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COMPACT MODULI OF K3 SURFACES WITH A NONSYMPLECTIC AUTOMORPHISM

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Abstract. We construct a modular compactification via stable slc pairs for the moduli spaces of K3 surfaces with a nonsymplectic group of automorphisms under the assumption that some combination of the fixed loci of automorphisms defines an effective big divisor, and prove that it is semitoroidal.

Contents

1. Introduction

Let X be a smooth K3 surface over the complex numbers. An automorphism σ of X is called *nonsymplectic* if it has finite order $n > 1$ and $\sigma^*(\omega_X) = \zeta_n \omega_X$, where $\omega_X \in H^{2,0}(X)$ is a nonzero 2-form and ζ_n is a primitive nth root of identity. By changing the generator of the cyclic group $\mu_n = \langle \sigma \rangle$ we can and will assume that $\zeta_n = \exp(2\pi i/n)$. It is well known that a K3 surface admitting such an automorphism is projective. The possibilities for the order n are the numbers whose Euler function satisfies $\varphi(n) \le 20$, with the single exception $n \ne 60$, see [MO98, Thm. 3].

In this paper we study compactification of moduli spaces of pairs (X, σ). But to begin with, the automorphism group $\mathrm{Aut}(X, \sigma)$, i.e. those automorphisms of X commuting with σ, may be infinite. To fix this, we usually additionally assume:

$(\exists g \ge 2)$ The fixed locus $\mathrm{Fix}(\sigma)$ contains a curve C_1 of genus $g \ge 2$.

By looking at the μ_n-action on the tangent space of any fixed point, it is easy to see that $\mathrm{Fix}(\sigma)$ is a disjoint union of several smooth curves and points. The Hodge index theorem implies that at most one of the fixed curves has genus $g \ge 2$. Alternatively, σ could fix one or two curves of genus $g = 1$. All other fixed curves are isomorphic to $\mathbb{P}^1$.

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Under the ($\exists g \geq 2$) assumption, the group $\operatorname{Aut}(X, \sigma)$ is finite. The opposite is almost true. For example let $n = 2$, i.e. σ is an involution. Generically, σ^* fixes the Neron-Severi lattice $S_X \subset H^2(X, \mathbb{Z})$ and acts as multiplication by (-1) on the lattice $T_X = S_X^\perp$ of transcendental cycles. Then $\operatorname{Aut}(X, \sigma) = \operatorname{Aut}(X)$. Deformation classes of such K3 surfaces (X, σ) are classified by the primitive 2-elementary hyperbolic sublattices $S \subset L_{K3}$. By Nikulin [Nik79b] there are 75 cases, uniquely determined by certain invariants (g, k, δ). Among them 51 satisfy ($\exists g \geq 2$). The only case when $|\operatorname{Aut}(X)| < \infty$ but ($\exists g \geq 2$) fails is $(g, k, \delta) = (1, 9, 1)$, which is a one-dimensional family of K3 surfaces of Picard rank 19, mirror to degree 2 K3 surfaces. In the case $(g, k, \delta) = (2, 1, 0)$, one has $|\operatorname{Aut}(X)| = \infty$ but the set $\operatorname{Fix}(\sigma)$ consists of two elliptic curves, so ($\exists g \geq 2$) does not hold.

The moduli stack of smooth quasipolarized K3 surfaces is notoriously non-separated, as is the moduli stack of smooth K3s with a nonsymplectic automorphism. For a fixed isometry $\rho \in O(L_{K3})$ of order n, there exists the moduli stack and moduli space of smooth K3 surfaces "of type ρ": those pairs (X, σ) where the action of σ^* on $H^2(X, \mathbb{Z})$ can be modeled by ρ. We construct this moduli space in Section 2. The maximal separated quotient of F_ρ is $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$, where $\mathbb{D}_\rho$ is a symmetric Hermitian domain of type IV if $n = 2$ or a complex ball if $n > 2$, Γ_ρ is an arithmetic group, and $\Delta_\rho \subset \mathbb{D}_\rho$ is a union of Heegner divisors.

Assuming ($\exists g \geq 2$), the space $F_\rho^{\text{ade}} := (\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ is the coarse moduli space for the K3 surfaces $\overline{X}$ with ADE singularities, obtained from the smooth K3 surfaces X by contracting the (-2)-curves perpendicular to the component C_1 with $g \geq 2$ in $\operatorname{Fix}(\sigma)$. The stack of such ADE K3 surfaces is separated.

Our main goal is to construct a geometrically meaningful, Hodge-theoretic compactification of the moduli space F_ρ^{ade}, under the assumption ($\exists g \geq 2$). Let $R = C_1$, $\varphi_{|mR|} \colon X \to \overline{X}$ be the contraction as above, and $\overline{R}$ be the image of R. Then for any $0 < \epsilon \ll 1$ the pair $(\overline{X}, \epsilon\overline{R})$ is a stable pair with semi log canonical singularities. The theory of KSBA moduli spaces (see [Kol21] for the general case or [AET19, ABE20] for the much easier special case needed here) gives a moduli compactification

$$F_\rho^{\text{ade}} \hookrightarrow \overline{F}_\rho^{\text{slc}}$$

to a space of stable pairs with automorphism. Our main theorem states:

Theorem (Theorem 3.26). *Up to normalization, $\overline{F}_\rho^{\text{slc}}$ is a semitoroidal compactification of $\mathbb{D}_\rho/\Gamma_\rho$.*

Semitoroidal compactifications were introduced by Looijenga [Loo03b] as a common generalization of the Baily-Borel and toroidal compactifications of arithmetic quotients of Hermitian symmetric domains, associated to the groups $U(1, n)$ or $O(2, n)$. As a corollary, the family of ADE K3 surfaces with an automorphism extends along the inclusion $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho \hookrightarrow \mathbb{D}_\rho/\Gamma_\rho$.

The proof applies a modified form of one of the main theorems of [AE21] about "recognizable divisors." An ample divisor R on the generic K3 surface in F_ρ is called *recognizable* if it extends uniquely to a divisor R_0 on any Kulikov surface X_0—these are K-trivial, reduced normal crossings surfaces $X_0 = \cup V_i$ which admit a one-parameter smoothing $X_0 \hookrightarrow X$ into F_ρ with smooth total space X. We prove that the $g \geq 2$ component of the fixed locus on (X, σ) is recognizable. The proof hinges on the fact that R_0 lies in the union of the locus of indeterminacy and the fixed locus of a rigid nonsymplectic birational automorphism of X_0.

As we point out in Section 5, the results also extend to the more general situation of a symmetry group $G \subset \operatorname{Aut} X$ which is not purely symplectic.

The cases $n = 2, 3, 4, 6$ are of the most interest. If $n \neq 2, 3, 4, 6$ then the space $\mathbb{D}_\rho/\Gamma_\rho$ is already compact, see [Mat16] or Corollary 3.15.

K3 surfaces with an involution were classified by Nikulin in [Nik79b]. K3s with a nonsymplectic automorphism of prime order $p \geq 3$ we classified by Artebani, Sarti, and Taki in [AS08, AST11]. The case $n = 4$ was treated by Artebani-Sarti in [AS15] and the case $n = 6$ by Dillies in [Dil09, Dil12].

We note three cases where our KSBA, semitoroidal compactification $\overline{F}_\rho^{\mathrm{slc}}$ is computed in complete detail: Alexeev-Engel-Thompson [AET19] for the case of K3 surfaces of degree 2, generically double covers of $\mathbb{P}^2$, forthcoming work of Deopurkar-Han [DH22] which treats a 9-dimensional ball quotient for $n = 3$, and work of Moon-Schaffler [MS21], which studies a 5-dimensional example for $n = 4$.

The paper is organized as follows. In Section 2, we set up the moduli theory of K3 surfaces with a nonsymplectic automorphism. In Section 3, we define the stable pair compactifications and prove the main Theorem 3.26. In Section 4, we relate K3 surfaces with nonsymplectic automorphisms to their quotients $Y = \overline{X}/\mu_n$, and the KSBA compactification of F_ρ with the KSBA compactification of the moduli spaces of log del Pezzo pairs $(Y, \frac{n-1+\epsilon}{n} B)$. In Section 5 we extend the results in two ways: to K3 surfaces with a finite group of symmetries $G \subset \operatorname{Aut} X$ that is not purely symplectic, and to more general choices of polarizing divisor.

Throughout, we work over the field of complex numbers.

2. Moduli of K3s with a nonsymplectic automorphism

2A. **Notations.** A *lattice* L is a finitely generated, free abelian group with a nondegenerate $\mathbb{Z}$-valued symmetric bilinear form. It is *unimodular* if the bilinear form identifies $L^* = L$, and has a *signature* (m, n) if $L \otimes \mathbb{R} \cong \mathbb{R}^{m,n}$. Let $L = H^{\oplus 3} \oplus E_8^{\oplus 2}$ be a fixed copy of the unique even, unimodular lattice of signature $(3, 19)$, where $H = \mathrm{II}_{1,1}$ corresponds to the bilinear form $b(x, y) = xy$ and E_8 is the unique negative-definite even unimodular lattice of rank 8. For any smooth K3 surface X the cohomology lattice $H^2(X, \mathbb{Z})$ is isometric to L.

Denote by $S = S_X$ the Neron-Severi lattice $\operatorname{Pic}(X) = \operatorname{NS}(X)$. By the Lefschetz $(1,1)$-theorem, it equals $(H^{2,0}(X))^\perp \cap H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$. We have $H^{2,0}(X) = \mathbb{C}\omega_X$ for some nowhere-vanishing holomorphic two-form ω_X. If X is projective, then S_X is nondegenerate of signature $(1, r_X - 1)$. In this case, its orthogonal complement $T_X = (S_X)^\perp \subset H^2(X, \mathbb{Z})$ is the *transcendental lattice*, of signature $(2, 20 - r_X)$. The *Kähler cone* $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ is the set of classes of Kähler forms on X; it is an open convex cone.

Theorem 2.1 (Torelli Theorem for K3 surfaces, [PSS71])**.** *The isomorphisms $\sigma\colon X' \to X$ are in bijection with the isometries $\sigma^*\colon H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$ satisfying the conditions $\sigma^*(H^{2,0}(X)) = H^{2,0}(X')$ and $\sigma^*(\mathcal{K}_X) = \mathcal{K}_{X'}$.*

For any lattice H, a *root* is a vector $\delta \in H$ with $\delta^2 = -2$. The set of all roots is denoted by H_{-2}. The Weyl group $W(H)$ is the group generated by reflections $v \mapsto v + (v, \delta)\delta$ for $\delta \in H_{-2}$. It is a normal subgroup of the isometry group $O(H)$.

2B. **Moduli of marked unpolarized K3s.** The basic reference here is [ast85]. Let X be a K3 surface. A *marking* is an isometry $\phi\colon H^2(X,\mathbb{Z}) \to L$. Let

$$\mathbb{D} := \mathbb{P}\{x \in L_{\mathbb{C}} \mid x \cdot x = 0,\ x \cdot \bar{x} > 0\}, \quad \dim \mathbb{D} = 20.$$

There exists a fine moduli space $\mathcal{M}$ of *marked K3 surfaces* and a period map $\pi\colon \mathcal{M} \to \mathbb{D}$, $(X, \phi) \mapsto \phi(H^{2,0}(X)) \in \mathbb{D}$. In fact, $\mathcal{M}$ is a non-Hausdorff 20-dimensional complex manifold, with two isomorphic connected components interchanged by negating ϕ. The period map π is étale and surjective.

