

CONSTRUCTING THE OSELEDETS DECOMPOSITION WITH SUBSPACE GROWTH ESTIMATES

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ABSTRACT. The semi-invertible version of Oseledets’ multiplicative ergodic theorem providing a decomposition of the underlying state space of a random linear dynamical system into fast and slow spaces is deduced for a strongly measurable cocycle on a separable Banach space. This work is a much shorter means of obtaining this general version of the theorem, using measurable growth estimates on subspaces for linear operators combined with a modified version of Kingman’s subadditive ergodic theorem.

1. INTRODUCTION

The multiplicative ergodic theorem is a fundamental tool in the study of linear dynamical systems. Given a Banach space X write $\mathcal{B}(X)$ for the space of bounded linear operators on X . Let Ω be a Lebesgue probability space. Given an ergodic system $\sigma : \Omega \rightarrow \Omega$ and function $\mathcal{L} : \Omega \rightarrow \mathcal{B}(X)$, one may compose copies of \mathcal{L} along orbits and investigate long term behaviour of any $x \in X$. Such an \mathcal{L} is referred to as a *cocycle*, and is said to be *forward-integrable* if $\log^+ \|\mathcal{L}\| \in L^1(\Omega)$. Iteratively set $\mathcal{L}_\omega^{(0)} = 1_X : X \rightarrow X$, and $\mathcal{L}_\omega^{(n)} := \mathcal{L}_{\sigma^{n-1}\omega} \circ \mathcal{L}_\omega^{(n-1)}$ for $n \in \mathbb{N}$. The multiplicity property $\mathcal{L}_\omega^{(m+n)} = \mathcal{L}_{\sigma^m\omega}^{(m)} \circ \mathcal{L}_\omega^{(n)}$ is clear. If \mathcal{L} is required to be invertible then $\mathcal{L}_\omega^{(n)} = \mathcal{L}_{\sigma^n\omega}^{-1} \circ \cdots \circ \mathcal{L}_{\sigma^{-1}\omega}^{-1}$ for $n < 0$ extends the cocycle to all $n \in \mathbb{Z}$. If \mathcal{L}^{-1} is forward-integrable then \mathcal{L} is said to be *backward-integrable*. One may seek ways of describing X in terms of long term behaviour of vectors under $\mathcal{L}_\omega^{(n)}$ as $n \rightarrow \infty$. Given any normed space V write $\mathbb{S}_V = \{x \in V : \|x\| = 1\}$ and $\mathbb{B}_V = \{x \in V : \|x\| < 1\}$. Oseledets [12] proved the following in 1965:

Theorem 1. *Let Ω be a Lebesgue probability space and $\sigma : \Omega \rightarrow \Omega$ be an invertible measure preserving transformation. Let $\mathcal{L} : \Omega \rightarrow GL_d(\mathbb{R})$ be forward and backward integrable, where GL_d denotes the invertible d -dimensional matrices. Then there are measurable numbers $\lambda_i(\omega), i \in \{1, \dots, r_\omega\}$ and a direct sum decomposition of invariant measurable subspaces $\mathbb{R}^d = \bigoplus_i E_i(\omega)$ such that for $x \in \mathbb{S}_{E_i(\omega)}$,*

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} x\| = \pm \lambda_i(\omega).$$

This limit converges uniformly in x .

In a setting where all invertibility assumptions are dropped the conclusions are weaker. The following flag decomposition is proven in the work of Raghunathan [14]:

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Theorem 2. *Let Ω be a Lebesgue probability space and $\sigma : \Omega \rightarrow \Omega$ be a (not necessarily invertible) measure preserving transformation. Let $\mathcal{L} : \Omega \rightarrow \text{Mat}_d(\mathbb{R})$ satisfy $\log^+ \|\mathcal{L}\| \in L^1(\Omega)$ where Mat_d denotes the d -dimensional matrices. Then there are measurable numbers $\lambda_i(\omega), i \in \{1, \dots, r_\omega\}$ and a flag of invariant measurable subspaces $\mathbb{R}^d = V_1(\omega) > V_2(\omega) > \dots > V_{r_\omega}(\omega)$ such that for $x \in V_i(\omega) \setminus V_{i+1}(\omega)$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}x\| = \lambda_i(\omega).$$

These results were subsequently extended to more general settings: by Ruelle to compact operators on Hilbert spaces [16], by Mañé to compact operators on Banach spaces with additional continuity assumptions [11] and by Thieullen to quasicompact operators [17].

