0.$$

Hence we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & HF^+(Y_0(K), \mathfrak{t}_k) & \longrightarrow & HF^\delta(Y_0(K), \mathfrak{t}_k) & \longrightarrow & HF^+(Y_0(K), \mathfrak{t}_k) \longrightarrow 0 \\ & & \downarrow (\overline{\psi^+})_* & & \downarrow (\psi^\delta)_* & & \downarrow (\overline{\psi^+})_* \\ 0 & \longrightarrow & H_*(MC(v_k^+ + h_k^+)) & \longrightarrow & H_*(MC(v_k^\delta + h_k^\delta)) & \longrightarrow & H_*(MC(v_k^+ + h_k^+)) \longrightarrow 0. \end{array}$$

Using the fact that  $\psi^\delta$  is a quasi-isomorphism, we see that

$$(\overline{\psi^+})_* : HF^+(Y_0(K), \mathfrak{t}_k) \rightarrow H_*(MC(v_k^+ + h_k^+))$$

is both injective and surjective.  $\square$

Given  $k \in \mathbb{Z}$  and  $n > 0$ , let

$$\mathcal{X}_k = \{ \mathfrak{r} \in \text{Spin}^c(W'_n(K)) \mid \langle c_1(\mathfrak{r}), \widehat{\varphi}_n \rangle \equiv 2k - n \pmod{2n}, \quad \mathfrak{r}|Y = \mathfrak{s} \}.$$

For the case when  $c_1(\mathfrak{s})$  is nontorsion, we need Proposition 3.5.

**Proposition 3.5.** *Suppose that  $n \geq \max\{2|k| + g(F), 2g(F)\}$ ,  $\mathfrak{r} \in \mathcal{X}_k \setminus \{\mathfrak{r}_k, \mathfrak{r}_k\}$ , then the map  $F_{W'_n(K), \mathfrak{r}}^-$  factorizes through the map*

$$U^{n-2|k|-g} : HF^-(Y_n(K), \mathfrak{s}_t) \rightarrow HF^-(Y_n(K), \mathfrak{s}_t).$$

Moreover, if  $c_1(\mathfrak{s})$  is nontorsion, then  $F_{W'_n(K), \mathfrak{r}}^+$  factorizes through

$$U^{n-2|k|-g} : HF^+(Y_n(K), \mathfrak{s}_t) \rightarrow HF^+(Y_n(K), \mathfrak{s}_t).$$

Our method of proving Proposition 3.5 is taken from [22, Theorem 3.1].

Let  $CF^{\leq 0}(Z, \mathbf{u})$  be the subcomplex of  $CF^\infty(Z, \mathbf{u})$  which consists of  $[\mathbf{x}, i]$ ,  $i \leq 0$ . This chain complex is clearly isomorphic to  $CF^-(Z, \mathbf{u})$  via the  $U$ -action. Ozsváth and Szabó [19] defined a natural transformation

$$F_{Y \# Z, \mathfrak{t} \# \mathbf{u}}^\circ : HF^\circ(Y, \mathfrak{t}) \otimes HF^{\leq 0}(Z, \mathbf{u}) \rightarrow HF^\circ(Y \# Z, \mathfrak{t} \# \mathbf{u}).$$

Suppose that  $W : Y_1 \rightarrow Y_2$  is a cobordism equipped with a  $\text{Spin}^c$  structure  $\mathfrak{S} \in \text{Spin}^c(W)$ ,  $\mathfrak{S}|_{Y_i} = \mathfrak{t}_i$ . Then there is a commutative diagram [19, Proposition 4.4]

$$(8) \quad \begin{array}{ccc} HF^\circ(Y_1, \mathfrak{t}_1) \otimes HF^{\leq 0}(Z, \mathbf{u}) & \xrightarrow{F_{Y_1 \# Z, \mathfrak{t}_1 \# \mathbf{u}}^\circ} & HF^\circ(Y_1 \# Z, \mathfrak{t}_1 \# \mathbf{u}) \\ \downarrow F_{W, \mathfrak{S}}^\circ \otimes \text{id} & & \downarrow F_{W \# (Z \times [0, 1]), \mathfrak{S} \# \mathbf{u}}^\circ \\ HF^\circ(Y_2, \mathfrak{t}_2) \otimes HF^{\leq 0}(Z, \mathbf{u}) & \xrightarrow{F_{Y_2 \# Z, \mathfrak{t}_2 \# \mathbf{u}}^\circ} & HF^\circ(Y_2 \# Z, \mathfrak{t}_2 \# \mathbf{u}) \end{array}$$

*Proof of Proposition 3.5.* As in the proof of [22, Theorem 3.1], we will first do a model computation. Let  $N = \nu(\widehat{F}_n) \setminus B^4$  be a punctured neighborhood of  $\widehat{F}_n \subset W'_n(K)$ . Let  $V = \#^{2g}(S^1 \times S^2)$ , then  $N$  is a cobordism from  $S^3$  to  $V_{-n}(B_g)$ , where  $B_g \subset V$  is the genus  $g$  Borromean knot.

Recall from [23, Section 9] that

$$HFK^\infty(V, B_g) \cong \mathbb{Z}[U, U^{-1}] \otimes \Lambda^*(\mathbb{Z}^{2g})$$

is generated by  $\mathbf{1} \otimes \theta$  as a module over  $\mathbb{Z}[U, U^{-1}] \otimes \Lambda^*(\mathbb{Z}^{2g})$ . Here

$$\theta \in \widehat{HF}(V) \cong \Lambda^*(\mathbb{Z}^{2g})$$

is the generator with the highest Maslov grading. The double filtration level of  $\mathbf{1} \otimes \theta$  is  $(0, g)$ , and the double filtration level of  $HFK^\infty(V, B_g)$  is supported in the set

$$\{(i, j) \in \mathbb{Z}^2 \mid |i - j| \leq g\}.$$

The map  $F_{N, \mathfrak{r}|N}^{\leq 0}$  is the composition of two maps: The first map sends  $HF^{\leq 0}(S^3)$  isomorphically to  $\mathbb{Z}[U] \otimes \theta \subset HF^{\leq 0}(V)$ , the second map is induced by the two-handle cobordism  $V \rightarrow V_{-n}(B_g)$ . Since we assume  $n \geq 2g(F)$ , by [23, Remark 4.3], we can apply [23, Theorem 4.1, Corollary 4.2]. In particular, when  $\mathfrak{r} = \mathfrak{r}_k$ , the second map is represented by a map

$${}^b v_k^{\leq 0} : CFK^\infty(V, B_g)\{i \leq 0\} \rightarrow CFK^\infty(V, B_g)\{i \leq 0 \text{ or } j \leq -k\},$$

and  $({}^b v_k^{\leq 0})_*$  maps  $\mathbb{Z}[U] \otimes \theta$  isomorphically to  $\mathbb{Z}[U] \otimes \theta \subset HFK^\infty(V, B_g)$ .