For a period point $x \in \mathbb{D}$, the vector space $(\mathbb{C}x \oplus \mathbb{C}\bar{x}) \cap L_{\mathbb{R}} \subset L_{\mathbb{C}}$ is positive definite of rank 2 and its orthogonal complement $x^{\perp} \cap L_{\mathbb{R}}$ has signature $(1, 19)$. Let

$$\{v \in x^{\perp} \cap L_{\mathbb{R}} \mid v^2 > 0\} = P_x \sqcup (-P_x)$$

be the two connected components of the set of positive square vectors. Then the fiber $\pi^{-1}(x)$ is identified with the set of connected components $\mathcal{C}$ of

$$\big(P_x \sqcup (-P_x)\big) \setminus \cup_{\delta} \delta^{\perp} \text{ for } \delta \in (x^{\perp} \cap L)_{-2}. \tag{1}$$

Namely, an open chamber $\mathcal{C}$ is identified with the Kähler cone $\mathcal{K}_X$ of the corresponding marked K3 surface X via the marking ϕ. The connected components are permuted by the reflections and $\pm\,\mathrm{id}$, and $\pi^{-1}(x)$ is a torsor under the group $\mathbb{Z}_2 \times W_x$, where $W_x = W(x^{\perp} \cap L)$. Since $x^{\perp} \cap L_{\mathbb{R}}$ is hyperbolic, the group and the fiber $\pi^{-1}(x)$ may be infinite. For a general point $x \in \mathbb{D}$, the lattice $x^{\perp} \cap L$ has no roots and the fiber $\pi^{-1}(x)$ consists of two points, one in each connected component of $\mathcal{M}$.

2C. **Moduli of ρ-marked and ρ-markable K3 surfaces with automorphisms.** Fix $\rho \in O(L)$ an isometry of order $n > 1$ and consider a K3 surface X with a non-symplectic automorphism σ of order n.

Definition 2.2. A *ρ-marking* of (X, σ) is an isometry $\phi : H^2(X,\mathbb{Z}) \to L$ such that $\sigma^* = \phi^{-1} \circ \rho \circ \phi$. We say that (X, σ) is *ρ-markable* if it admits a ρ-marking.

A family of ρ-marked surfaces is a smooth morphism $f\colon (\mathcal{X}, \sigma_B) \to B$ with an automorphism $\sigma_B\colon \mathcal{X} \to \mathcal{X}$ over B, together with an isomorphism of local systems $\phi_S\colon R^2 f_* \underline{\mathbb{Z}} \to L \otimes \underline{\mathbb{Z}}_B$ such that every fiber is a K3 surface with a ρ-marking. A family $f\colon (\mathcal{X}, \sigma_B) \to B$ is ρ-markable if such an isomorphism exists locally in complex-analytic topology on B.

We define the moduli stacks $\mathcal{M}_\rho$ of ρ-marked, resp. $\mathcal{F}_\rho$ of ρ-markable K3 surfaces by taking $\mathcal{M}_\rho(B)$, resp. $\mathcal{F}_\rho(B)$ to be the groupoids of such families over a base B.

Definition 2.3. Define $L_{\mathbb{C}}^{\zeta_n}$ to be the eigenspace of $x \in L_{\mathbb{C}}$ such that $\rho(x) = \zeta_n x$ and define the subdomain $\mathbb{D}_\rho := \mathbb{P}(L_{\mathbb{C}}^{\zeta_n}) \cap \mathbb{D} \subset \mathbb{D}$. Define $\Gamma_\rho \subset O(L)$ as the group of changes-of-marking: $\Gamma_\rho := \{\gamma \in O(L) \mid \gamma \circ \rho = \rho \circ \gamma\}$.

Definition 2.4. Let the *generic transcendental lattice* $T_\rho := L_{\mathbb{C}}^{\mathrm{prim}} \cap L$ be the intersection of L with the sum of all primitive eigenspaces of ρ, and let the *generic Picard lattice* be $S_\rho = (T_\rho)^{\perp}$. Let $L^G = \mathrm{Fix}(\rho) \subset S_\rho$ be classes in L fixed by ρ. (We write $G = \langle \rho \rangle$ to avoid the notation L^ρ.)

The ζ_n-eigenspaces $L_{\mathbb{C}}^{\zeta_n}$ and $T_{\rho,\mathbb{C}}^{\zeta_n}$ coincide, and for any K3 surface with a ρ-marking, the two fixed sublattices $\phi\colon (S_X)^G = H^2(X,\mathbb{Z})^G \xrightarrow{\sim} L^G$ are identified.

For there to exist a ρ-markable algebraic K3 surface, the signature of T_ρ must be $(2,\ell)$ for some ℓ, as there is necessarily a vector of positive norm fixed by σ^* (the sum of a σ^*-orbit of an ample class). The converse is also true.

When $n=2$, we have that $\mathbb{D}_\rho \subset \mathbb{P}(T_{\rho,\mathbb{C}})$ is (two copies of) the type IV domain associated to the lattice T_ρ. When $n \geq 3$, the condition that $x \cdot x = 0$ is vacuous on $\mathbb{D}_\rho$ because $x \cdot y = 0$ for eigenvectors x, y of ρ with non-conjugate eigenvalue. Thus,

$$\mathbb{D}_\rho = \mathbb{P}\{x \in T_{\rho,\mathbb{C}}^{\zeta_n} \mid x \cdot \bar{x} > 0\}$$

is a complex ball, a type I domain. The Hermitian form $x \cdot \bar{y}$ on $T_{\rho,\mathbb{C}}^{\zeta_n}$ necessarily has signature $(1,\ell)$ for some ℓ for there to exist a ρ-markable K3 surface.

Definition 2.5. The *discriminant locus* is $\Delta_\rho := (\cup_\delta \delta^\perp) \cap \mathbb{D}_\rho$ ranging over all roots δ in $(L^G)^\perp$.

Remark 2.6. Sections 10 and 11 of [DK07] contain a construction of the moduli space of K3 surfaces with a nonsymplectic automorphism, based on the moduli of lattice-polarized K3s. We give an alternative construction for two reasons:

1. [DK07] relies on [Dol96, Thm. 3.1], which has an inaccuracy, see [AE21].
2. lattice-polarized K3 surfaces include the data of an isometry $\mathrm{Fix}(\sigma^*) \to L^G$.

Because of (2), the coarse space in [DK07] is a finite-to-one, rather than one-to-one, parameterization of pairs (X,σ). In practice, these differences are quite minor, and the proofs of Lemma 2.7 and Theorem 2.10 below closely follow the arguments of Dolgachev-Kondo [DK07, Thms. 11.2, 11.3].

Lemma 2.7. *Let $\rho \in O(L)$ be an isometry of order $n > 1$. Then*

1. *A marking $\phi\colon H^2(X,\mathbb{Z}) \to L$ defines a ρ-marking, i.e. defines an automorphism σ with $\sigma^* = \phi^{-1} \circ \rho \circ \phi$ iff the period $x = \pi((X,\phi))$ lies in $\mathbb{D}_\rho \setminus \Delta_\rho$ and there exists an ample line bundle $\mathcal{L}_h$ on X with $h = \phi(\mathcal{L}_h) \in L^G$.*
2. *For a point $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ the set of ρ-marked K3s with this period is a torsor over the group $\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x)$.*

Proof. We have $\rho(x) = \zeta_n x \neq x$. For any $h \in L^G$ one has $\rho(h) = h$, which implies that $h \cdot x = 0$. Thus, $L^G \perp x$ and $(S_X)^G \simeq L^G$.

One must necessarily have $x \in \mathbb{D}_\rho$ for $a := \phi^{-1} \circ \rho \circ \phi$ to be a Hodge-isometry acting on $H^{2,0}(X)$ by multiplication by ζ_n. Then by the Torelli theorem, a is induced by an automorphism of X iff $a(\mathcal{K}_X) = \mathcal{K}_X$. By averaging, a preserving the Kähler cone is equivalent to having an a-invariant Kähler class $\mathcal{L}_h \in \mathcal{K}_X \cap H^2(X,\mathbb{Z})$. Since $L^G \perp x$, one has $\mathcal{L}_h \perp \omega_X$ and so $\mathcal{L}_h \in S_X$ defines an ample line bundle.

If $x \perp \delta$ for some root $\delta \in (L^G)^\perp$ then $\mathcal{L}_\delta = \phi^{-1}(\delta) \in \mathrm{Pic}(X)$ and either $\mathcal{L}_\delta$ or $\mathcal{L}_\delta^{-1}$ is effective. For the line bundle $\mathcal{L}_h$ as above, one has both $\mathcal{L}_h \cdot \mathcal{L}_\delta = 0$ because $h \perp \delta$ and $\mathcal{L}_h \cdot \mathcal{L}_\delta \neq 0$ because $\mathcal{L}_h$ is ample. Contradiction.

On the other hand, let $x \in \mathbb{D}_\rho \setminus \Delta_\rho$. Then $L^G \not\subset \cup_\delta \delta^\perp$ for $\delta \in (x^\perp \cap L)_{-2}$. Thus, there exists a chamber $\mathcal{C}$ in $P_x \setminus \cup_\delta \delta^\perp$ such that $\mathcal{C} \cap L^G \neq \emptyset$. Let (X,ϕ) be the K3 surface corresponding to this chamber. Then there exists $h \in \mathcal{C} \cap L^G$ and by the second paragraph, the marking ϕ is a ρ-marking. This proves (1).