The case where the underlying vector space is a separable Banach space was first presented by Lian and Lu [9]. Their monograph obtains the decomposition result assuming almost-everywhere injectivity of the cocycle in a separable Banach space.

The injectivity condition isn't necessary to obtain a decomposition: Froyland, Lloyd and Quas [10] demonstrated that as long as σ is invertible the space may still be written as a sum of fast and slow spaces. That paper dealt with the finite dimensional case, but in [3] González-Tokman and Quas presented a generalisation to a separable Banach space. Actually the claim that the slow space is measurable was made in that paper, but as discussed in Horan's thesis [5], the infinite-codimensional slow spaces may fail to be measurable - Example 35 in the appendix is given for convenience. González-Tokman and Quas give a subsequent shorter proof [4] explicitly in the separable dual case, using intuitive notions of volume growth that this work partially builds upon. The argument to obtain spaces for $n > 1$ relies on a notion of fibres over a measurable base space, examining cocycles whose domain varies depending on ω , a geometrically compelling argument for which not all details are supplied. For X with a separable dual, the appendix of [3] explains that it follows that $\mathcal{G}^k X$ must be separable, whence problems such as in Example 35 won't arise, and a separable dual is a sufficient condition for measurability of slow spaces. Another proof with an emphasis on volume growth is the noninvertible result of Blumenthal in [1] where a flag is obtained with no separability assumptions but with a much stronger uniform measurability condition for the cocycle.

Given measurable spaces A and B write $\mathcal{M}(A \rightarrow B)$ for the space of measurable functions from A to B . Let \mathcal{F} be the Borel sigma algebra induced by the strong operator topology on X . The space \mathcal{SM} of strongly measurable functions consists of measurable maps with respect to this choice of sigma algebra:

$$\mathcal{SM}(A \rightarrow \mathcal{B}(X)) = \mathcal{M}(A \rightarrow (\mathcal{B}(X), \mathcal{F})).$$

Write $\mathcal{G}_k X = \{V \leq X : \dim V = k\}$ for the *Grassmannian* of subspaces of dimension k . Given $T \in \mathcal{B}(X)$ define the slowest growth of vectors in a given subspace under T :

$$g(T, V) = g_T(V) = \inf_{x \in \mathbb{S}_V} \|Tx\|, \rho_k T = \sup_{V \in \mathcal{G}_k X} g_T(V).$$

The ρ_k are called Bernstein numbers in the work of Pietsch [13], which gives an overview of similar kinds of statistics in Banach spaces. Given a strongly measurable forward-integrable cocycle \mathcal{L} on a Lebesgue probability space there are decreasing sequences $(\mu_i)_{i \in \mathbb{N}}$ and λ_i of invariant functions

$$\mu_k = \lim_{n \rightarrow \infty} \frac{1}{n} \rho_k \mathcal{L}_\omega^{(n)}, \lambda_1 = \mu_1, \lambda_{i+1} = \mu_{\inf\{t: \mu_t < \lambda_i\}}.$$

While there are countably many μ_i there may only be finitely many λ_i 's, referred to henceforth as the Lyapunov exponents. The main result, an extension of the semi-invertible result to separable Banach spaces, may now be stated:

Theorem 3. *Let $(\Omega, \sigma, X, \mathcal{L})$ be a strongly measurable forward-integrable random linear dynamical system with (Ω, σ) an ergodic invertible map on a Lebesgue probability space and having Lyapunov exponents $(\mu_i)_{i=1}^\infty$ and $(\lambda_i)_{i=1}^L$, where $1 \leq L \leq \infty$. Then for each $0 \leq l < L$ there is a direct sum decomposition into equivariant spaces $X = (\bigoplus_{i \leq l} E_i(\omega)) \oplus V_{l+1}(\omega)$ with the $E_l : \Omega \rightarrow \mathcal{G}_{m_l} X$ measurable, $m_l \in \mathbb{N}$ and having*

$$\frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}|_{E_i(\omega)}\|, \frac{1}{n} \log \inf\{\|\mathcal{L}_\omega^{(n)}x\| : x \in \mathbb{S}_{E_i(\omega)}\} \rightarrow \lambda_i,$$

and

$$V_l(\omega) = \{x \in X : \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}x\| \leq \lambda_l\}.$$