Suppose that  $\mathfrak{r} \in \mathcal{X}_k \setminus \{\mathfrak{r}_k, \mathfrak{r}_k\}$ , then  $\mathfrak{r}$  is equal to  $\mathfrak{r}_k + l\text{PD}[\widehat{F}_n]$  for  $l \neq 0, -1$ . The grading difference of  $F_{N, \mathfrak{r}_k|N}^{\leq 0}(\mathbf{1})$  and  $F_{N, (\mathfrak{r}_k + l\text{PD}[\widehat{F}_n])|N}^{\leq 0}(\mathbf{1})$  is

$$\begin{aligned} & \frac{c_1^2(\mathfrak{r}_k|N) - c_1^2((\mathfrak{r}_k + l\text{PD}[\widehat{F}_n])|N)}{4} \\ &= \frac{-(2k - n)^2 + (2k - n - 2ln)^2}{4n} = l(ln + n - 2k). \end{aligned}$$

So  $F_{N, \mathfrak{r}|N}^{\leq 0}(\mathbf{1})$  is contained in the same Maslov grading level as

$$U^{\frac{l(l+1)n}{2} - lk} \otimes \theta,$$

whose double filtration level is

$$\left(-\frac{l(l+1)n}{2} + lk, -\frac{l(l+1)n}{2} + lk + g\right).$$

In  $HF^\infty(V, B_g)$ , the Maslov grading is equal to  $i + j$  up to an overall translation, hence the double filtration level of  $F_{N, \mathfrak{r}|N}^{\leq 0}(\mathbf{1})$  is contained in the range

$$\left(-\frac{l(l+1)n}{2} + lk + m, -\frac{l(l+1)n}{2} + lk + g - m\right), \quad m = 0, 1, \dots, g.$$

Any element in  $HF^\infty(V, B_g)$  with the above filtration levels can be obtained from

$$U^{\frac{l(l+1)n}{2} - lk - g} \otimes \theta,$$

whose double filtration level is

$$\left(-\frac{l(l+1)n}{2} + lk + g, -\frac{l(l+1)n}{2} + lk + 2g\right),$$

by applying the action of an element in  $\mathbb{Z}[U] \otimes \Lambda^*(H_1(V))$ . Since  $l \neq 0, -1$  and  $n \geq 2|k|$ , we have

$$\frac{l(l+1)n}{2} - lk - g \geq n - 2|k| - g.$$

So the map  $F_{N, \mathfrak{r}|N}^{\leq 0}$  factorizes through

$$U^{n-2|k|-g} : HF^{\leq 0}(S^3) \rightarrow HF^{\leq 0}(S^3).$$

As in the proof of [22, Theorem 3.1], we can decompose the cobordism  $W'_n(K)$  as the composition of two cobordisms

$$W_1 : Y_n(K) \rightarrow V_{-n}(B_g) \# Y_n(K)$$

and

$$W_2 : V_{-n}(B_g) \# Y_n(K) \rightarrow Y,$$

where  $W_1$  is a “timewise” connected sum of  $N$  and  $Y_n(K) \times [0, 1]$ . By (8), we have

$$\begin{array}{ccc} HF^-(S^3) \otimes HF^{\leq 0}(Y_n(K), \mathfrak{s}_t) & \xrightarrow{F_{S^3 \# Y_n(K), \mathfrak{s}_t}^-} & HF^-(Y_n(K), \mathfrak{s}_t) \\ \downarrow F_{N, \mathfrak{r}|N}^- \otimes \text{id} & & \downarrow F_{W_1, \mathfrak{r}|N \# \mathfrak{s}_t}^- \\ HF^-(V_{-n}(B_g)) \otimes HF^{\leq 0}(Y_n(K), \mathfrak{s}_t) & \xrightarrow{F_{V_{-n}(B_g) \# Y_n(K), \mathfrak{s}_t}^-} & HF^-(V_{-n}(B_g) \# Y_n(K), \mathfrak{s}_t). \end{array}$$

The horizontal map in the first row is surjective. Since  $F_{N, \mathfrak{r}|N}^{\leq 0}$  and thus  $F_{N, \mathfrak{r}|N}^-$  factorize through  $U^{n-2|k|-g}$ ,  $F_{W_1, \mathfrak{r}|W_1}^-$  also factorizes through  $U^{n-2|k|-g}$ . By Lemma 2.1,  $F_{W'_n(K), \mathfrak{r}}^-$  factorizes through  $F_{W_1, \mathfrak{r}|W_1}^-$ , so  $F_{W'_n(K), \mathfrak{r}}^-$  factorizes through  $U^{n-2|k|-g}$ .

When  $c_1(\mathfrak{s})$  is nontorsion, using the natural short exact sequence (1), we see that  $F_{W'_n(K), \mathfrak{r}}^+$  also factorizes through  $U^{n-2|k|-g}$ , since  $HF^+$  is just a submodule of  $HF^-$ .  $\square$

*Proof of Theorem 3.1 when  $c_1(\mathfrak{s})$  is nontorsion.* As before, Theorem 3.4 implies that  $CF^+(Y_0(K), \mathfrak{t}_k)$  is quasi-isomorphic to  $MC(f_3^+)$ , where

$$f_3^+ = \sum_{\mathfrak{r} \in \mathcal{X}_k} f_{W'_n(K), \mathfrak{r}}^+ : CF^+(Y_n(K), \mathfrak{s}_t) \rightarrow CF^+(Y, \mathfrak{s}).$$

Since  $c_1(\mathfrak{s})$  is nontorsion,  $H_*(A_k^+)$  is a finitely generated abelian group. So  $U^m|H_*(A_k^+) = 0$  when  $m$  is greater than a constant  $C_1$  independent of  $n$ . By Theorem 2.5, this implies that  $U^m|HF^+(Y_n(K), \mathfrak{s}_t) = 0$  when  $m > C_1$ . Proposition 3.5 implies that  $F_{W'_n(K), \mathfrak{r}}^+ = 0$  when  $\mathfrak{r} \in \mathcal{X}_k \setminus \{\mathfrak{r}_k, \mathfrak{r}_k\}$  and  $n > \max\{2|k| + g, 2g\} + C_1$ . Our conclusion follows by Theorem 2.5.  $\square$