Any surface with the same period x is isomorphic to X, but with a marking $\phi' = g \circ \phi$ for some $g \in \mathbb{Z}_2 \times W_x$. Then one has both $\sigma^* = \phi^{-1} \circ \rho \circ \phi$ and $\sigma^* = (\phi')^{-1} \circ \rho \circ \phi'$ iff $g \in \Gamma_\rho$. This proves (2). ☐

Lemma 2.8. *There exists a fine moduli space $\mathcal{M}_\rho$ of ρ-marked K3 surfaces with a nonsymplectic automorphism. $\mathcal{M}_\rho$ is an open subset of $\pi^{-1}(\mathbb{D}_\rho \setminus \Delta_\rho)$.*

Proof. The points of $\mathcal{M}$ over $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ are chambers $\mathcal{C}$ as in Equation (1). By Lemma 2.7, one has $\mathcal{C} \in \mathcal{M}_\rho$ iff $\mathcal{C} \cap L^G \neq \emptyset$. This is an open condition. □

The restriction of $\pi\colon \mathcal{M} \to \mathbb{D}$ gives the period map $\pi_\rho\colon \mathcal{M}_\rho \to \mathbb{D}_\rho \setminus \Delta_\rho$. The general fiber of π_ρ is a torsor over $\Gamma_\rho \cap (\mathbb{Z}_2 \times W(S_\rho))$. Thus, $\mathcal{M}_\rho$ is not separated iff there exists $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ such that $\Gamma_\rho \cap W_x \supsetneq \Gamma_\rho \cap W(S_\rho)$. This indeed happens:

Example 2.9. Consider the 9-dimensional family of μ_3-covers of $\mathbb{P}^1 \times \mathbb{P}^1$ branched in a curve B of bidegree $(3,3)$, studied by Kondō [Kon02]. In this case,

$$S_\rho = L^G = (\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1))(3) = H(3) \quad \text{and} \quad T_\rho = (L^G)^\perp = H \oplus H(3) \oplus E_8^2.$$

Let $\overline{Y}$ be a degeneration of the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ to a quadratic cone and $\overline{X} \to \overline{Y}$ be the μ_3-cover branched in a curve $\overline{B} \in |\mathcal{O}_{\overline{Y}}(3)|$ not passing through the apex. Let $Y = \mathbb{F}_2$ and X be the minimal resolutions of $\overline{Y}$ and $\overline{X}$. The $\mathbb{P}^1$-fibration on Y gives an elliptic fibration on X, and the preimage of the (-2)-section of Y is a union of three disjoint (-2)-sections e, σe, $\sigma^2 e$ on X, cyclically permuted by the automorphism σ. The invariant sublattice $S_X^\sigma = (\mathrm{Pic}(\mathbb{F}_2))(3) = H(3)$ is generated by f and $f' = f + \sum_{i=0}^2 \sigma^i e$.

The only (-2)-curves on X are $\sigma^i e$ and they do not lie in $S_\rho^\perp$. Thus, once we fix a marking ϕ, the period x of X will be in $\mathbb{D}_\rho \setminus \Delta_\rho$. The reflections w_i in the roots $\rho^i\phi(e)$ commute. Their product $w = w_0 w_1 w_2$ is non-trivial: on L^G it acts as the reflection that interchanges $\phi(f)$ and $\phi(f')$. It is easy to check that $w \in \Gamma_\rho$. So $\Gamma_\rho \cap W_x \neq 1$ and $W(L^G) = 1$.

Thus, the map $\mathcal{M}_\rho \to \mathbb{D}_\rho \setminus \Delta_\rho$ is not separated in this case. Locally it looks like the "line with doubled origin" $\mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1 \to \mathbb{A}^1$ times $\mathbb{A}^8$. Here is another way to see the same. The positive cone P in $H(3)_\mathbb{R}$ is the unique Weyl chamber for the Weyl group $W(H(3)) = 1$; its rays are $\phi(f)$ and $\phi(f')$. The hyperplane $\phi(e)^\perp$ cuts it in half. The intersections of the Weyl chambers $\mathcal{C} \subset P_x \setminus \cup\delta^\perp$ of Equation 1 with P are either halves of P.

Theorem 2.10. *The moduli stack $\mathcal{F}_\rho$ of ρ-markable K3 surfaces with nonsymplectic automorphism has coarse moduli space $F_\rho = \mathcal{M}_\rho/\Gamma_\rho$. There is a bijective period map $F_\rho \to (\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ and the separated quotient F_ρ^{sep} of the coarse space is $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$. The generic inertia of $\mathcal{F}_\rho$ is the group*

$$K_\rho := \ker(\Gamma_\rho \to \mathrm{Aut}(\mathbb{D}_\rho))/\Gamma_\rho \cap (\mathbb{Z}_2 \times W(S_\rho)).$$

Proof. The statement is immediate from the definitions and the above two lemmas, by quotienting the period map π_ρ. The points of $\pi_\rho^{-1}(x)$ are permuted by Γ_ρ and thus they are identified in the Γ_ρ-quotient. The bijectivity of the quotiented period map follows.

For ρ to correspond to a K3 surface with a nonsymplectic automorphism, S_ρ must have signature $(1, r-1)$ for some r, and T_ρ must have signature $(2, 20-r)$. The action of Γ_ρ on the type IV domain $\mathbb{D}(T_\rho)$ factors through $O(T_\rho)$ and is therefore properly discontinuous. Thus, the effective action of Γ_ρ on $\mathbb{D}_\rho$ is properly discontinuous, and so $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ is makes sense as a complex-analytic space. (It is also quasiprojective by Baily-Borel.)

The last statement follows from Lemma 2.7(2) by noting that for a generic $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ one has $x^\perp \cap L = S_\rho$. □

Remark 2.11. Even though the map to $(\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ in Theorem 2.10 is bijective, the coarse moduli space of F_ρ is a non-separated algebraic space when $\mathcal{M}_\rho$

is not separated. This is very similar to the algebraic space obtained by dividing a line with doubled origin $\mathbb{A}^1 \cup_{\mathbb{A}^1 \setminus 0} \mathbb{A}^1$ by the involution $z \to -z$ exchanging the two origins. The quotient is a non-separated algebraic space admitting a bijective morphism to $\mathbb{A}^1 = \mathbb{A}^1/\pm$.

The separated quotient F_ρ^{sep} is a stack $[\mathbb{D}_\rho \setminus \Delta_\rho :_W \Gamma_\rho]$ which can be locally constructed near $x \in \mathbb{D}_\rho \setminus \Delta_\rho$ by first taking a coarse quotient by the normal subgroup $\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x) \trianglelefteq \mathrm{Stab}_x(\Gamma_\rho)$ and then taking the stack quotient by $\mathrm{Stab}_x(\Gamma_\rho)/\Gamma_\rho \cap (\mathbb{Z}_2 \times W_x)$. See [AE21, Rem. 2.36].

Proposition 2.12. *Suppose $\sigma \in \mathrm{Aut}(X)$ fixes a curve R of genus at least 2, i.e. the assumption $(\exists g \geq 2)$ holds. Then $\mathrm{Aut}(X, \sigma)$ is finite.*

Proof. Let $h \in \mathrm{Aut}(X, \sigma)$ be an automorphism of X satisfying $h \circ \sigma = \sigma \circ h$. Then h permutes the fixed components of σ. Since there is at most one component R of genus $g \geq 2$, we conclude $h(R) = R$. Hence $h \in \mathrm{Aut}(X, \mathcal{O}_X(R))$, a finite group. □

Note that generic stabilizer K_ρ from Theorem 2.10 is never the trivial group, as $\rho \in K_\rho$ is a nontrivial element. As this is the automorphism group of a generic element $(X, \sigma) \in F_\rho$, if $(\exists g \geq 2)$ holds then K_ρ is finite by Proposition 2.12.

Example 2.13. Consider the double cover $\pi\colon X \to \mathbb{P}^2$ branched over a smooth sextic B. There is a nonsymplectic involution σ switching the two sheets of X, acting on $H^2(X, \mathbb{Z})$ by fixing $h = c_1(\pi^*\mathcal{O}(1))$ and negating $h^\perp$. Choosing a model ρ for the action of σ^* on cohomology, we have $S_\rho = \langle 2 \rangle$ and $T_\rho = \langle -2 \rangle \oplus H^{\oplus 2} \oplus E_8^{\oplus 2}$ are the $(+1)$- and (-1)-eigenspaces, respectively.

The divisor $\Delta_\rho/\Gamma_\rho \subset \mathbb{D}_\rho/\Gamma_\rho = F_2$ has two irreducible components corresponding to Γ_ρ-orbits of roots $\delta \in (T_\rho)_{-2}$. Such an orbit is uniquely determined by the divisibility (1 or 2) of $\delta \in T_\rho^*$. The case where the divisibility is 2 corresponds to when B acquires a node. Then there is an involution σ on the minimal resolution of the double cover $X \to \overline{X} \to \mathbb{P}^2$, but $\sigma^*(\delta) = \delta$, $\sigma^*(h) = h$ and the $(+1, -1)$-eigenspaces of σ^* have dimensions $(2, 20)$. Thus, no ρ-marking can be extended over a family $\mathcal{X} \to C$ with central fiber X and general fiber as above.

When the divisibility of δ is 1, $\mathbb{P}^2$ degenerates to $\mathbb{F}_4^0 = \mathbb{P}(1,1,4)$ and the minimal resolution of the double cover $X \to \overline{X} \to \mathbb{F}_4^0$ is an elliptic K3 surface with σ the elliptic involution. Again the eigenspaces have dimension profile $(2, 20)$ and so (X, σ) is not ρ-markable for the ρ as above.

3. Stable pair compactifications

3A. **Complete moduli of stable slc pairs.** We refer the reader to [ABE20, Sec. 2B] and [AE21, Sec. 7D] for a detailed discussion of stable K3 surface pairs and their compactified moduli. A pair (X, Δ) consisting of a projective variety X and a $\mathbb{Q}$-Weil divisor Δ is *stable* if:

(1) the pair (X, Δ) has semi log canonical singularities, and
(2) the divisor $\omega_X + \Delta$ is $\mathbb{Q}$-Cartier and ample.

In our context, we will have $X = S$ a Gorenstein surface with $\omega_S \simeq \mathcal{O}_S$ and we will take $\Delta = \epsilon D$ for a small rational number ϵ, with D an ample Cartier divisor. Thus (2) holds, and for ϵ small enough, condition (1) will reduce to the statement that S itself has semi log canonical singularities with D containing no log canonical centers. In fact, for a fixed D^2 there exists ϵ_0 so that if a pair $(S, \epsilon D)$ is stable in the above definition for some ϵ then it is stable for any $0 < \epsilon \leq \epsilon_0$.

Definition 3.1. A *stable (Calabi-Yau) surface pair* is a pair $(S, \epsilon D)$, where

(1) S is a connected, reduced, projective Gorenstein surface S with $\omega_S \simeq \mathcal{O}_S$ which has semi log canonical singularities.
(2) D is an effective ample Cartier divisor on S that does not contain any log canonical centers of S.

The application to K3 surfaces is the observation that for any K3 surface $\overline{X}$ with ADE singularities and an effective ample divisor $\overline{R}$, the pair $(\overline{X}, \epsilon\overline{R})$ is stable. Indeed, $\omega_{\overline{X}} \simeq \mathcal{O}_{\overline{X}}$ and the surface $\overline{X}$ has canonical singularities—which is much better than semi log canonical—and there are no log centers.

As usual, let F_{2d} denote the moduli space of polarized K3 surfaces $(\overline{X}, \overline{L})$ with ADE singularities and ample primitive line bundle $\overline{L}$ of degree $\overline{L}\cdot\overline{L} = 2d$, and let $P_{2d,m} \to F_{2d}$ denote the moduli space of pairs $(\overline{X}, \epsilon\overline{R})$ with an effective divisor $\overline{R} \in |m\overline{L}|$. Then the main result for K3 surfaces is the following:

Theorem 3.2. *Stable Calabi-Yau surface pairs with bounded D^2 and fixed $\epsilon < \epsilon_0$ form an algebraic Deligne-Mumford moduli stack $\mathcal{M}^{\mathrm{slc}}$, whose coarse moduli space M^{slc} is proper.*

The closure $\overline{P}^{\mathrm{slc}}_{2d,m}$ of $P_{2d,m}$ in M^{slc} is projective and provides a compactification of $P_{2d,m}$ to a moduli space of stable surface pairs.