The projection $\Pi : X \rightarrow \bigoplus_{j < i} E_j$ parallel to V_i is strongly measurable and tempered, that is to say, $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\Pi_{\sigma^n \omega}\| = 0$ almost surely. There is a nontrivial decomposition, $L \geq 2$, exactly when $\nu = \lim_{n \rightarrow \infty} \mu_n < \lambda_1$.

The choice of construction method for the Lyapunov exponents is a fundamental aspect to each proof: in [14] μ_i are written simultaneously using finite dimensional singular value decomposition, while by contrast in [9] the first Lyapunov exponent is defined according to the asymptotic growth rate of $\|\mathcal{L}_\omega^{(n)}\|$ followed by the Lyapunov exponents being first described as the asymptotic growth rate for nonzero elements of the fast spaces in the statement of the theorem. While singular value decomposition is no longer available in these contexts, some construction must be written down that may be viewed as reminiscent of a proof of the singular value decomposition of a transformation. The work presented here explicitly relies on a notion of singular values for arbitrary elements of $\mathcal{B}(X)$. Given a $T \in \mathcal{B}(X)$, sufficient conditions on these singular values for contracting fast-growing regions of $\mathcal{G}_k X$ under the action $V \mapsto TV$ are derived. A modified version of Kingman's subadditive theorem is established and used to guarantee the asymptotic growth rates of the singular values of $\mathcal{L}_\omega^{(n)}$ over appropriate discrete time intervals.

The results from past papers stated thus far have not required ergodicity. Throughout the rest of this paper, ergodicity will be assumed, simplifying the classification of invariant functions, although all results as is typical may be formulated with this assumption dropped. In this Banach space setting, the main theorem yields in a trichotomous classification of forward-integrable cocycles on separable Banach spaces: one of

- \mathcal{L} fails to be quasicompact, with $\mu_i = \nu$ for all $i \in \mathbb{N}$ - no fast spaces may be detected
- \mathcal{L} is quasicompact, with finitely many finite dimensional fast spaces growing at rates $\lambda_1 > \dots > \lambda_L > \lambda_{L+1} = \nu$
- \mathcal{L} is quasicompact, with a countable sequence of finite dimensional fast spaces growing at rates $\lambda_1 > \lambda_2 > \dots \rightarrow \nu$.

These three possibilities just correspond to situations where there are no, finitely many or countably many fast spaces. The number ν is an alternative choice to the typical index of compactness $\kappa = \lim_{n \rightarrow \infty} \frac{1}{n} \|\mathcal{L}_\omega^{(n)}\|_c$ that proves simpler to work with in this proof. In the appendix it is verified that these quantities are equal, so that the final result represents an extension of the result of Lian and Lu with

the injectivity assumption dropped, avoiding measurability or domain concerns. In particular, the advantages of the current work are in its brevity, the preciseness of the conclusion and the use of an intuitive geometric perspective.

2. THE GRASSMANNIAN

When discussing subspaces of X , we may also consider the space of subspaces of X they lie in:

Definition 4. Given vector spaces $U, V \leq X$ with $U \oplus V = X$, write $\Pi_{U|V}$ for the projection defined by $\Pi_{U|V}(u+v) = u$ for any $u \in U$ and $v \in V$. The Grassmannian is defined as the set

$$\begin{aligned} \mathcal{G}X &= \{V \text{ closed } \leq X : \text{there exists a projection } X \xrightarrow{\Pi} V \text{ with } \|\Pi\| < \infty\} \\ &= \{V \text{ closed } \leq X : \text{there exists } W \leq X \text{ with } \|\Pi_{V|W}\| < \infty\}. \end{aligned}$$

$\mathcal{G}X$ may be metrised by any of a few equivalent choices of distance between spaces, such as the Hausdorff distance between unit spheres. $\mathcal{G}_k X \subset \mathcal{G}X$ has already been defined. Write $\mathcal{G}^k X = \{V \leq X : \text{codim } V = k\} \subseteq \mathcal{G}X$.