*Proof of Theorem 3.3.* We use the fact that  $\underline{CF}^+(Y_0(K), \mathfrak{t}_k; \mathbb{Z}[T, T^{-1}])$  is quasi-isomorphic to  $MC(\underline{f}_3^+)$ , where

$$\underline{f}_3^+ = \sum_{\mathfrak{r} \in \mathcal{X}_k} T^{\frac{1}{n}\langle \mathfrak{r} - \mathfrak{r}_k, \widehat{\varphi}_n \rangle} f_{W'_n(K), \mathfrak{r}}^+ : CF^+(Y_n(K), \mathfrak{s}_t)[T, T^{-1}] \rightarrow CF^+(Y, \mathfrak{s})[T, T^{-1}].$$

The corresponding long exact sequence can be found in [21, Theorem 9.23] and the last formula in [24, Section 3]. The fact that the two chain complexes are quasi-isomorphic can be proved in a similar way as Theorem 3.4. The rest of the argument is the same as in the proof of Theorem 3.1.  $\square$

#### 4. THE ZERO SURGERY

In this section, we will prove our main results. For simplicity, we will use coefficients in a fixed field  $\mathbb{F}$  for Heegaard Floer homology throughout this section. We may choose  $\mathbb{F} = \mathbb{F}_2$  since we only prove Theorem 3.4 with  $\mathbb{F}_2$  coefficients in Appendix A, but other fields can also be used if we assume Theorem 3.4 with  $\mathbb{Z}$  coefficients.

##### 4.1. A rank inequality.

**Lemma 4.1.** *Suppose that  $Y, K, F, X, G$  satisfy the same condition as in Theorem 1.5, then*

$$\text{im}(v_k^+)_* \supset \text{im}(h_k^+)_*,$$

when  $k \geq g(G)$ . Moreover, if  $G$  is a disk, then

$$(v_0^+)_* = (h_0^+)_*.$$

*Proof.* Without loss of generality, we may assume  $k = g(G)$ , since we can always increase the genus of  $G$ . Let  $W = W'_n(K) \# N\overline{CP}^2$ ,  $\Sigma \subset W$  be the closed oriented surface obtained by capping off  $-G$  with the cocore of the 2-handle in  $W'_n(K)$ , and  $\mathfrak{S}$  be the connected sum of  $\mathfrak{r}_k$  with  $N$  copies of  $\mathfrak{t}_1$ , where  $\mathfrak{t}_1 \in \text{Spin}^c(\overline{CP}^2)$  satisfies that  $c_1(\mathfrak{t}_1)$  generates  $H^2(\overline{CP}^2)$ . It follows from (7) that

$$\langle c_1(\mathfrak{r}_k \# N\mathfrak{t}_1), [\Sigma] \rangle = -2k - n, \quad \mathfrak{r}_k \# N\mathfrak{t}_1 = \mathfrak{r}_k \# N\mathfrak{t}_1 - \text{PD}[\Sigma].$$

Our conclusion follows from Theorems 2.5 and 2.2 and the blow-up formula [26, Theorem 1.4].  $\square$

Proposition 4.2 is an analogue of [23, Corollary 4.5].

**Proposition 4.2.** *Let  $K$  be a null-homologous knot in  $Y$ ,  $\varphi$  be a Seifert homology class,  $\mathfrak{s} \in \text{Spin}^c(Y)$ . Let*

$$d = \max \left\{ i \in \mathbb{Z} \mid \widehat{HF}K(Y, K, \mathfrak{s}, \varphi, i) \neq 0 \right\}.$$

If

$$(9) \quad \text{im}(v_{d-1})_* \supset \text{im}(h_{d-1})_*,$$

and one of the following conditions holds:

- $c_1(\mathfrak{s})$  is nontorsion and  $d \geq 1$ ,
- $HF^+(Y, \mathfrak{s}) = 0$ ,
- $d > 1$ ,

then

$$\text{rank } HF^+(Y_0(K), \mathfrak{t}_{d-1}) \geq \text{rank } \widehat{HFK}(Y, K, \mathfrak{s}, \varphi, d).$$

*Proof.* As in the proof of [23, Corollary 4.5], there is an exact triangle

$$(10) \quad \begin{array}{ccc} H_*(A_{d-1}^+) & \xrightarrow{(v_{d-1}^+)_*} & H_*(B^+) \\ \uparrow & \swarrow & \\ H_*(C\{-1, d-1\}) & & \end{array}$$

For simplicity, denote  $(v_{d-1}^+)_* = v$ ,  $(h_{d-1}^+)_* = h$ . Then  $\text{im } h \subset \text{im } v$  by (9). It follows from (10) that

$$(11) \quad \text{rank } \widehat{HFK}(Y, K, \mathfrak{s}, \varphi, d) = \text{rank ker } v + \text{rank coker } v.$$

By Theorem 3.1,

$$(12) \quad \text{rank } HF^+(Y_0(K), \mathfrak{t}_{d-1}) = \text{rank ker}(v + h) + \text{rank coker}(v + h).$$

*Case 1.*  $c_1(\mathfrak{s})$  is nontorsion and  $d \geq 1$ .

In this case both  $H_*(A_{d-1}^+)$  and  $H_*(B^+)$  have finite ranks. It follows from (11) and (12) that

$$\begin{aligned} \text{rank } \widehat{HFK}(Y, K, \mathfrak{s}, \varphi, d) &= \text{rank } H_*(A_{d-1}^+) + \text{rank } H_*(B^+) - 2 \text{rank im } v, \\ \text{rank } HF^+(Y_0(K), \mathfrak{t}_{d-1}) &= \text{rank } H_*(A_{d-1}^+) + \text{rank } H_*(B^+) - 2 \text{rank im}(v + h). \end{aligned}$$

Our conclusion follows from (9).

*Case 2.*  $HF^+(Y, \mathfrak{s}) = 0$ .

Since  $H_*(B^+) = 0$ , the right hand sides of both (11) and (12) are  $\text{rank } H_*(A_{d-1}^+)$ .

*Case 3.*  $d > 1$ .

In this case we may assume  $c_1(\mathfrak{s})$  is torsion, then there is an absolute  $\mathbb{Q}$ -grading on  $C$ . Let  $\rho : \text{im } v \rightarrow H_*(A_{d-1}^+)$  be a homogeneous homomorphism of  $\mathbb{F}$ -vector spaces such that

$$v \circ \rho = \text{id}|_{\text{im } v}.$$

By (9), the map  $\text{id} + \rho h : H_*(A_{d-1}^+) \rightarrow H_*(A_{d-1}^+)$  is well-defined. Clearly,  $\text{id} + \rho h$  maps  $\text{ker}(v + h)$  to  $\text{ker } v$ .