Proof. See [ABE20, Sec. 2B]. □

3B. **Stable pair compactification of F^{sep}_ρ.** To apply Theorem 3.2 and construct a stable pair compactification in the present context, we must choose an ample divisor on any K3 surface $(X, \sigma) \in F_\rho$.

Definition 3.3. A *canonical choice of polarizing divisor* for F_ρ is a relatively big and nef divisor R on the universal ρ-markable K3 surface.

Suppose that for each surface $(X, \sigma) \in F_\rho$ assumption $(\exists g \geq 2)$ holds, i.e. the fixed locus $\mathrm{Fix}(\sigma)$ contains a component C_1 of genus $g \geq 2$, as well as possibly several smooth rational curves C_i and some isolated points. In fact, it suffices that a single $(X, \sigma) \in F_\rho$ satisfies assumption $(\exists g \geq 2)$ because the genus of C_1 is constant in a family of smooth K3 surfaces with nonsymplectic automorphism. Then $R = C_1$ gives a canonical choice of polarizing divisor for F_ρ.

Let $\pi\colon X \to \overline{X}$ be the contraction to an ADE K3 surface so that the divisor $\overline{R} := \pi(C_1)$ is ample; it has degree $\overline{R}^2 = 2g(C_1) - 2 > 0$. If $\overline{R} \in |m\overline{L}|$ for a primitive $\overline{L}$ then $(\overline{X}, \overline{L}) \in F_{2d}$ and the pair $(\overline{X}, \epsilon\overline{R}) \in P_{2d,m}$.

Definition 3.4. Define a map $\psi\colon F_\rho \to P_{2d,m}$ as follows. Pointwise, it sends (X, σ) to $(\overline{X}, \epsilon\overline{R})$. In every flat family $f\colon \mathcal{X} \to B$ of K3 surfaces with automorphism, the sheaf $\mathcal{O}_{\mathcal{X}}(\mathcal{R})$ is relatively big and nef. Since $R^i\mathcal{L}^d = 0$ for $i > 0$, $d > 0$, it gives a contraction to a flat family $\overline{f}\colon (\overline{\mathcal{X}}, \overline{\mathcal{R}}) \to B$. This induces the map on moduli.

Lemma 3.5. *The map $\psi\colon F_\rho \to P_{2d,m}$ defined above induces an injective map $F^{\mathrm{sep}}_\rho \to \mathrm{im}(\psi)$.*

Proof. The map ψ factors through the separated quotient of F_ρ because $P_{2d,m}$ is separated. Now suppose there is an isomorphism of pairs $\overline{f}\colon (\overline{X}_1, \epsilon\overline{R}_1) \to (\overline{X}_2, \epsilon\overline{R}_2)$ inducing an isomorphism of the minimal resolutions $f\colon (X_1, R_1) \to (X_2, R_2)$. Consider the morphism $\varphi = \sigma_1^{-1} f^{-1} \sigma_2 f$. Then φ is a *symplectic* automorphism of X_1 fixing the curve R_1 pointwise. Since φ preserves $\mathcal{O}_{X_1}(R_1)$, it has finite order. By

[Nik79a] the fixed set of a nontrivial finite order symplectic K3 automorphism is finite. Thus, $\varphi = \mathrm{id}$ and so f automatically preserves the group action. So, (X, σ) is uniquely recovered by $(\overline{X}, \overline{R})$. $\square$

Remark 3.6. F_ρ^{sep} has a moduli interpretation as the space F_ρ^{ade} of ADE K3 surfaces $(\overline{X}, \overline{\sigma})$ with an automorphism, such that $\mathrm{Fix}(\overline{\sigma})$ is ample and the minimal resolution $(X, \sigma) \to (\overline{X}, \overline{\sigma})$ is ρ-markable.

Definition 3.7. Let $Z = \mathrm{im}(\psi)$ and let $\overline{Z}$ be its closure in $\overline{P}^{\mathrm{slc}}_{2d,m}$, with reduced scheme structure. The stable pair compactification

$$F_\rho^{\mathrm{sep}} = F_\rho^{\mathrm{ade}} \hookrightarrow \overline{F}_\rho^{\mathrm{slc}}$$

is defined as the normalization of $\overline{Z}$.

In particular, $\overline{F}_\rho^{\mathrm{slc}}$ is normal by definition. Points correspond to the pairs $(\overline{X}, \epsilon\overline{R})$, possibly degenerate, with some finite data.

3C. **Kulikov degenerations of K3 surfaces.** A basic tool in the study of degenerations of K3 surfaces is Kulikov models. We use them in the argument below, so we briefly recall the definition.

Let $(C, 0)$ denote the germ of a smooth curve at a point $0 \in C$ and let $C^* = C \setminus 0$. Let $X^* \to C^*$ be an algebraic family of K3 surfaces.

Definition 3.8. A *Kulikov model* $X \to (C, 0)$ is an extension of $X^* \to C^*$ for which X is a smooth algebraic space, $K_X \sim_C 0$, and X_0 has reduced normal crossings. We say the X is *Type I, II, or III*, respectively, depending on whether X_0 is smooth, has double curves but no triple points, or has triple points, respectively. We call the central fiber X_0 of such a family a *Kulikov surface*.

Notation 3.9. We capitalize "Type" I, II, III for Kulikov models and use lowercase "type" I, IV for Hermitian symmetric domains.

A key result on the degenerations of K3 surfaces is the theorem of Kulikov [Kul77] and Persson-Pinkham [PP81]:

Theorem 3.10. *Let $Y^* \to C^*$ be a family of algebraic K3 surfaces. Then there is a finite base change $(C', 0) \to (C, 0)$ and a sequence of birational modifications of the pull back $Y' \dashrightarrow X$ such that X has smooth total space, $K_X \sim_{C'} 0$, and X_0 has reduced normal crossings.*

We recall some fundamental results about Kulikov models. The primary reference is [FS86]. Let $T : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$ denote the Picard-Lefschetz transformation associated to an oriented simple loop in C^* enclosing 0. Since X_0 is reduced normal crossings, T is unipotent. Let

$$N := \log T = (T - I) - \tfrac{1}{2}(T - I)^2 + \cdots$$

be the logarithm of the monodromy.

Theorem 3.11 ([FS86][Fri84])**.** *Let $X \to (C, 0)$ be a Kulikov model. We have that*

if X is Type I, then $N = 0$,
if X is Type II, then $N^2 = 0$ but $N \neq 0$,
if X is Type III, then $N^3 = 0$ but $N^2 \neq 0$.

The logarithm of monodromy is integral, and of the form $Nx = (x \cdot \lambda)\delta - (x \cdot \delta)\lambda$ *for* $\delta \in H^2(X_t, \mathbb{Z})$ *a primitive isotropic vector, and* $\lambda \in \delta^\perp/\delta$ *satisfying*

$$\lambda^2 = \#\{\textit{triple points of } X_0\}.$$

When $\lambda^2 = 0$*, its imprimitivity is the number of double curves of* X_0*.*

Thus, the Types I, II, III of Kulikov model are distinguished by the behavior of the *monodromy invariant* λ: either $\lambda = 0$, $\lambda^2 = 0$ but $\lambda \neq 0$, or $\lambda^2 \neq 0$ respectively.

Definition 3.12. Let $J \subset H^2(X_t, \mathbb{Z})$ denote the primitive isotropic lattice $\mathbb{Z}\delta$ in Type III or the saturation of $\mathbb{Z}\delta \oplus \mathbb{Z}\lambda$ in Type II.

3D. **Baily-Borel compactification.** Let N be a lattice of signature $(2, \ell)$, together with an isometry $\rho \in O(N)$ of finite order n, such that all eigenvalues of ρ on $N_\mathbb{C}$ are primitive nth roots of unity, and $N_\mathbb{C}^{\zeta_n}$ contains a vector x of positive Hermitian norm $x \cdot \bar{x}$. This is the situation which arises for a nonsymplectic automorphism of an algebraic K3 surface, with $N = T_\rho$. We have a type IV domain

$$\mathbb{D}_N = \mathbb{P}\{x \in N_\mathbb{C} \mid x \cdot x = 0,\ x \cdot \bar{x} > 0\}.$$

For $n = 2$ one has $\mathbb{D}_\rho = \mathbb{D}_N$. For $n > 2$ one has a type I subdomain of $\mathbb{D}_N$

$$\mathbb{D}_\rho = \mathbb{P}\{x \in N_\mathbb{C}^{\zeta_n} \mid x \cdot \bar{x} > 0\}.$$

$\mathbb{D}_\rho$ admits the action of the arithmetic group $\widetilde{\Gamma}_\rho := \{\gamma \in O(N) \mid \gamma \circ \rho = \rho \circ \gamma\}$. Fix a finite index subgroup $\Gamma \subset \widetilde{\Gamma}_\rho$.

Recall that $\mathbb{D}_N$ and $\mathbb{D}_\rho$ embed into their compact duals $\mathbb{D}_N^c$, $\mathbb{D}_\rho^c$, which are defined by dropping the condition that $x \cdot \bar{x} > 0$. Define $\overline{\mathbb{D}}_N \subset \mathbb{D}_N^c$, $\overline{\mathbb{D}}_\rho \subset \mathbb{D}_\rho^c$ as their topological closures. One has a well known description of the rational boundary components of $\mathbb{D}_N$, see e.g. [Loo03b].

Definition 3.13. A *rational boundary component* of $\mathbb{D}_N$ is an analytic subset $B_J \subset \overline{\mathbb{D}}_N$ of the form:

(1) $\mathbb{P}J_\mathbb{C} \setminus \mathbb{P}J_\mathbb{R} \subset \overline{\mathbb{D}}_N$ for $\operatorname{rk} J = 2$ a primitive isotropic sublattice of N,
(2) $\mathbb{P}J_\mathbb{C} \in \overline{\mathbb{D}}_N$ for $\operatorname{rk} J = 1$ a primitive isotropic sublattice of N.

The rational boundary components of $\mathbb{D}_\rho$ are intersections of $B'_J = B_J \cap \overline{\mathbb{D}}_\rho$.

One defines the *rational closure* of $\mathbb{D}_\rho$ to be $\mathbb{D}_\rho^{\mathrm{bb}} := \mathbb{D}_\rho \cup_J B'_J$ with a horoball topology at the boundary. Then the *Baily-Borel compactification* of $\mathbb{D}_\rho/\Gamma$ is (at least topologically) $\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{bb}} := \mathbb{D}_\rho^{\mathrm{bb}}/\Gamma$.

The space $\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{bb}}$ was shown to have the structure of a projective variety by Baily-Borel [BB66]. For type IV domains $\mathbb{D}_N = \mathbb{D}_\rho$ when $n = 2$, the boundary components (1) are isomorphic to $\mathbb{H} \sqcup (-\mathbb{H})$ and the boundary components (2) are points. For $n > 2$, the boundary components of the type I domain $\mathbb{D}_\rho$ are points. If $\operatorname{rk} J = 2$ then a point $[x] \in B_J$ corresponds to the elliptic curve $E_x = J_\mathbb{C}/(J + \mathbb{C}x)$.