Lemma 5. *Suppose that X is a Banach space. Then*

- *If X is separable then $\mathcal{G}_k X$ is separable and complete.*
- *If Ω is a measurable space, $E \in \mathcal{M}(\Omega \rightarrow \mathcal{G}_k X)$ and $A \in \mathcal{SM}(\Omega \rightarrow \mathcal{B}(X))$ then the pointwise pushforward $\omega \mapsto A(\omega)E(\omega)$ is measurable.*

Proof. Completeness is established in section 2.1 of chapter IV of [6]. The pushforward result is established in corollary B.13 of [3]. □

Lemma 6. *If we take $B = \mathcal{B}(X)$ then we have the following characterisation of the space of strongly measurable functions:*

$$\mathcal{SM}(\Omega \rightarrow \mathcal{B}(X)) = \left\{ \mathcal{L}: \Omega \rightarrow \mathcal{B}(X) : \begin{array}{l} \text{for each } x \in X, \\ (\omega \mapsto \mathcal{L}_\omega x) \in \mathcal{M}(\Omega \rightarrow X) \end{array} \right\}.$$

Proof. Lemma A.4 of [3] checks the equivalence of these conditions. □

The following is readily verified:

Lemma 7. *Let $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$. Then*

$$g_T(V)g_S(TV) \leq g_{S \circ T}(V) \leq \min\{\|T|_V\|g_S(TV), g_T(V)\|S\|\}.$$

In Banach space the product of the ρ_i 's may be shown to be equivalent to a submultiplicative notion of fastest volume growth:

Lemma 8. *Define $D_k \in \mathcal{M}(\mathcal{B}(X) \rightarrow \mathbb{R})$ by*

$$D_k(T) = \sup_{u_1, \dots, u_k \in \mathbb{S}X} \prod_{i=1}^k d(\text{span}\{Tu_j\}_{j < i}, Tu_i).$$

Then D_k is submultiplicative, and there are constants $c_k < C_k$ such that

$$c_k D_k \leq \prod_{i=1}^k \rho_i \leq C_k D_k.$$

Proof. See [4]. □

We make use of the following straightforward construction:

Lemma 9. *Let Ω be a measurable space. Suppose that $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ for some measurable space X . Then if we can write down a measurable set $S_n \subseteq \Omega$ for each $n \in \mathbb{N}$ such that $\bigcup_n S_n = \Omega$ then there is an associated measurable $N: \Omega \rightarrow \mathbb{N}$ given by $N_\omega = \inf\{n : \omega \in S_n\}$. Further, the map $\omega \mapsto x_{N_\omega}$ is measurable.*

This principle is applied to check measurability in the context of spaces which are separable.

3. MEASURABLE STATISTICS FOR LINEAR OPERATORS

Lemma 10. *Let X be a normed space. Then $g: \mathcal{B}(X) \times \mathcal{G}_k X \rightarrow [0, \infty)$ is continuous.*

Proof. Let $\epsilon > 0$. Here use the following metric inducing the product topology:

$$d((S, V), (T, W)) = \|S - T\| + d(V, W).$$

First note that g is continuous at $(0, V)$ for any V since $g(0, V) = 0$ and if $d(V, W) + \|T - 0\| < \epsilon$ then $g(T, W) < \epsilon$. Otherwise, the projection onto the sphere $p_S T = \frac{1}{\|T\|} T$ is continuous. Let $\epsilon > 0$. Since $g(T, V) = \|T\|g(p_S T, V)$ for nonzero T , it suffices to check g is continuous on $\mathbb{S}_{\mathcal{B}(X)} \times \mathcal{G}_k X$. Let

$$(S, V), (T, W) \in \mathbb{S}_{\mathcal{B}(X)} \times \mathcal{G}_k X$$

be such that

$$d((S, V), (T, W)) = \|S - T\| + d(V, W) < \epsilon.$$

Since V and W are finite dimensional their spheres are compact and we may choose an $x \in \mathbb{S}_V$ with $\|Sx\| = g(S, V)$. Choose a $y \in \mathbb{S}_W$ with $\|x - y\| < \epsilon$. Then

$$\begin{aligned} g(T, W) &\leq \|Ty\| \leq \|T - S\| + \|S\|\|y - x\| + \|Sx\| \\ &< \epsilon + \epsilon + g(S, V). \end{aligned}$$