Since  $d > 1$ , the grading shift of  $h$  is strictly less than the grading shift of  $v$ , so the grading shift of  $\rho h$  is negative. As the grading of  $H_*(A_{d-1}^+)$  is bounded from below, for any  $x \in H_*(A_{d-1}^+)$ ,  $(\rho h)^m(x) = 0$  when  $m$  is sufficiently large. The map

$$\text{id} - \rho h + (\rho h)^2 - (\rho h)^3 + \cdots : H_*(A_{d-1}^+) \rightarrow H_*(A_{d-1}^+)$$

is well-defined, and it maps  $\text{ker } v$  to  $\text{ker}(v + h)$ . This map is the inverse to  $\text{id} + \rho h : \text{ker}(v + h) \rightarrow \text{ker } v$ , so

$$\text{rank ker}(v + h) = \text{rank ker } v.$$

Since  $\text{im}(v + h) \subset \text{im } v$ ,

$$\text{rank coker}(v + h) \geq \text{rank coker } v.$$

Our conclusion holds by (11), (12). □

**4.2. The proof of Theorem 1.5 when  $g(F) > 1$ .** In this subsection, we will prove Theorem 1.5 in the case  $g(F) > 1$ . If  $g(F) = 1$ ,  $G$  has to be a disk, and this case will be treated in Section 4.4.

*Proof of Theorem 1.5 when  $g(F) > 1$ .* We first prove Property (G1). Since  $F$  is taut, we get  $\widehat{HFK}(Y, K, [F], g(F)) \neq 0$  by Theorem 2.8. Using Lemma 4.1 and Proposition 4.2, we see that  $HF^+(Y_0(K), [\widehat{F}], g(F) - 1) \neq 0$ , so  $\widehat{F}$  is also taut.

Next, we prove Property (G2). Suppose that  $Y_0(K)$  is a surface bundle over  $S^1$ , with  $S$  being the fiber in the homology class  $\widehat{\varphi}$ . By Property (G1),  $g(S) > 0$ , so  $Y_0(K)$  is irreducible. If  $Y \setminus K$  is reducible, let  $P \subset Y \setminus K$  be an essential sphere. Since  $Y_0(K)$  is irreducible,  $P$  must bound a ball in  $Y_0(K)$ , and this ball must contain the dual knot of  $K$ . This is not possible because the dual knot of  $K$  is not null-homologous in  $Y_0(K)$ . So  $Y \setminus K$  is irreducible.

Since  $Y_0(K)$  fibers over  $S^1$  with fiber in the homology class  $[\widehat{F}]$ ,  $\widehat{F}$  is isotopic to a fiber. (A reference for this folklore result can be found in the last sentence of the first paragraph in [31, Section 3].) We have  $HF^+(Y_0(K), [\widehat{F}], g(F) - 1) \cong \mathbb{F}$  by [22, Theorem 5.2]. Using Lemma 4.1 and Proposition 4.2, we see that

$$\text{rank } \widehat{HFK}(Y, K, [F], g(F)) \leq 1.$$

However, we already know  $\widehat{HFK}(Y, K, [F], g(F)) \neq 0$ , so  $\widehat{HFK}(Y, K, [F], g(F)) \cong \mathbb{F}$ , and our conclusion follows from Theorem 2.9.  $\square$

**4.3. The proof of Theorem 1.7.** Let  $Y, K, X$  be as in Theorem 1.7, and let  $G \subset X$  be the disk bounded by  $K$ . Without loss of generality, we will assume  $K$  is not the unknot. There is a natural map

$$\pi : X \rightarrow Y$$

which is the composition of the pinching map  $X \rightarrow Y \times [0, 1]$  with the projection  $Y \times [0, 1] \rightarrow Y$ . Suppose that  $\varphi_0 = [\pi(G)] \in H_2(Y, K)$  is the homology class of the immersed disk  $\pi(G)$ , and  $\varphi \in H_2(Y, K)$  is a Seifert homology class for  $K$ . Let  $\widehat{\varphi}_0, \widehat{\varphi} \in H_2(Y_0(K))$  be the extensions of  $\varphi_0, \varphi$ . Given a  $\text{Spin}^c$  structure  $\mathfrak{s} \in \text{Spin}^c(Y)$ , let  $\mathfrak{t}_k \in \text{Spin}^c(Y_0(K))$  be the extension of  $\mathfrak{s}|(Y \setminus K)$  with

$$\langle c_1(\mathfrak{t}_k), \widehat{\varphi}_0 \rangle = 2k,$$

and let  $\xi_k \in \text{Spin}^c(Y, K)$  be the relative  $\text{Spin}^c$  structure whose underlying  $\text{Spin}^c$  structure is  $\mathfrak{s}$ , and

$$\langle c_1(\xi_k), \varphi_0 \rangle = 2k + 1.$$

Let  $C = CFK^\infty(Y, K, \xi_0, \varphi_0)$ .

**Lemma 4.3.** *With the above notation, we have*

$$\langle c_1(\xi_k), \varphi \rangle - \langle c_1(\xi_l), \varphi \rangle = \langle c_1(\xi_k), \varphi_0 \rangle - \langle c_1(\xi_l), \varphi_0 \rangle = 2k - 2l.$$

*Proof.* Since  $\varphi - \varphi_0$  is represented by a closed surface in  $Y \setminus K$ , we have

$$\begin{aligned} \langle c_1(\xi_k), \varphi \rangle - 2k - 1 &= \langle c_1(\xi_k), \varphi - \varphi_0 \rangle \\ &= \langle c_1(\mathfrak{s}), \varphi - \varphi_0 \rangle \end{aligned}$$

is independent of  $k$ .  $\square$

**Lemma 4.4.** *Let  $F$  be a taut Seifert-like surface for a nontrivial knot  $K$ , and let*

$$\Xi_{\max} = \{ \xi \in \underline{\text{Spin}}^c(Y, K) \mid \langle c_1(\xi), [F] \rangle = 2 + \chi_-(F) \}.$$

*Suppose  $\mathfrak{s} \in \text{Spin}^c(Y)$  is the underlying  $\text{Spin}^c$  structure of some  $\xi \in \Xi_{\max}$ . If  $HF^+(Y, \mathfrak{s}) \neq 0$ , then  $\langle c_1(\xi), \varphi_0 \rangle \geq 3$ .*