Lemma 3.14. *If* $n > 2$*, for each boundary component* B'_J *we necessarily have* $\operatorname{rk} J = 2$ *and* $n \in \{3, 4, 6\}$*, and* $x \in B'_J$ *corresponds to the elliptic curve with* $j(E_x) = 0$ *if* $n = 3$ *or* 6*, and with* $j(E_x) = 1728$ *if* $n = 4$*.*

Proof. If B'_J is boundary component of $\mathbb{D}_\rho$ then $N_\mathbb{C}^{\zeta_n} \cap J_\mathbb{C} \neq 0$. Since J is defined over $\mathbb{Z}$ and $\zeta_n \notin \mathbb{R}$, then $N_\mathbb{C}^{\bar{\zeta}_n} \cap J_\mathbb{C} \neq 0$ as well. This implies that $\operatorname{rk} J = 2$ and

$$J_\mathbb{C} = J_\mathbb{C}^{\zeta_n} \oplus J_\mathbb{C}^{\bar{\zeta}_n}.$$

Thus, $\rho(J_{\mathbb{C}}) = J_{\mathbb{C}}$, implying that $\rho(J) = J$. Additionally, $\rho\big|_J \in \mathrm{GL}(J) \cong \mathrm{GL}_2(\mathbb{Z})$ necessarily has order n. Thus, $n \in \{3, 4, 6\}$. For a point $[x] \in B'_J$ one has $x \in N_{\mathbb{C}}^{\zeta_n}$ and so $\mu_n \subset \mathrm{Aut}(E_x)$. This determines $j(E_x)$. □

Corollary 3.15. *If $n \neq 2, 3, 4, 6$ then the rational closure of $\mathbb{D}_\rho$ is simply $\mathbb{D}_\rho$ itself. So $\mathbb{D}_\rho/\Gamma$ is already compact.*

The following is a well-known consequence of Schmid's nilpotent orbit theorem.

Proposition 3.16. *Let $X^* \to C^*$ be a degeneration of a ρ-markable K3 surfaces over a punctured analytic disk C^*. A lift of the period mapping $\widetilde{C^*} \cong \mathbb{H} \to \mathbb{D}_\rho$ approaches the Baily-Borel cusp B_J as $\mathrm{Im}(\tau) \to \infty$, where J is the monodromy lattice in $H^2(X_t, \mathbb{Z})$, cf. Definition 3.12. When $\mathrm{rk}(J) = 2$, the limiting point $x \in B_J$ corresponds to an elliptic curve E_x isomorphic to any double curve of the central fiber X_0 of a Kulikov model $X \to C$.*

Corollary 3.17. *If $n \neq 2, 3, 4, 6$, any degeneration of $(X, \sigma) \in F_\rho$ has Type I. If $n \in \{3, 4, 6\}$, any degeneration of $(X, \sigma) \in F_\rho$ has Type I or II.*

The last statement was also proved by Matsumoto [Mat16] using different techniques. His proof also holds in some prime characteristics.

3E. **Semitoroidal compactifications.** Semitoroidal compactifications of arithmetic quotients $\mathbb{D}/\Gamma$ for type IV Hermitian symmetric domains $\mathbb{D}$ were defined by Looijenga [Loo03b] (where they were called "semitoric"). They simultaneously generalize toroidal and Baily-Borel compactifications of $\mathbb{D}/\Gamma$. The case of the complex ball $\mathbb{D}$ (a type I symmetric Hermitian domain) is comparatively trivial. The semitoroidal compactifications in this case are implicit in [Loo03a, Loo03b]. We quickly overview the construction in both cases now.

Definition 3.18. A Γ-*admissible semifan* $\mathfrak{F}$ consists of the following data:

When $n = 2$, it is a convex, rational, locally polyhedral decomposition $\mathfrak{F}_J$ of the rational closure $\mathcal{C}^+(J^\perp/J)$ of the positive norm vectors, for all rank 1 primitive isotropic sublattices $J \subset N$, such that:

(1) $\{\mathfrak{F}_J\}_{J\subset N}$ is Γ-invariant. In particular, a fixed $\mathfrak{F}_J$ is invariant under the natural action of $\mathrm{Stab}_J(\Gamma)$ on $\mathcal{C}^+(J^\perp/J)$.
(2) A compatibility condition of the $\{\mathfrak{F}_J\}_{J\subset N}$ along any primitive isotropic lattice $J' \subset N$ of rank 2 holds, see Definition 3.19.

When $n > 2$, the data is much simpler: It consists, for each primitive isotropic sublattice $J \subset N$ satisfying $J_{\mathbb{C}} \cap N_{\mathbb{C}}^{\zeta_n} \neq \emptyset$, of a primitive sublattice $\mathfrak{F}_J \subset J^\perp/J$ such that the collection $\{\mathfrak{F}_J\}$ is Γ-invariant.

Definition 3.19. Let $J' \subset N$ be primitive isotropic of rank 2. We say that the collection $\{\mathfrak{F}_J\}_{J\subset N}$ is *compatible along* J' if, given any primitive sublattice $J \subset J'$ of rank 1, the kernel of the hyperplanes of $\mathfrak{F}_J$ containing J'/J, when intersected with $(J')^\perp/J \subset J^\perp/J$ and then descended to $(J')^\perp/J'$, cut out a fixed sublattice $\mathfrak{F}_{J'} \subset (J')^\perp/J'$ which is independent of J.

In both the $n = 2$ and $n > 2$ cases, we use the same notation $\mathfrak{F} := \{\mathfrak{F}_J\}_{J\subset N}$ even though J ranges over rank 1 isotropic sublattices when $n = 2$ and ranges over rank 2 isotropic sublattices when $n > 2$.

In the type IV case, Looijenga constructs a compactification $\mathbb{D}/\Gamma \hookrightarrow \overline{\mathbb{D}/\Gamma}^{\mathfrak{F}}$ for any Γ-admissible semifan $\mathfrak{F}$, so consider the type I case. By Lemma 3.14 we may restrict to $n \in \{3, 4, 6\}$. There is a $\mathbb{Z}[\zeta_n]$-lattice

$$Q := (N \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_n])^{\zeta_n} \subset N_{\mathbb{C}}^{\zeta_n} = Q_{\mathbb{C}}$$

on which Hermitian form $x \cdot \overline{y}$ defines a $\mathbb{Z}[\zeta_n]$-valued Hermitian pairing of signature $(1, \ell)$ for some ℓ. Any element of $\widetilde{\Gamma}_\rho$ (in particular, any element of Γ) preserves Q and the Hermitian form on it. The converse also holds. Thus $\Gamma \subset U(Q)$ is a finite index subgroup of the group of unitary isometries of Q and $\Gamma_{\mathbb{R}} = U(Q_{\mathbb{C}}) = U(1, \ell)$. The boundary components $B_J = \mathbb{P}(J_{\mathbb{C}}^{\zeta_n})$ are then projectivizations of the isotropic $\mathbb{Z}[\zeta_n]$-lines $K \subset Q$. Here $K_{\mathbb{C}} = J_{\mathbb{C}}^{\zeta_n}$. Choose a generator $k \in K$. Then any $[x] \in \mathbb{D}_\rho \subset \mathbb{P}Q_{\mathbb{C}}$ has a unique representative $x \in Q_{\mathbb{C}}$ for which $k \cdot x = 1$. This realizes $\mathbb{D}_\rho$ as a tube domain in the affine hyperplane $V_k := \{k \cdot x = 1\} \subset Q_{\mathbb{C}}$. Concretely, it is the "upper half-space model" of complex-hyperbolic space. Choosing some isotropic $k' \in Q_{\mathbb{C}}$ for which $k' \cdot k = 1$, any element $x \in V_k$ can be written uniquely as $x = k' + x_0 + ck$ for some $x_0 \in \{k, k'\}^\perp$ and $c \in \mathbb{C}$. The image of $\mathbb{D}_\rho$ is exactly those x satisfying $2\mathrm{Re}(c) > -x_0 \cdot \bar{x}_0$.

Let $U_K \subset P_K := \mathrm{Stab}_K(\Gamma)$ be the unipotent subgroup of the parabolic stabilizer (i.e. U_K acts on K, $K^\perp/K$, and $Q/K^\perp$ by the identity). Then U_K acts on V_k by translations. The fibration $\mathbb{D}_\rho \to K_{\mathbb{C}}^\perp/K_{\mathbb{C}}$ sending $x \mapsto x_0 \bmod K_{\mathbb{C}}$ is a fibration of right half-planes. The action of U_K fibers over the action of a translation subgroup $\overline{U}_K \subset K^\perp/K$ on $K_{\mathbb{C}}^\perp/K_{\mathbb{C}}$ and thus, there is a fibration

$$\mathbb{D}_\rho/U_K \to (K_{\mathbb{C}}^\perp/K_{\mathbb{C}})/\overline{U}_K =: A_K$$

over an abelian variety. The fibers are quotients of the right half-planes with coordinate c by the $\mathbb{Z}$-action of a purely imaginary translation. This realizes $\mathbb{D}_\rho/U_K$ as a punctured holomorphic disc bundle over A_K.

Definition 3.20. $\mathbb{D}_\rho/U_K$ is the *first partial quotient* associated to the Baily-Borel cusp K. The extension of this punctured disc bundle to a disc bundle

$$\overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}} \to A_K$$

for a given K is called the *toroidal extension at the cusp K*.

We identify the divisor at infinity, i.e. the zero section of the disc bundle, with the abelian variety A_K itself.

Construction 3.21. The *toroidal compactification* of $\mathbb{D}_\rho/\Gamma$ is constructed as follows: Let Γ_K be the finite group defined by the exact sequence

$$0 \to U_K \to \mathrm{Stab}_K(\Gamma) \to \Gamma_K \to 0.$$

For each cusp K, take the quotient the toroidal extension

$$V_K := \overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}}/\Gamma_K \supset \mathbb{D}_\rho/\mathrm{Stab}_K(\Gamma).$$

A well-known theorem states that there exists a horoball neighborhood N_K of $\mathbb{P}K_{\mathbb{C}}$ in $\mathbb{D}_\rho^{\mathrm{bb}}$ such that $(N_K \setminus \mathbb{P}K_{\mathbb{C}})/\mathrm{Stab}_K(\Gamma) \hookrightarrow \mathbb{D}_\rho/\Gamma$ injects. Thus, we can glue a neighborhood of the boundary $A_K/\Gamma_K \subset V_K$ to $\mathbb{D}_\rho/\Gamma$, ranging over all Γ-orbits of cusps K. The result is the toroidal compactification $\overline{\mathbb{D}_\rho/\Gamma}^{\mathrm{tor}}$.

The boundary divisors of $\overline{\mathbb{D}_\rho/\Gamma}^{\rm tor}$ are in bijection with Γ-orbits of isotropic $\mathbb{Z}[\zeta_n]$-lines $K \subset Q$ and the boundary divisor is isomorphic to A_K/Γ_K, where Γ_K acts by a subgroup of the finite group $U(K^\perp/K)$. There is a morphism

$$\overline{\mathbb{D}_\rho/\Gamma}^{\rm tor} \to \overline{\mathbb{D}_\rho/\Gamma}^{\rm bb}$$

which contracts each boundary divisor to a point. As such, the normal bundle of the boundary divisor is anti-ample. Passing to a finite index subgroup $\Gamma_0 \subset \Gamma$, we can assume that Γ_K is trivial for all cusps K and the anti-ampleness still holds. This proves that the normal bundle to $A_K \subset \overline{\mathbb{D}_\rho/U_K}^{\rm can}$ in the first partial quotient is anti-ample.