By the same argument swapping (S, V) and (T, W) , then $g(S, V) < g(T, W) + 2\epsilon$ and continuity (in fact, uniform continuity) on $\mathbb{S}_{\mathcal{B}(X)} \times \mathcal{G}_k X$ is clear. Thus g is continuous on $\mathcal{B}(X) \times \mathcal{G}_k X$ as required. \square

Corollary 11. *The quantities $\rho_k : \mathcal{B}(X) \rightarrow \mathbb{R}$ are measurable functions.*

Proof. The ρ_k may now be written as the supremum of the functions $\{T \mapsto g(T, V) : V \in \mathcal{G}_k X\}$ which may be written as the supremum of a countable family since $\mathcal{G}_k X$ is separable. \square

Lemma 12. *Grassmannian contraction estimates: Let $T \in \mathcal{B}(X)$ with $\rho_{k+1}(T) < \rho_k(T)$ and $\Theta \in (\rho_{k+1}T, \rho_k T)$. Suppose that $V, W \in \mathcal{G}_k X$ are choices of fast-growing spaces:*

$$g_T(V), g_T(W) > \Theta.$$

Then

$$d(TV, TW) < \frac{2}{1 - \frac{\rho_{k+1}T}{\Theta}} \frac{\rho_{k+1}T}{\Theta}.$$

In particular, if $\Theta > 2\rho_{k+1}T$ then

$$d(TV, TW) < 4 \frac{\rho_{k+1}T}{\Theta}.$$

Proof. If $V = W$ there is nothing to prove so assume otherwise. Let $a \in V \setminus W$. Then $W \oplus \text{span}\{a\} \in \mathcal{G}_{k+1}X$ so that there is some $a - b \in W \oplus \text{span}\{a\}$ with $b \in W$ and $g_T(a - b) \leq \rho_{k+1}T$. Set $c = a - b$. By fastness $\|a\| \leq \frac{\|Ta\|}{\Theta}$ and $\|b\| \leq \frac{\|Tb\|}{\Theta}$.

$$\begin{aligned} \|Tc\| &\leq \|c\|\rho_{k+1}T \leq (\|a\| + \|b\|)\rho_{k+1}T \\ &\leq \frac{\|Ta\| + \|Tb\|}{\Theta} \rho_{k+1}T \\ &\leq \frac{2\|Ta\| + \|Tc\|}{\Theta} \rho_{k+1}T, \end{aligned}$$

so that

$$\|Tc\| < \frac{2}{1 - \frac{\rho_{k+1}T}{\Theta}} \frac{\rho_{k+1}T}{\Theta} \|Ta\|.$$

Then since a isn't in the kernel then $d(\frac{Ta}{\|Ta\|}, TW) \leq \frac{\|Tc\|}{\|Ta\|} < \frac{2}{1 - \frac{\rho_{k+1}T}{\Theta}} \frac{\rho_{k+1}T}{\Theta}$. Since the choice of a was arbitrary,

$$d(TV, TW) < \frac{2}{1 - \frac{\rho_{k+1}T}{\Theta}} \frac{\rho_{k+1}T}{\Theta}.$$

□

Lemma 13. *Let $k \in \mathbb{N}$. Then there is a measurable $U : \mathcal{G}_kX \rightarrow \mathcal{G}_{k-1}X$ such that $U(V) < V$.*

Proof. Let $\{U_1, U_2, \dots\}$ be dense in $\mathcal{G}_{k-1}X$. Suppose $V \in \mathcal{G}_kX$. Let $s_0(V) = 1$ and

$$s_i(V) = \inf\{s : \sup_{v \in \mathbb{S}_{U_s}} d(v, V) < 2^{-i} \text{ and } d(U_s, U_{s_{i-1}(V)}) < 2^{1-i}\} \text{ for } i > 0.$$