*Proof.* Gluing  $\pi(-G)$  to  $F$ , we get an immersed closed surface with  $\chi_- = \chi_-(F) - 1$  in  $Y$  representing the homology class  $\varphi - \varphi_0$ . By Gabai's theorem that the singular Thurston norm is equal to the Thurston norm [3, Corollary 6.18], the Thurston norm of  $\varphi - \varphi_0 \in H_2(Y)$  is at most  $\chi_-(F) - 1$ . Since  $HF^+(Y, \mathfrak{s}) \neq 0$ , it follows from the adjunction inequality that

$$\begin{aligned} \chi_-(F) - 1 &\geq \langle c_1(\mathfrak{s}), \varphi - \varphi_0 \rangle \\ &= \langle c_1(\xi), \varphi - \varphi_0 \rangle \\ &= \chi_-(F) + 2 - \langle c_1(\xi), \varphi_0 \rangle. \end{aligned}$$

Hence  $\langle c_1(\xi), \varphi_0 \rangle \geq 3$ . □

*Proof of Theorem 1.7 for Property (G1).* Suppose that  $F$  is a taut Seifert-like surface for  $K$ . Let  $\Xi_{\max}$  be as in Lemma 4.4, then there exists  $\xi \in \Xi_{\max}$  with  $\widehat{HFK}(Y, K, \xi) \neq 0$  by Theorem 2.8. Let  $\mathfrak{s} \in \text{Spin}^c(Y)$  be the underlying  $\text{Spin}^c$  structure of  $\xi$ . By Lemma 4.3, if  $\mathfrak{s}$  is the underlying  $\text{Spin}^c$  structure of another  $\eta \in \underline{\text{Spin}}^c(Y, K)$  with  $\widehat{HFK}(Y, K, \eta) \neq 0$ , then

$$\langle c_1(\xi) - c_1(\eta), \varphi_0 \rangle = \langle c_1(\xi) - c_1(\eta), \varphi \rangle \geq 0.$$

Suppose that  $\langle c_1(\xi), \varphi_0 \rangle = 2d + 1$ , then the above inequality means that

$$(13) \quad d = \max \left\{ i \in \mathbb{Z} \mid \widehat{HFK}(Y, K, \mathfrak{s}, \varphi_0, i) \neq 0 \right\}.$$

If  $\mathfrak{s}, d$  satisfy one of the three conditions in Proposition 4.2, it follows from Proposition 4.2 and (13) that  $HF^+(Y_0(K), \mathfrak{t}_{d-1}) \neq 0$ . If  $\mathfrak{s}, d$  do not satisfy any of the three conditions in Proposition 4.2, then  $c_1(\mathfrak{s})$  is torsion, and  $d = 1$  by Lemma 4.4. Hence  $\mathfrak{t}_{d-1}$  is also torsion, so we also have  $HF^+(Y_0(K), \mathfrak{t}_{d-1}) \neq 0$ .

We have

$$\begin{aligned} \langle c_1(\mathfrak{t}_{d-1}), [\widehat{F}] \rangle &= \langle c_1(\mathfrak{t}_{d-1}), [\widehat{F}] - \widehat{\varphi}_0 \rangle + \langle c_1(\mathfrak{t}_{d-1}), \widehat{\varphi}_0 \rangle \\ &= \langle c_1(\xi), [F] - \varphi_0 \rangle + 2d - 2 \\ &= \chi_-(F) + 2 - (2d + 1) + 2d - 2 \\ &= \chi_-(F) - 1 \\ (14) \quad &= \chi_-(\widehat{F}). \end{aligned}$$

Since  $HF^+(Y_0(K), \mathfrak{t}_{d-1}) \neq 0$ , it follows from Theorem 2.7 that  $\widehat{F}$  is Thurston norm minimizing.

In order to show that  $\widehat{F}$  is taut, we only need to prove that if  $T \subset \widehat{F}$  is a torus component, then  $[T]$  is not represented by a sphere. If  $F$  is disconnected, then  $b_1(Y) > 0$ , and this case follows from Proposition 1.10, which will be proved in Section 4.4. If  $F$  is connected, then  $\chi_-(F) = 1$ , and this case will also be proved in Section 4.4. □



*Proof of Theorem 1.7 for Property (G2).* Without loss of generality, we may assume  $K$  is not the unknot. As in the proof in Section 4.2,  $Y \setminus K$  is irreducible.

If  $g(S) > 1$ , by [22, Theorem 5.2], there exists  $\mathfrak{t} \in \text{Spin}^c(Y_0(K))$  which is the unique  $\text{Spin}^c$  structure such that  $HF^+(Y_0(K), \mathfrak{t}) \neq 0$  and  $\langle c_1(\mathfrak{t}), \widehat{\varphi} \rangle = 2g(S) - 2$ . Moreover,  $HF^+(Y_0(K), \mathfrak{t}) \cong \mathbb{F}$ .

Suppose  $\xi \in \text{Spin}^c(Y, K)$  satisfies that  $\langle c_1(\xi), \varphi \rangle = 2g(S) + 1$ . Let  $\mathfrak{s} \in \text{Spin}^c(Y)$  be the underlying  $\text{Spin}^c$  structure of  $\xi$ , and let  $\mathfrak{r} \in \text{Spin}^c(Y_0(K))$  be the extension of  $\xi$ .

*Claim.* If  $\widehat{HFK}(Y, K, \xi) \neq 0$ , then  $\mathfrak{t} = \mathfrak{r} - \text{PD}[\mu]$ , and  $\widehat{HFK}(Y, K, \xi) \cong \mathbb{F}$ .

If  $\widehat{HFK}(Y, K, \xi) \neq 0$ , let  $d \in \mathbb{Z}$  be defined by

$$\langle c_1(\xi), \varphi_0 \rangle = 2d + 1.$$

By Property (G1), there exists no  $\eta \in \text{Spin}^c(Y, K)$  such that  $\langle c_1(\eta), \varphi \rangle > 2g(S) + 1$ . It follows from Lemma 4.3 that (13) holds.

If  $\mathfrak{s}, d$  satisfy one of the three conditions in Proposition 4.2, then

$$HF^+(Y_0(K), \mathfrak{t}_{d-1}) \neq 0.$$

As we have computed in (14),  $\langle c_1(\mathfrak{t}_{d-1}), \widehat{\varphi} \rangle = 2g(S) - 2$ , hence we must have  $\mathfrak{t}_{d-1} = \mathfrak{t}$ . It follows from Proposition 4.2 that  $\widehat{HFK}(Y, K, \xi) \cong \mathbb{F}$ . Our claim follows.