Using [Gra62] one shows that a divisor in a smooth analytic space, isomorphic to an abelian variety and with anti-ample normal bundle, can be contracted along any abelian subvariety. In particular, for any sub-$\mathbb{Z}[\zeta_n]$-lattice $\mathfrak{F}_K \subset K^\perp/K$, there is a contraction

$$\overline{\mathbb{D}_\rho/U_K}^{\rm can} \to \overline{\mathbb{D}_\rho/U_K}^{\mathfrak{F}_K}$$

which is an isomorphism away from the boundary divisor and contracts exactly the translates of the abelian subvariety $\mathrm{im}(\mathfrak{F}_K)_\mathbb{C} \subset A_K$.

To construct $\overline{\mathbb{D}_\rho/\Gamma}^{\mathfrak{F}}$, we glue $\overline{\mathbb{D}_\rho/U_K}^{\mathfrak{F}_K}/\Gamma_K$ to $\mathbb{D}_\rho/\Gamma$ along a punctured analytic open neighborhood of the boundary component K. This is only possible if the action of Γ_K on $\overline{\mathbb{D}_\rho/U_K}^{\rm can}$ descends along the above contraction. The condition in Definition 3.18 ensures that the collection $\mathfrak{F} = \{\mathfrak{F}_K\}$ is Γ-invariant. So an individual $\mathfrak{F}_K$ is Γ_K-invariant and the Γ_K action descends. Thus, we have constructed the semitoroidal compactification.

Remark 3.22. A feature of the construction is that one can pull back a semifan $\mathfrak{F}$ for a type IV domain to any type I subdomain, and there will be a morphism between the corresponding semitoric compactifications.

3F. **Recognizable divisors.** We recall the main new concept "recognizability" introduced in [AE21]. We slightly modify the definition as necessary for moduli spaces of K3 surfaces with ρ-markable automorphism:

Definition 3.23. A canonical choice of polarizing divisor R for F_ρ is *recognizable* if for every Kulikov surface X_0 of Type I, II, or III, there is a divisor $R_0 \subset X_0$ which is (up to the action of $\mathrm{Aut}^0(X_0)$) the flat limit of the R_t, $t \neq 0$ on *any* smoothing into ρ-markable K3 surfaces $X \to (C, 0)$, $C^* \subset F_\rho$.

We use the term "smoothing" to mean specifically a Kulikov model $X \to (C, 0)$. Roughly, Definition 3.23 amounts to saying that the canonical choice R can also be made on any Kulikov surface, including smooth K3s, so long it appears as a limit of ρ-markable surfaces.

Theorem 3.24. *If R is recognizable, then $\overline{F}_\rho^{\rm slc}$ is a semitoroidal compactification of F_ρ for a unique semifan $\mathfrak{F}_R$.*

Proof. The proof for type IV domains, i.e. when $n = 2$, is a direct application of [AE21, Thm. 1.2]. So we restrict our attention to the type I case $n > 2$, which is ultimately much simpler.

First, we show that $\overline{F}_\rho^{\rm slc}$ contains $\mathbb{D}_\rho/\Gamma_\rho$. Let $\mathcal{M}_\rho^*$ be the closure of the moduli space of ρ-marked K3 surfaces $\mathcal{M}_\rho$ in the space of all marked K3 surfaces $\mathcal{M}$ and let $F_\rho^* = \mathcal{M}_\rho^*/\Gamma_\rho$ be the quotient. Given any smooth K3 surface $X_0 \in F_\rho^* \setminus F_\rho$

recognizability implies that the universal family $(\mathcal{X}^*, \mathcal{R}^*) \to F_\rho$ extends over F_ρ^* by the same argument as [AE21, Prop. 6.3]: There is a preferred set-theoretic extension of the divisor $\mathcal{R}^*$ over X_0 by the divisor $R_0 \subset X_0$ certifying recognizability. This set-theoretic extension is actually algebraic because it is algebraic along any arc $(C, 0) \subset F_\rho^*$ and F_ρ^* is normal. Then, the argument of Lemma 3.5 gives a morphism $(F_\rho^*)^{\rm sep} = \mathbb{D}_\rho/\Gamma_\rho \to P_{2d,m}$.

Because $\overline{F}_\rho^{\rm slc}$ is the normalized closure of the image of $F_\rho^{\rm sep} = (\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ it is also the normalized closure of the image of $(F_\rho^*)^{\rm sep} = \mathbb{D}_\rho/\Gamma_\rho$. Noting that $\mathbb{D}_\rho/\Gamma_\rho$ is already normal completes the proof of the theorem when $n \neq 3, 4, 6$ by Corollary 3.15 and shows that $\overline{F}_\rho^{\rm slc}$ compactifies $\mathbb{D}_\rho/\Gamma_\rho$ when $n \in \{3, 4, 6\}$.

Consider the toroidal extension $\overline{\mathbb{D}_\rho/U_K}^{\rm can}$ (see Def. 3.20) at the cusp K, of the first partial quotient. Recognizability implies:

Lemma 3.25. *There is a family of pairs $(\mathcal{X}, \mathcal{R}) \to \overline{\mathbb{D}_\rho/U_K}^{\rm can}$ enjoying the following properties:*

(1) *the fiber over any point $0 \in A_K$ in the abelian variety forming the boundary divisor is a Type II Kulikov surface X_0 and the fiber over any point in $\mathbb{D}_\rho/U_K$ is a smooth K3 surface.*
(2) *$\mathcal{R}$ is a relatively big and nef extension of the canonical choice of polarizing divisor R, which contains no singular strata of any fiber.*
(3) *The period map (extended to the Type II Kulikov surfaces) is the identity.*

Proof of Lemma 3.25. Let $\mathbb{D}_N \supset \mathbb{D}_\rho$ be the type IV domain as in Section 3D. Let $U_J \subset O(N)$ be the unipotent stabilizer of the rank 2 isotropic $\mathbb{Z}$-lattice $J \subset N$ which corresponds to the rank 1 isotropic $\mathbb{Z}[\zeta_n]$-lattice $K \subset Q$.

There is a toroidal extension $\mathbb{D}_N/U_J \hookrightarrow \overline{\mathbb{D}_N/U_J}^{\rm can}$ of the unipotent quotient of the associated type IV domain, see e.g. [AE21, Prop. 4.16]: roughly, $\mathbb{D}_N/U_J$ embeds into a line bundle over $J^\perp/J \otimes_{\mathbb{Z}} \widetilde{\mathcal{E}}$ where $\widetilde{\mathcal{E}}$ is the universal elliptic curve over $\mathbb{H} \sqcup (-\mathbb{H})$. The toroidal extension is defined as the closure of the image in a projective line bundle. The eigenspace $\mathbb{D}_\rho/U_K$ sits inside the line bundle as the inverse image of

$$K^\perp/K \otimes_{\mathbb{Z}[\zeta_n]} E \subset J^\perp/J \otimes_{\mathbb{Z}} \widetilde{\mathcal{E}},$$

where E is the elliptic curve admitting an action of ζ_n (note that $K = J$ but with the additional structure of a $\mathbb{Z}[\zeta_n]$-lattice). This embedding arises from functoriality: The toroidal compactification of a type I subdomain inside of a type IV domain can be constructed by simply taking its closure in any toroidal compactification of the type IV domain.

Let $C^* \to F_\rho$ be a one-parameter degeneration whose monodromy lattice (Definition 3.12) is the rank 2 lattice J. Then, possibly after a finite base change, there is a Kulikov model $\pi\colon (X, R) \to (C, 0)$ with R extending as a relatively big and nef divisor containing no strata of any fiber. Furthermore, the image of 0 in $\overline{F}_\rho^{\rm slc}$ (the unique stable limit of the family C^*) can be computed as the central fiber of $\mathrm{Proj}_C \bigoplus_{n \geq 0} \pi_* \mathcal{O}_X(nR)$, see [AE21, Sec. 3C].

Let $L = \mathcal{O}_X(R)$. Then [AE21, Prop. 5.42] states that the polarized Kulikov model $(X, L) \to (C, 0)$ can be extended to a family of Kulikov models

$$(\mathcal{X}^+, \mathcal{L}^+) \to \overline{\mathbb{D}_N/U_J}^{\rm can}$$

with $\mathcal{L}^+$ a relatively big and nef line bundle. Of course, R does not extend in a natural way to all of $\mathcal{X}^+$ because the subdomain $\mathbb{D}_\rho/U_K$ of K3 surfaces with automorphisms has smaller dimension than $\mathbb{D}_N/U_J$. But we can define

$$(\mathcal{X},\mathcal{R}) \to \overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}}$$

as the closure of the universal family of pairs $(\mathcal{X}^*,\mathcal{R}^*) \to \mathbb{D}_\rho/U_K$ in the restriction of the family $\mathcal{X}^+$ to the type I subdomain.

The arguments of [AE21, Sec. 6] now apply essentially verbatim to say that $\mathcal{R}$ is a relatively big and nef divisor, and contains no strata of any fiber. The key point is that recognizability ensures the existence of a set-theoretic extension $R_0' \subset X_0'$ of $\mathcal{R}^*$ to any Type II Kulikov surface X_0' over the boundary. This set-theoretic extension is easily shown to be algebraic by considering one-parameter families. Additionally, we have that

(1) $R_0 \subset X_0$ is big and nef, containing no strata and
(2) $\mathcal{R}$ extends $R_0 \subset X_0$.

We may conclude that $\mathcal{L}^+\big|_{\mathcal{X}} = \mathcal{O}_{\mathcal{X}}(\mathcal{R})$ is relatively big and nef and also $\mathcal{R}$ contains no strata of any fiber [AE21, Prop. 6.9]. □

We now complete the proof of Theorem 3.24.

From Lemma 3.25, we get a classifying map $q\colon \overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}} \to \overline{F}_\rho^{\mathrm{slc}}$ by passing to the relative stable model $(\overline{\mathcal{X}},\epsilon\overline{\mathcal{R}})$ of the family $(\mathcal{X},\mathcal{R})$. The map q factors through the quotient by $\Gamma_K = \mathrm{Stab}_K(\Gamma)/U_K$ because all points in the Γ_K-orbit of a general fiber these represent the same point in F_ρ^{ade}. Applying this argument to all Γ-orbits of cusps K, we conclude that there is a descended morphism

$$p\colon \overline{\mathbb{D}_\rho/\Gamma_\rho}^{\mathrm{tor}} \to \overline{F}_\rho^{\mathrm{slc}}.$$

Consider the restriction $q\big|_{A_K}$ of q to the boundary divisor $A_K \subset \overline{\mathbb{D}_\rho/U_K}^{\mathrm{can}}$ and let $A_K \to Z_K \to q(A_K)$ be its Stein factorization. The normal image Z_K of an abelian variety A_K with connected fibers is necessarily an abelian variety, with the map being the quotient by an abelian subvariety. This abelian subvariety corresponds to a primitive sublattice $\mathfrak{F}_K \subset K^\perp/K$. Furthermore, $\mathfrak{F}_K$ is Γ_K-invariant because q descends to p.