In other words, $U_{s_i(V)}$ is the first member of the dense collection to fall in the nonempty open set

$$B_{2^{1-i}}(U_{s_{i-1}}) \cap \bigcup_{W \in \mathcal{G}_{k-1}X : W < V} B_{2^{-i}}(W).$$

Then U_{s_i} is Cauchy and so the pointwise limit of measurable functions $U_{s_i(V)}$ is convergent to some measurable $U(V) \in \mathcal{G}_{k-1}X$. Further, $\sup_{v \in \mathbb{S}_{U(V)}} d(v, V) \leq 2^{-i}$ for any i , so $U(V) < V$. □

Lemma 14. *Let $k \in \mathbb{N}$. Let $U \in \mathcal{M}(\mathcal{G}_kX \rightarrow \mathcal{G}_{k-1}X)$ with $U(V) < V$ as guaranteed by Lemma 13. Then there is a measurable $b : \mathcal{G}_kX \rightarrow \mathbb{S}_X$ such that $U \oplus \text{span}\{b\}$ is the identity on \mathcal{G}_kX .*

Proof. Let $\{x_1, x_2, \dots\}$ be dense in \mathbb{S}_X . Let $t_0(V) = 1$ and

$$t_i(V) = \inf\{t : d(U(V) \oplus \text{span}\{x_t\}, V) < 2^{-t} \text{ and } \|x_t - x_{t_{i-1}}\| < 2^{1-i}\} \text{ for } i > 0.$$

Again we get that x_{t_i} converges pointwise to some measurable b with the desired properties. □

Applying this result recursively we obtain the following:

Lemma 15. *Let X be a separable Banach space. Then for every $k > 0$ there is a measurable map $b : \mathcal{G}_kX \rightarrow \mathbb{S}_X^k$ such that for each $V \in \mathcal{G}_kX$,*

$$V = \text{span}\{b_i(V)\}_{i=1}^k.$$

Lemma 16. *Let $V \in \mathcal{G}^k X$ with bounded projection $\Pi : X \rightarrow V$. Let $T \in \mathcal{B}(X)$. Then $\rho_{k+l}(T) \leq \rho_l(T \circ \Pi)$.*

Proof.

$$\begin{aligned} \rho_{k+l}(T) &= \sup_{U \in \mathcal{G}_{k+l} X} g_T(U) \leq \sup_{U \in \mathcal{G}_{k+l} X} g_T(U \cap V) \\ &= \sup_{U \in \mathcal{G}_{\leq k+l, \geq l} V} g_T(U) = \sup_{U \in \mathcal{G}_l V} g_{T|_V}(U) \\ &= \sup_{U \in \mathcal{G}_l V} g_{T \circ \Pi}(U) \leq \rho_l(T \circ \Pi). \end{aligned}$$

□

Lemma 17. *Let $X = E \oplus V = E' \oplus V'$ with corresponding projections $\Pi : X \rightarrow V \in \mathcal{G}^k X$ and $\Pi' : X \rightarrow V' \in \mathcal{G}^k X$. Let $T \in \mathcal{B}(X)$ such that $\Pi' \circ T = T \circ \Pi$, and suppose further that $g_T(E) > \|T|_V\|$. Then $\rho_l(T \circ \Pi) \leq 4\|\Pi\|\|\Pi'\|\rho_{l+k}(T)$.*

Proof. Make use of the inequality $\rho_l(T \circ \Pi) \leq \|\Pi\|\rho_l(T|_V)$. Let $U \in \mathcal{G}_l V$ with $g_{T \circ \Pi}(U) > e^{-\epsilon}\rho_l(\Pi \circ T|_V) = e^{-\epsilon}\rho_l(T|_V)$. Let $x \in \mathbb{S}_{U \oplus E}$. Observe that $\|\Pi'Tx\| \leq \|\Pi'\|\|Tx\|$ and similar for $1 - \Pi'$ so that