If  $\mathfrak{s}, d$  do not satisfy any of the three conditions in Proposition 4.2, Lemma 4.4 implies that  $d = 1$  and  $c_1(\mathfrak{s})$  is torsion, we have

$$\begin{aligned} 2g(S) + 1 &= \langle c_1(\xi), \varphi \rangle \\ &= \langle c_1(\xi), \varphi - \varphi_0 \rangle + \langle c_1(\xi), \varphi_0 \rangle \\ &= \langle c_1(\mathfrak{s}), \varphi - \varphi_0 \rangle + 2d + 1 \\ &= 3, \end{aligned}$$

a contradiction to our assumption that  $g(S) > 1$ . This proves our claim.

Let  $F$  be a taut Seifert-like surface for  $K$  with  $[F] = \varphi$ . By Property (G1),  $\widehat{F}$  is taut in  $Y_0(K)$ . Since  $[\widehat{F}] = \widehat{\varphi}$  is the homology class of the fiber,  $\widehat{F}$  must be isotopic to the fiber. (See the proof in Section 4.2 for a reference for this result.) In particular,  $F$  is connected. Our theorem in this case follows from the claim and Theorem 2.9.

The case  $g(S) = 1$  will be treated in Section 4.4. □

**4.4. The  $\chi_-(F) = 1$  case.** In this subsection, we will prove the  $\chi_-(F) = 1$  case of Theorem 1.7, which also implies the  $g(F) = 1$  case of Theorem 1.5. We first give a proof of Proposition 1.10.

*Proof of Proposition 1.10.* Let  $M = Y \setminus K$ . Since  $b_1(Y) > 0$ , there exists a closed, oriented, connected surface  $S$  in the interior of  $M$ , such that  $S$  is taut in  $M$ . Notice that for the  $\infty$  slope on  $K$ , the core of the surgery solid torus in  $Y_\infty(K) = Y$ , which is  $K$ , is null-homotopic. Using a theorem of Lackenby [11, Theorem A.21], which is a stronger version of the main result in [4], we conclude that each 2-sphere in  $Y_0(K)$  bounds a rational homology ball. Hence  $Y_0(K)$  does not have an  $S^1 \times S^2$  connected summand. □

To prove the  $\chi_-(F) = 1$  case of Theorem 1.7, we use the argument in [2]. We will use twisted coefficients in the Novikov ring  $\Lambda = \mathbb{F}[[T, T^{-1}]]$ .

**Proposition 4.5.** *Let  $Y, K, X$  be as in Theorem 1.7, and let  $G \subset X$  be the disk bounded by  $K$ . Suppose that  $K$  is nontrivial and  $F$  is a genus-1 Seifert surface for  $K$ , then there does not exist an embedded sphere  $P \subset Y_0(K)$  such that  $[P] \cdot [\mu] \neq 0$ , where  $\mu$  is the meridian of  $K$ .*

*Proof.* We will consider

$$\underline{HF}^+(Y_0(K); \Lambda) := \underline{HF}^+(Y_0(K); \mathbb{F}[[T, T^{-1}]]) \otimes_{\mathbb{F}[[T, T^{-1}]]} \Lambda.$$

As in [16], if there exists such a sphere  $P$ , then  $\underline{HF}^+(Y_0(K); \Lambda) = 0$ .

By Lemma 4.1, we have  $(v_0^+)_* = (h_0^+)_*$ . It follows from Theorem 3.3 that  $(v_0^+)_*$  is an isomorphism, which would imply that  $\widehat{HFK}(Y, K, [F], 1) = 0$  by the exact triangle (11), where  $d = 1$ . This contradicts Theorem 2.8.  $\square$

*Proof of Theorem 1.7 when  $\chi_-(F) = 1$ .* We first prove Property (G1). Without loss of generality, we assume that  $Y \setminus K$  is irreducible.

Since  $\chi_-(F) = 1$  and  $Y \setminus K$  is irreducible, every component of  $\widehat{F}$  is a torus. To prove Property (G1), we just need to show that  $Y_0(K)$  contains no nonseparating 2-spheres. This follows from Proposition 4.5 when  $b_1(Y) = 0$ , and from Proposition 1.10 when  $b_1(Y) > 0$ .

Now we prove Property (G2). Suppose that  $Y_0(K)$  is a torus bundle over  $S^1$ , with fiber in the homology class  $\widehat{\varphi}$ . As argued in the last subsection,  $\varphi$  is represented by a genus-1 Seifert surface  $F$ , since we already know Property (G1) in this case. By [1],  $\underline{HF}^+(Y_0(K); \Lambda) \cong \Lambda$ . By Lemma 4.1 and Theorem 3.3, we see that the homology of the mapping cone of

$$(1 + T)(v_0^+) : A_0^+[[T, T^{-1}]] \rightarrow B^+[[T, T^{-1}]]$$

is  $\Lambda$ . Hence the homology of the mapping cone of

$$v_0^+ : A_0^+ \rightarrow B^+$$

is  $\mathbb{F}$ . Using (10), we get  $\widehat{HFK}(Y, K, [F], 1) \cong \mathbb{F}$ , so  $K$  is fibered with fiber  $F$  by Theorem 2.9.  $\square$

### 5. KNOTS IN TORUS BUNDLES

In this section, we will prove Theorem 1.12. Let  $Y$  be a  $T^2$ -bundle over  $S^1$ ,  $K \subset Y$  be a null-homologous knot. Let  $\gamma \subset Y \setminus K$  be a loop which intersects a torus fiber exactly once transversely, and  $\mu \subset Y \setminus K$  be a meridian of  $K$ . As in Section 3, the evaluation over  $[\gamma] + [\mu]$  defines a ring homomorphism  $\mathbb{Z}[H^1(\cdot)] \rightarrow \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[[T, T^{-1}]]$ . Using this evaluation, we will get the corresponding twisted Heegaard Floer homology  $\underline{HF}^\circ(\cdot; \Lambda)$  with coefficients in the Novikov ring  $\Lambda$ . The same argument as in [24] and [13] shows that twisted knot Floer homology in this case detects the genus of a knot and whether a knot is fibered. See also [16, Section 3.3].

*Proof of Theorem 1.12.* Without loss of generality, we assume  $K$  is nontrivial. Let  $\varphi \in H_2(Y, K)$  be a Seifert homology class, and let

$$d = \max \left\{ i \in \mathbb{Z} \mid \widehat{HFK}(Y, K, \varphi, i) \neq 0 \right\}.$$

Then  $\widehat{HFK}(Y, K, \varphi, d; \Lambda)$  is isomorphic to the mapping cone of

$$\underline{v}_{d-1}^+ : \underline{A}_{d-1}^+ \rightarrow \underline{B}^+,$$

and  $\underline{HF}^+(Y_0(K), \widehat{\varphi}, d-1; \Lambda)$  is isomorphic to the mapping cone of

$$\underline{v}_{d-1}^+ + \underline{h}_{d-1}^+ : \underline{A}_{d-1}^+ \rightarrow \underline{B}^+.$$

Here  $\underline{A}_{d-1}^+$  and  $\underline{B}^+$  are twisted chain complexes.