Thus, the sublattices $\mathfrak{F}_K$ define a Γ_ρ-admissible semifan and the curves contracted by p are exactly the same as the curves contracted by the map

$$\overline{\mathbb{D}_\rho/\Gamma_\rho}^{\mathrm{tor}} \to \overline{\mathbb{D}_\rho/\Gamma_\rho}^{\mathfrak{F}_R}.$$

The result follows from the normality of $\overline{F}_\rho^{\mathrm{slc}}$ and Zariski's main theorem. This argument is quite similar to the type IV case [AE21, Thm. 7.18]. □

3G. **The main theorem.**

Theorem 3.26. *Assuming* ($\exists g \geq 2$)*,* $R = C_1$ *is recognizable for* F_ρ*. The stable pair compactification* $\overline{F}_\rho^{\mathrm{slc}}$ *is a semitoroidal compactification of* $\mathbb{D}_\rho/\Gamma_\rho$*.*

Proof. By Theorem 3.24, the second statement follows from the first. Let $(X,R) \to (C,0)$ be a Kulikov model with a flat family of divisors $R \subset X$ for which

(1) there is an automorphism σ on $X^* \to C^*$ making $(X_t,\sigma_t) \in F_\rho$ for $t \neq 0$,
(2) $R_t \subset \mathrm{Fix}(\sigma_t)$ is the fixed component of genus at least 2 for $t \neq 0$, and
(3) $R_0 = \lim_{t\to 0} R_t$.

By [AE21, Prop. 6.12], it suffices to show: For any deformation of the smoothing of X_0 into F_ρ that keeps the isomorphism type of X_0 constant, the limiting curve $R_0 \subset X_0$ does not deform, up to $\mathrm{Aut}^0(X_0)$.

The automorphism σ on the generic fiber of any smoothing defines a birational automorphism of X. Any two Kulikov models are related by an automorphism followed by a sequence of Atiyah flops of Types 0, I, II along curves in X_0 which are either (-2)-curves or (-1)-curves on component(s) of X_0. As such, there are only countably many curves in X_0 along which it is possible to make an Atiyah flop, and this continues to be the case after a flop is made. Thus, up to conjugation by $\mathrm{Aut}^0(X_0)$, there are only countably many possibilities for the birational automorphism $\sigma_0 := \sigma|_{X_0} : X_0 \dashrightarrow X_0$.

Hence if $X_0 \hookrightarrow X$ and $X_0 \hookrightarrow \widetilde{X}$ are (deformation-equivalent) smoothings into F_ρ as above, we have $\widetilde{\sigma}_0 = \psi \circ \sigma_0 \circ \psi^{-1}$ for some $\psi \in \mathrm{Aut}^0(X_0)$.

Let $\{A_j\}$ be the countable set of curves in X_0 along which σ_0 can be indeterminate. Any such curve A_j is $\mathrm{Aut}^0(X_0)$-invariant. Let $A = \cup_j A_j$ be their union. Clearly, the limit divisor R_0 is contained in the union of $A \cup S$ where S is the closure of the fixed locus of σ_0 in its locus of determinacy. Similarly, $\widetilde{R}_0$ is contained in $A \cup \widetilde{S}$ and $\sigma_0(P) = P$ if and only if $\widetilde{\sigma}_0(\psi(P)) = \psi(P)$. Since the smoothing $\widetilde{X}$ is a deformation of the smoothing X and the limiting divisor of R varies continuously, we conclude that $\widetilde{R}_0 = \psi(R_0)$ and therefore R is recognizable. □

Proposition 3.27. *Any element* $(\overline{X}, \epsilon\overline{R}) \in \overline{F}_\rho^{\mathrm{slc}}$ *has an automorphism* $\overline{\sigma} \in \mathrm{Aut}(\overline{X})$. *Furthermore,* $\overline{R} = \mathrm{Fix}(\overline{\sigma})$ *and* $\overline{\sigma}^*$ *acts on* $H^0(\overline{X}, \omega_{\overline{X}}) \cong \mathbb{C}$ *by multiplication by* ζ_n.

Proof. As noted in Remark 3.6, any point in $F_\rho^{\mathrm{sep}} = (\mathbb{D}_\rho \setminus \Delta_\rho)/\Gamma_\rho$ corresponds to a pair $(\overline{X}, \overline{\sigma})$ of an ADE K3 surface with automorphism, for which $\overline{R} = \mathrm{Fix}(\overline{\sigma})$ is ample and the minimal resolution is ρ-markable. Then any boundary point $(\overline{X}_0, \epsilon\overline{R}_0) \in \overline{F}_\rho^{\mathrm{slc}}$ is a stable limit of such ADE K3 surface pairs $f : (\overline{X}, \epsilon\overline{R}) \to C$.

Since $\overline{R}_t$ is $\overline{\sigma}_t$-invariant and the canonical model is unique, $\overline{X}$ admits an automorphism $\overline{\sigma}$ whose fixed locus contains $\overline{R}_0$. In fact, $\mathrm{Fix}(\overline{\sigma}_0) = \overline{R}_0$: $\mathrm{Fix}(\overline{\sigma})$ is a Cartier divisor, and thus forms a flat family of divisors containing $\overline{R}$. But $\mathrm{Fix}(\overline{\sigma}_0)$ already contains the flat limit $\overline{R}_0$. The statement about $\omega_{\overline{X}_0}$ follows from the fact that $f_*\omega_{\overline{X}/C}$ is invertible (by Base Change and Cohomology, since $R^1 f_*\omega_{\overline{X}/C} = 0$) and $\overline{\sigma}_t^*$ acts by ζ_n on the generic fiber of this line bundle. □

4. Moduli of quotient surfaces

We refer the reader to [Kol13] for the notions appearing in the following definitions. The pair (Y, Δ) is called demi-normal if X satisfies Serre's S_2 condition, has double normal crossing singularities in codimension 1, and $\Delta = \sum d_i D_i$ is an effective Weil $\mathbb{Q}$-divisor with $0 < d_i \le 1$ not containing any components of the double crossing locus of Y.

The following is [Kol13, Prop. 2.50(4)], using our adopted notations.

Proposition 4.1. *Étale locally, there is a one-to-one correspondence between*

(a) *Local demi-normal pairs* $(y \in Y, \frac{n-1}{n}B)$ *of index* n*, i.e. such that the divisor* $nK_Y + (n-1)B$ *is Cartier.*

(b) *Local demi-normal pairs* ($\widetilde{y} \in \widetilde{Y}$) *such that* $K_{\widetilde{Y}}$ *is Cartier, with a* μ_n*-action that is free on a dense open subset, and such that the induced action on* $\omega_{\widetilde{Y}} \otimes \mathbb{C}(\widetilde{y})$ *is faithful.*

Moreover, the pair $(Y, \frac{n-1}{n}B)$ *is slc iff* $\widetilde{Y}$ *is slc.*

The variety $\widetilde{Y}$ is called the local index-1 cover of the pair $(Y, \frac{n-1}{n}B)$. [Kol13, Sec. 2] also gives a global construction.

Theorem 4.2. *Let* $(\overline{X}, \epsilon\overline{R}) \in \overline{F}_\rho^{\mathrm{slc}}$ *and let* $\pi \colon \overline{X} \to Y = \overline{X}/\mu_n$ *be the quotient map with the branch divisor* $B = f(\overline{R})$. *Then*

(1) $nK_Y + (n-1)B \sim 0$,
(2) B *and* $-K_Y$ *are ample* $\mathbb{Q}$*-Cartier divisors,*
(3) *the pair* $(Y, \frac{n-1+\epsilon}{n}B)$ *is stable for any rational* $0 < \epsilon \ll 1$, *i.e. it has slc singularities and the* $\mathbb{Q}$*-divisor* $K_Y + \frac{n-1+\epsilon}{n}B$ *is ample.*

Vice versa, for a pair (Y, B) *satisfying the above conditions, its index-*1 *cover* $\overline{X}$ *with the ramification divisor* $\overline{R}$ *satisfies:*

(1) $K_{\overline{X}} \sim 0$ *and the* μ_n*-action on* $\overline{X}$ *is nonsymplectic,*
(2) $\overline{R}$ *is* $\mathbb{Q}$*-Cartier,*
(3) *the pair* $(\overline{X}, \epsilon\overline{R})$ *is stable for any rational* $0 < \epsilon \ll 1$.

Proof. This follows from Proposition 4.1 and the formulas

$$\pi^*(B) = n\overline{R}, \qquad \pi^*\left(K_Y + \frac{n-1+\epsilon}{n}B\right) = K_{\overline{X}} + \epsilon\overline{R}.$$

□

Corollary 4.3. *The coarse moduli space* $\overline{F}_\rho^{\mathrm{slc}}$ *coincides with the normalization of the KSBA compactification of the irreducible component in the moduli space of log canonical pairs* $(Y, \frac{n-1+\epsilon}{n}B)$ *of log del Pezzo surfaces* Y *with* $(n-1)B \in |-nK_Y|$ *in which a generic surface is a quotient of a K3 surface with a nonsymplectic automorphism of type* ρ. *The stack for the former is a* μ_n*-gerbe over the stack for the latter.*

For the proof, we note that a small deformation of a K3 surface is a K3 surface.

Example 4.4. The KSBA compactification of the moduli space of K3 surfaces of degree 2, with the ramification divisor R as the recognizable divisor, is studied in detail in [AET19]. By Corollary 4.3, it coincides with Hacking's compactification [Hac04] of the moduli space of pairs $(\mathbb{P}^2, \frac{1+\epsilon}{2}B_6)$ of plane sextic curves.

5. Extensions

The results of this paper are easily extended to the case of an action by an arbitrary finite group G for which there is some $g \in G$ with $g^*\omega_X \neq \omega_X$ and to more general divisors defined by group actions. Most of the changes amount to introducing more cumbersome notations.

5A. **A general nonsymplectic group of automorphisms.**

Definition 5.1. Let X be a smooth K3 surface and $\sigma\colon G \subset \operatorname{Aut} X$ be a finite symmetry group. The action of G on $H^{2,0}(X) = \mathbb{C}\omega_X$ gives an exact sequence

$$0 \to G_0 \to G \xrightarrow{\alpha} \mu_n \to 1, \qquad \mu_n \subset \mathbb{C}^*.$$

One says that G is nonsymplectic (or not purely symplectic) if $G \neq G_0$, i.e. $\alpha \neq 1$.

We now extend the results of Section 2 directly to this more general setting.

Definition 5.2. Fix a finite subgroup $\rho\colon G \to O(L)$ and a nontrivial character $\chi\colon G \to \mathbb{C}^*$. Let $(X, \sigma\colon G \to \operatorname{Aut} X)$ be a K3 surface with a nonsymplectic automorphism group.

A (ρ,χ)-*marking* of (X,σ) is an isometry $\phi : H^2(X,\mathbb{Z}) \to L$ such that for any $g \in G$ one has $\phi \circ \sigma(g)^* = \rho(g) \circ \phi$ and such that the character $\alpha\colon G \to \mathbb{C}^*$ induced by σ coincides with χ. We say that (X,σ) is ρ-*markable* if it admits a ρ-marking.