$$\begin{aligned} \|Tx\| &\geq \max\left\{\frac{\|\Pi'Tx\|}{\|\Pi'\|}, \frac{\|(1-\Pi')Tx\|}{\|1-\Pi'\|}\right\} \geq \max\left\{\frac{g_{T\Pi}(U)\|\Pi x\|}{\|\Pi'\|}, \frac{g_T(E)\|(1-\Pi)x\|}{\|1-\Pi'\|}\right\} \\ &\geq \frac{1}{1+\|\Pi'\|} \max\{g_{T\Pi}(U)\|\Pi x\|, g_T(E)\|1 - \Pi x\|\} \\ &\geq \frac{e^{-\epsilon}\rho_l(T|_V)}{1+\|\Pi'\|} \max\{\|\Pi x\|, \|(1 - \Pi)x\|\} \geq \frac{e^{-\epsilon}\rho_l(T|_V)}{4\|\Pi'\|} \geq \frac{e^{-\epsilon}\rho_l(T \circ \Pi)}{4\|\Pi'\|\|\Pi\|}. \end{aligned}$$

We may then conclude that $g_T(U) \geq \frac{e^{-\epsilon}\rho_l(T \circ \Pi)}{4\|\Pi'\|\|\Pi\|}$ with $\epsilon > 0$ arbitrary, so that $\rho_l(T \circ \Pi) \leq 4\rho_{l+k}(T \circ \Pi)\|\Pi'\|\|\Pi\|$ as required. □

4. A BALANCED SUBADDITIVE ERGODIC THEOREM

Consider a decomposition of the interval $[a, b) = [a, c) \sqcup [c, b) \subset \mathbb{Z}$. In discrete time, where these intervals represent different intervals in which dynamics may occur. Given a cocycle $\mathcal{L} : \Omega \rightarrow \mathcal{B}(X)$, for fixed ω and each $b \geq a \in \mathbb{Z}$ we may define the map $\mathcal{L}_{a \rightarrow b}(\omega) := \mathcal{L}_{\sigma^a \omega}^{(b-a)}$, which may be thought of as the evolution rule for X from time a to time b . Under this notation it is easy to see that for any $a < c < b \in \mathbb{Z}$, $\mathcal{L}_{c \rightarrow b} \circ \mathcal{L}_{a \rightarrow c} = \mathcal{L}_{a \rightarrow b}$. The inequality $\|\mathcal{L}_{a \rightarrow b}\| \leq \|\mathcal{L}_{a \rightarrow c}\|\|\mathcal{L}_{c \rightarrow b}\|$ holds. Taking log of each side, one obtains the following triangle inequality-like bound on the growth of points in X from time a to b :

$$f_{a \rightarrow b}(\omega) := \log \|\mathcal{L}_{a \rightarrow b}\| \leq \log \|\mathcal{L}_{a \rightarrow c}\| + \log \|\mathcal{L}_{c \rightarrow b}\| = f_{a \rightarrow c}(\omega) + f_{c \rightarrow b}(\omega).$$

Definition 18. Let $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}_0} \subseteq \mathcal{M}(\Omega \rightarrow \mathbb{R})$ satisfy

$$f_{m+n}(\omega) \leq f_m(\sigma^n \omega) + f_n(\omega).$$

Then \mathcal{F} is referred to as a subadditive family of measurable functions.

The following view is useful:

Definition 19. A subadditive family $\{f_n\}_{n \in \mathbb{N}_0}$ generates a stationary subadditive process $\{f_{a \rightarrow b} : a < b, a, b \in \mathbb{Z}\}$ and vice versa via the relation

$$f_{a \rightarrow b} := f_{b-a} \circ \sigma^a.$$

A subadditive process is a collection $\{f_{a \rightarrow b}\}_{a < b \in \mathbb{Z}} \subseteq \mathcal{M}(\Omega \rightarrow \mathbb{R})$ such that for all $a < c < b \in \mathbb{Z}$,

$$f_{a \rightarrow b} \leq f_{a \rightarrow c} + f_{c \rightarrow b},$$

and the stationarity condition is

$$f_{a+l \rightarrow b+l} = f_{a \rightarrow b} \circ \sigma^l.$$

As such both notations may be interchanged as appropriate. A similar formalism is outlined in [8].