Since  $([\gamma] + [\mu]) \cdot [\text{torus fiber}] \neq 0$ , by [1],

$$H_*(\underline{B}^+) = \underline{HF}^+(Y; \Lambda) \cong \Lambda,$$

which is a 1-dimensional vector space over the field  $\Lambda$ . So  $(\underline{v}_{d-1}^+)_*$  is either surjective or 0.

If  $(\underline{v}_{d-1}^+)_*$  is surjective, the same argument of counting dimensions as in Case 1 of Proposition 4.2 shows that

$$\text{rank}_\Lambda \underline{HF}^+(Y_0(K), \widehat{\varphi}, d-1; \Lambda) \geq \text{rank}_\Lambda \widehat{HFK}(Y, K, \varphi, d; \Lambda).$$

Hence Property G follows just as before.

If  $(\underline{v}_{d-1}^+)_*$  is 0, then  $(\underline{v}_0^+)_* = 0$  by (5). Since  $(\underline{v}_0^+)_*$  and  $(\underline{h}_0^+)_*$  have the same rank, we also have  $(\underline{h}_0^+)_* = 0$ . Thus  $(\underline{h}_{d-1}^+)_* = 0$  by (6). So both mapping cones are quasi-isomorphic to  $\underline{A}_{d-1}^+ \oplus \underline{B}^+$ , and Property G still follows.  $\square$

#### APPENDIX A. A PROOF OF THEOREM 3.4

In this appendix, we will sketch a proof of the version of Theorem 3.4 with  $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$  coefficients, following the strategy of the proof of [25, Theorem 4.5]. The reason that the proof only works over  $\mathbb{F}_2$  is that the key lemma in homological algebra (Lemma A.1) is only stated over  $\mathbb{F}_2$ . As remarked in a footnote in [25, Section 4], the proof can be carried over to  $\mathbb{Z}$  coefficients routinely. However, since  $\mathbb{F}_2$  coefficients are enough for the applications in this paper, we are satisfied with the current version.

Lemma A.1 in homological algebra is [25, Lemma 4.2].

**Lemma A.1.** *Let  $\{A_i\}_{i=1}^\infty$  be a collection of chain complexes of  $\mathbb{F}_2$ -vector spaces and let*

$$\{f_i : A_i \rightarrow A_{i+1}\}_{i=1}^\infty$$

*be a collection of chain maps satisfying the following two properties:*

(1)  *$f_{i+1} \circ f_i$  is chain homotopically trivial, by a chain homotopy*

$$H_i : A_i \rightarrow A_{i+2},$$

(2) *the map*

$$\phi_i = f_{i+2} \circ H_i + H_{i+1} \circ f_i : A_i \rightarrow A_{i+3}$$

*is a quasi-isomorphism.*

*Then the map*

$$\psi_i : A_i \rightarrow MC(f_{i+1})$$

*defined by*

$$\psi_i(a_i) = (f_i(a_i), H_i(a_i))$$

*is a quasi-isomorphism.*

Suppose that we have a pointed Heegaard  $(n+1)$ -tuple

$$(\Sigma, \xi^0, \xi^1, \dots, \xi^n, z)$$

satisfying certain admissibility conditions. There is a standard way to define a map

$$\mu_n : \bigotimes_{i=1}^n CF^+(\Sigma, \xi^{i-1}, \xi^i, z) \rightarrow CF^+(\Sigma, \xi^0, \xi^n, z)$$

by counting pseudo-holomorphic  $(n+1)$ -gons, where  $CF^+(\Sigma, \xi^i, \xi^j, z)$  is the Heegaard Floer chain complex constructed for the Heegaard diagram  $(\Sigma, \xi^i, \xi^j, z)$ . These maps satisfy a well-known generalized associativity property

$$(15) \quad \sum_{i+j=n+1} \sum_{\ell=1}^{n-j+1} \mu_i(a_1 \otimes \dots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \dots \otimes a_{\ell+j-1}) \otimes a_{\ell+j} \otimes \dots \otimes a_n) = 0.$$

Let

$$(\Sigma, \alpha, \beta, \gamma, \delta, w),$$

be the genus  $h$  Heegaard quadruple constructed for the pair  $(Y, K)$  in Section 3. We will consider a sequence of  $h$ -tuples of attaching curves  $\{\eta^i\}_{i=1}^\infty$ , where  $\eta^1 = \beta$  and  $\eta^{3i+1}$  ( $i > 0$ ) consists of small Hamiltonian translates of curves in  $\beta$ ,  $\eta^2 = \gamma$  and  $\eta^{3i+2}$  ( $i > 0$ ) consists of small Hamiltonian translates of curves in  $\gamma$ ,  $\eta^3 = \delta$  and  $\eta^{3i+3}$  ( $i > 0$ ) consists of small Hamiltonian translates of curves in  $\delta$ . Let

$$\eta^i = (\eta_1^i, \dots, \eta_h^i).$$

Note that  $(\Sigma, \eta^{3i+1}, \eta^{3i+2}, w)$  is a Heegaard diagram for  $\#^{h-1}S^1 \times S^2$ ,  $i \geq 0$ . Let  $\Theta_{3i+1, 3i+2}$  be the top generator of  $\widehat{CF}(\Sigma, \eta^{3i+1}, \eta^{3i+2}, w)$ . Similarly, define  $\Theta_{3i+3, 3i+4}$  to be the top generator of  $\widehat{CF}(\Sigma, \eta^{3i+3}, \eta^{3i+4}, w)$ ,  $i \geq 0$ .

The diagram  $(\Sigma, \eta^{3i+2}, \eta^{3i+3}, w)$  is a Heegaard diagram for  $(\#^{h-1}S^1 \times S^2) \# L(n, 1)$ ,  $i \geq 0$ . There are  $n$  generators of  $\widehat{CF}(\Sigma, \eta^{3i+2}, \eta^{3i+3}, w)$  which can be viewed as “top” generators, but only one of them corresponds to  $\mathfrak{t}_k$ . Let this generator be  $\Theta_{3i+2, 3i+3}$ . (More precisely, the pair of pants construction gives us a cobordism

$$X_{\alpha, \eta^{3i+2}, \eta^{3i+3}} : Y_0(K) \sqcup (\#^{h-1}S^1 \times S^2) \# L(n, 1) \rightarrow Y_n(K).$$

Then the  $\text{Spin}^c$  structure associated with  $\Theta_{3i+2, 3i+3}$  is the restriction of the  $\text{Spin}^c$  structure over  $X_{\alpha, \eta^{3i+2}, \eta^{3i+3}}$  that extends  $\mathfrak{t}_k$ .)