A family of (ρ,χ)-marked K3 surfaces is a smooth family $f\colon (\mathcal{X}, \sigma_B, \phi_B) \to B$ with a group of automorphisms $\sigma_B\colon G \to \operatorname{Aut}(\mathcal{X}/B)$ and with a marking $\phi_B\colon R^2 f_*\mathbb{Z} \to L \otimes \underline{\mathbb{Z}}_B$ such that every fiber is a (ρ,χ)-marked K3 surface.

A family of smooth ρ-markable K3 surfaces is a family $f\colon (\mathcal{X}, \sigma_B) \to B$ of K3 surfaces with a group of automorphisms over the base B which admits a ρ-marking analytically-locally on B. We define the moduli stacks $\mathcal{M}_{\rho,\chi}$ of (ρ,χ)-marked, resp. $\mathcal{F}_{\rho,\chi}$ of (ρ,χ)-markable K3 surfaces by taking $\mathcal{M}_{\rho,\chi}(B)$, resp. $\mathcal{F}_{\rho,\chi}(B)$ to be the groupoids of such families over B.

Definition 5.3. Define the vector space

$$L_{\mathbb{C}}^{\rho,\chi} = \{x \in L_{\mathbb{C}} \mid \rho(g)(x) = \chi(g)x\}$$

to be the intersection of the eigenspaces for each $g \in G$, and the period domain as

$$\mathbb{D}_{\rho,\chi} = \mathbb{P}\{x \in L_{\mathbb{C}}^{\rho,\chi} \mid x \cdot x = 0, x \cdot \bar{x} > 0\}$$

The first condition is redundant if there exists $g \in G$ with $\chi(g) \neq \pm 1$. Thus, $\mathbb{D}_\rho$ is a type IV domain if $|\chi(G)| = 2$ and a type I domain, a complex ball, if $|\chi(G)| > 2$.

The discriminant locus is $\Delta_\rho := \cup_\delta \delta^\perp \cap \Delta_\rho$ ranging over all roots δ in $(L^G)^\perp$, where $L^G = \{a \in L \mid \rho(g)(a) = a\}$ is the sublattice of L fixed by G.

Definition 5.4. Define $\Gamma_\rho := \{\gamma \in O(L) \mid \gamma \circ \rho = \rho \circ \gamma\}$.

Then the direct analogue of Lemma 2.7 and Theorem 2.10 is

Theorem 5.5. *For a fixed finite group $\rho\colon G \to O(L)$ with a nontrivial character $\chi\colon G \to \mathbb{C}^*$:*

1. *There exists a fine moduli space $\mathcal{M}_{\rho,\chi}$ of (ρ,χ)-marked K3 surfaces (X,σ,ϕ). It admits an étale period map $\pi_\rho\colon \mathcal{M}_{\rho,\chi} \to \mathbb{D}_{\rho,\chi} \setminus \Delta_\rho$. The fiber $\pi_\rho^{-1}(x)$ over a point $x \in \mathbb{D}_{\rho,\chi} \setminus \Delta_\rho$ is a torsor over $\Gamma_\rho \cap (\mathbb{Z}_2 \cap W_x)$.*
2. *The moduli stack $\mathcal{F}_{\rho,\chi}$ of ρ-markable K3 surfaces (X,σ) is obtained as a quotient of $\mathcal{M}_{\rho,\chi}$ by Γ_ρ. On the level of coarse moduli spaces, it admits a bijective map to $(\mathbb{D}_{\rho,\chi} \setminus \Delta_\rho)/\Gamma_\rho$.*

Proof. If the group G does not act purely symplectically, i.e. there exists $g \in G$ with $\rho(g)(x) \neq x$ then $L^G \perp x$ and $S_X^G \simeq L^G$. The rest of the proof of Lemma 2.7 works the same for any finite group. The proof of Theorem 2.10 goes through verbatim. ∎

5B. **More general polarizing divisors.** With a more general group action, there are more choices for the polarizing divisors. For a generic K3 surface X with a period $x \in \mathbb{D}_{\rho,\chi} \setminus \Delta_\rho$ we can consider any combination $\sum b_i B_i$ of curves B_i which are either fixed by some element $g \in G$ or are some of the (-2)-curves corresponding to the roots in the generic Picard lattice $(L_{\mathbb{C}}^{\rho,\chi})^\perp \cap L$ that generically gives a big and nef divisor on X. Theorem 3.26 extends to this situation with the same proof.

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References

[ABE20] Valery Alexeev, Adrian Brunyate, and Philip Engel, *Compactifications of moduli of elliptic K3 surfaces: stable pair and toroidal*, Geom. Topol. **26** (2022), no. 8, 3525–3588, DOI 10.2140/gt.2022.26.3525. MR4562567

[AE21] Valery Alexeev and Philip Engel, *Compact moduli of K3 surfaces*, Ann. of Math. (2) **198** (2023), no. 2, 727–789, DOI 10.4007/annals.2023.198.2.5. MR4635303

[AET19] Valery Alexeev, Philip Engel, and Alan Thompson, *Stable pair compactification of moduli of K3 surfaces of degree 2*, J. Reine Angew. Math. **799** (2023), 1–56, DOI 10.1515/crelle-2023-0011. MR4595306

[AS08] Michela Artebani and Alessandra Sarti, *Non-symplectic automorphisms of order 3 on K3 surfaces*, Math. Ann. **342** (2008), no. 4, 903–921, DOI 10.1007/s00208-008-0260-1. MR2443767

[AS15] Michela Artebani and Alessandra Sarti, *Symmetries of order four on K3 surfaces*, J. Math. Soc. Japan **67** (2015), no. 2, 503–533, DOI 10.2969/jmsj/06720503. MR3340184

[ast85] *Géométrie des surfaces K3: modules et périodes*, Société Mathématique de France, Paris, 1985, Papers from the seminar held in Palaiseau, October 1981–January 1982, Astérisque No. 126 (1985). MR785216 (87h:32052)

[AST11] Michela Artebani, Alessandra Sarti, and Shingo Taki, *K3 surfaces with non-symplectic automorphisms of prime order*, Math. Z. **268** (2011), no. 1-2, 507–533, DOI 10.1007/s00209-010-0681-x. With an appendix by Shigeyuki Kondō. MR2805445

[BB66] W. L. Baily Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528, DOI 10.2307/1970457. MR216035

[DH22] Anand Deopurkar and Changho Han, *Stable quadrics, admissible covers, and Kondō's sextic K3 surfaces*, In preparation, 2022.

[Dil09] Jimmy Dillies, *Order 6 non-symplectic automorphisms of K3 surfaces*, `arXiv:0912.5228`, 2009.

[Dil12] Jimmy Dillies, *On some order 6 non-symplectic automorphisms of elliptic K3 surfaces*, Albanian J. Math. **6** (2012), no. 2, 103–114. MR3009163

[DK07] Igor V. Dolgachev and Shigeyuki Kondō, *Moduli of K3 surfaces and complex ball quotients*, Arithmetic and geometry around hypergeometric functions, Progr. Math., vol. 260, Birkhäuser, Basel, 2007, pp. 43–100, DOI 10.1007/978-3-7643-8284-1_3. MR2306149

[Dol96] I. V. Dolgachev, *Mirror symmetry for lattice polarized K3 surfaces*, J. Math. Sci. **81** (1996), no. 3, 2599–2630, DOI 10.1007/BF02362332. Algebraic geometry, 4. MR1420220

[Fri84] Robert Friedman, *A new proof of the global Torelli theorem for K3 surfaces*, Ann. of Math. (2) **120** (1984), no. 2, 237–269, DOI 10.2307/2006942. MR763907

[FS86] Robert Friedman and Francesco Scattone, *Type III degenerations of K3 surfaces*, Invent. Math. **83** (1986), no. 1, 1–39, DOI 10.1007/BF01388751. MR813580

[Gra62] Hans Grauert, *Über Modifikationen und exzeptionelle analytische Mengen* (German), Math. Ann. **146** (1962), 331–368, DOI 10.1007/BF01441136. MR137127

[Hac04] Paul Hacking, *Compact moduli of plane curves*, Duke Math. J. **124** (2004), no. 2, 213–257, DOI 10.1215/S0012-7094-04-12421-2. MR2078368

[Kol13] János Kollár, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács, DOI 10.1017/CBO9781139547895. MR3057950

[Kol21] János Kollár, *Families of varieties of general type*, Cambridge Tracts in Mathematics, vol. 231, Cambridge University Press, Cambridge, 2023. With the collaboration of Klaus Altmann and Sándor J. Kovács. MR4566297

[Kon02] Shigeyuki Kondō, *The moduli space of curves of genus 4 and Deligne-Mostow's complex reflection groups*, Algebraic geometry 2000, Azumino (Hotaka), Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, Tokyo, 2002, pp. 383–400, DOI 10.2969/aspm/03610383. MR1971521

[Kon20] Shigeyuki Kondō, *$K3$ surfaces*, EMS Tracts in Mathematics, vol. 32, EMS Publishing House, Berlin, [2020] ©2020. Translated from the Japanese original by the author. MR4321993

[Kul77] Vik. S. Kulikov, *Degenerations of $K3$ surfaces and Enriques surfaces* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 5, 1008–1042, 1199. MR506296

[Loo03a] Eduard Looijenga, *Compactifications defined by arrangements. I. The ball quotient case*, Duke Math. J. **118** (2003), no. 1, 151–187, DOI 10.1215/S0012-7094-03-11816-5. MR1978885

[Loo03b] Eduard Looijenga, *Compactifications defined by arrangements. II. Locally symmetric varieties of type IV*, Duke Math. J. **119** (2003), no. 3, 527–588, DOI 10.1215/S0012-7094-03-11933-X. MR2003125

[Mat16] Yuya Matsumoto, *Degeneration of K3 surfaces with non-symplectic automorphisms*, Rend. Semin. Mat. Univ. Padova **150** (2023), 227–245, DOI 10.4171/rsmup/123. MR4651699

[MO98] Natsumi Machida and Keiji Oguiso, *On $K3$ surfaces admitting finite non-symplectic group actions*, J. Math. Sci. Univ. Tokyo **5** (1998), no. 2, 273–297. MR1633933

[MS21] Han-Bom Moon and Luca Schaffler, *KSBA compactification of the moduli space of K3 surfaces with a purely non-symplectic automorphism of order four*, Proc. Edinb. Math. Soc. (2) **64** (2021), no. 1, 99–127, DOI 10.1017/S001309152100002X. MR4249842

[Nik79a] V. V. Nikulin, *Finite groups of automorphisms of Kählerian $K3$ surfaces* (Russian), Trudy Moskov. Mat. Obshch. **38** (1979), 75–137. MR544937

[Nik79b] V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238. MR525944

[PP81] Ulf Persson and Henry Pinkham, *Degeneration of surfaces with trivial canonical bundle*, Ann. of Math. (2) **113** (1981), no. 1, 45–66, DOI 10.2307/1971133. MR604042

[PSS71] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, *Torelli's theorem for algebraic surfaces of type* K3 (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530–572. MR284440

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