Now we define the map

$$f_i : CF^+(\Sigma, \alpha, \eta^i, w) \rightarrow CF^+(\Sigma, \alpha, \eta^{i+1}, w)$$

by

$$f_i(x) = \mu_2(x, \Theta_{i, i+1}).$$

It is a chain map by (15) and the fact that

$$(16) \quad \mu_1(\Theta_{i, i+1}) = 0.$$

We also define

$$H_i : CF^+(\Sigma, \alpha, \eta^i, w) \rightarrow CF^+(\Sigma, \alpha, \eta^{i+2}, w)$$

by

$$H_i(x) = \mu_3(x, \Theta_{i, i+1}, \Theta_{i+1, i+2}).$$

It is standard to check

$$(17) \quad f_{i+1} \circ f_i = \partial \circ H_i + H_i \circ \partial$$

using (15), (16) and the fact that

$$(18) \quad \mu_2(\Theta_{i,i+1}, \Theta_{i+1,i+2}) = 0.$$

We then define

$$G_i : CF^+(\Sigma, \alpha, \eta^i, w) \rightarrow CF^+(\Sigma, \alpha, \eta^{i+3}, w)$$

by

$$G_i(x) = \mu_4(x, \Theta_{i,i+1}, \Theta_{i+1,i+2}, \Theta_{i+2,i+3}).$$

Let

$$\sigma : CF^+(\Sigma, \alpha, \eta^i, w) \rightarrow CF^+(\Sigma, \alpha, \eta^{i+3}, w)$$

be the map defined by

$$\sigma(x) = \mu_2(x, \mu_3(\Theta_{i,i+1}, \Theta_{i+1,i+2}, \Theta_{i+2,i+3})).$$

By (16),  $\Theta_{i,i+1}, \Theta_{i+1,i+2}, \Theta_{i+2,i+3}$  are all cycles. It follows from (15) and (18) that  $\mu_3(\Theta_{i,i+1}, \Theta_{i+1,i+2}, \Theta_{i+2,i+3})$  is a cycle. So  $\sigma$  is a chain map. Using (15), (16) and (18), we get

$$(19) \quad \partial \circ G_i + G_i \circ \partial + f_{i+2} \circ H_i + H_{i+1} \circ f_i + \sigma = 0.$$

We claim that

$$(20) \quad \mu_3(\Theta_{i,i+1}, \Theta_{i+1,i+2}, \Theta_{i+2,i+3}) = \Theta_{i,i+3} + Uy,$$

where  $\Theta_{i,i+3}$  is the top generator of  $\widehat{CF}(\Sigma, \eta^i, \eta^{i+3}, w)$ , and  $y$  is some element in  $CF^+(\Sigma, \eta^i, \eta^{i+3}, w)$ .

Since  $\eta^{i+3}$  is a Hamiltonian translate of  $\eta^i$ ,  $\mu_2(\cdot, \Theta_{i,i+3})$  is a chain homotopy equivalence. The claim implies that

$$\sigma = \iota + U\rho,$$

where  $\iota$  is a chain homotopy equivalence, and  $U\rho$  is a chain map. (It is not clear whether  $\rho$  is a chain map.)

Let

$$\kappa : CF^+(\Sigma, \alpha, \eta^{i+3}, w) \rightarrow CF^+(\Sigma, \alpha, \eta^i, w)$$

be a chain map which is an inverse to  $\iota$  up to chain homotopy. Consider the chain map

$$\begin{aligned} \kappa' &= \kappa(\text{id} - U\rho\kappa + (U\rho\kappa)^2 - (U\rho\kappa)^3 + \cdots) \\ &= (\text{id} - \kappa U\rho + (\kappa U\rho)^2 - (\kappa U\rho)^3 + \cdots)\kappa. \end{aligned}$$

For any  $x \in CF^+(\Sigma, \alpha, \eta^{i+3}, w)$ ,  $U^N x = 0$  when  $N$  is sufficiently large. So  $\kappa'$  is well-defined, and we can compute

$$(\iota + U\rho)\kappa' \simeq \text{id}, \quad \kappa'(\iota + U\rho) \simeq \text{id}.$$

So  $\sigma = \iota + U\rho$  is a quasi-isomorphism. Now (19) implies that

$$f_{i+2} \circ H_i + H_{i+1} \circ f_i$$

is chain homotopic to a quasi-isomorphism, so it is a quasi-isomorphism as well. This finishes checking Condition (2) in Lemma A.1. Condition (1) in Lemma A.1 is just (17).

To finish the proof of Theorem 3.4, we only need to prove (20), which is essentially [25, Equation (11)]. In fact, as in [25], the proof reduces to the computation in a genus-1 surface.

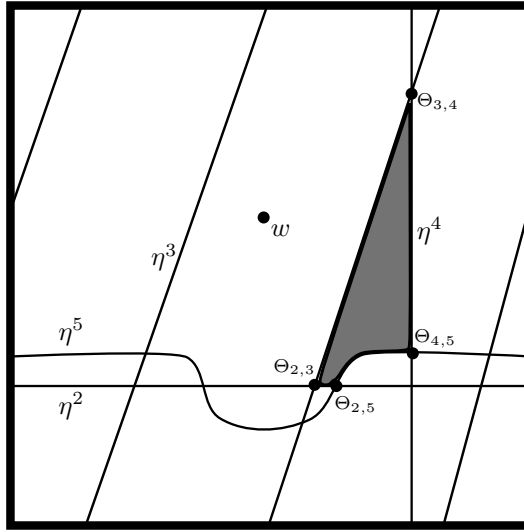


FIGURE 1. The domain of a holomorphic quadrilateral

There are two small triangles not containing  $w$  which contribute to

$$\mu_2(\Theta_{i,i+1}, \Theta_{i+1,i+2}).$$

(These two triangles cancel, which is part of the reason for (18).) The curve  $\eta^{i+3}$  cuts exactly one of these two triangles so as to get a positive quadrilateral with  $\Theta_{i,i+1}$  and  $\Theta_{i+1,i+2}$  being two of its vertices. The domain of this quadrilateral when  $i = 2$  is shown in Figure 1. All other quadrilaterals contributing to

$$\mu_3(\Theta_{i,i+1}, \Theta_{i+1,i+2}, \Theta_{i+2,i+3})$$

necessarily contain the base point  $w$ . So we have (20).

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