The Kingman theorem concerns subadditive families of functions, here a slight refinement when the underlying transformation is invertible will be required.

Theorem 20 (Kingman [7]). *Let $(\Omega, \sigma, \mathbb{P})$ be an ergodic system and let $\{f_n\}_{n \in \mathbb{N}} \subset L^1\Omega$ be subadditive. Then there is a constant $C \in [-\infty, \infty)$ such that*

$$\frac{1}{n}f_n(\omega) \rightarrow C = \lim_{n \rightarrow \infty} \frac{1}{n} \int f_n$$

pointwise almost everywhere.

In the case where σ is invertible, one easily obtains the following useful corollary:

Corollary 21. *Let $(\Omega, \sigma, \mathbb{P})$ be an invertible ergodic system on a Lebesgue probability space and let $\{f_n\}_{n \in \mathbb{N}} \subset L^1\Omega$ be subadditive. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n}f_n(\sigma^{-n}\omega) = \lim_{n \rightarrow \infty} \frac{1}{n}f_n(\omega)$$

pointwise almost everywhere.

Proof. Simply set $g_n = f_n \circ \sigma^{-n}$. Then g_n is subadditive with respect to σ^{-1} :

$$\begin{aligned} g_{m+n}(\omega) &= f_{m+n}(\sigma^{-(m+n)}\omega) \leq f_m(\sigma^{-(m+n)}\omega) + f_n(\sigma^{m-(m+n)}\omega) \\ &= g_m(\sigma^{-n}\omega) + g_n(\omega), \end{aligned}$$

so that certainly there exists some L such that $\frac{1}{n}f_n \circ \sigma^{-n} \rightarrow L$. It remains to check that this limit and $\frac{1}{n}f_n(\omega) \rightarrow L'$ coincide. Let $\epsilon > 0$. Then there exists an N such that

$$\mathbb{P}(\omega : \frac{1}{n}f_n(\omega) \in (L' - \epsilon, L' + \epsilon)) > \frac{1}{2}$$

and

$$\mathbb{P}(\omega : \frac{1}{n}g_n(\omega) = \frac{1}{n}f_n(\sigma^{-n}\omega) \in (L - \epsilon, L + \epsilon)) > \frac{1}{2},$$

so that the set

$$\{\omega : \frac{1}{n}f_n(\sigma^n\omega) \in (L - \epsilon, L + \epsilon)\} \cap \{\omega : \frac{1}{n}f_n(\omega) \in (L' - \epsilon, L' + \epsilon)\}$$

has positive measure, whence $|L - L'| < 2\epsilon$. ϵ was arbitrary though, so that $L = L'$. □

Kingman's original result may be modified in a few directions. One may very rapidly obtain the following, which may be found in [2]:

Lemma 22. *Let $\{f_n\}_{n \in \mathbb{N}} \subseteq L^1\Omega$ be a subadditive family of functions. Then*

$$\frac{1}{n}f_{n \rightarrow 2n} \rightarrow C = \lim_{n \rightarrow \infty} \frac{1}{n} \int f_n.$$

Proof. The lower bound is immediately clear, since

$$\frac{1}{n} f_{n \rightarrow 2n}(\omega) \geq \frac{1}{n} (f_{0 \rightarrow 2n} - f_{0 \rightarrow n}) \rightarrow 2C - C = C \text{ as } n \rightarrow \infty.$$

On the other hand, let $\epsilon > 0$, then since $\frac{1}{n} \int f_n \rightarrow C$ there exists some N such that for all $n \geq N$, $\frac{1}{n} \int f_n \leq C + \epsilon$. Given some fixed $n \geq N$ and some $j < N$, set $j = \lfloor \frac{n}{N} \rfloor$ and break up the interval $[0, n) = [n, n + j) \sqcup [n + j, n + j + N), \dots \sqcup [n + j + (t - 1)N, n + j + \lfloor \frac{n}{N} \rfloor N) \sqcup [n + j + \lfloor \frac{n}{N} \rfloor N, 2n)$. Applying subadditivity in this manner

$$\begin{aligned} f_{n \rightarrow 2n} &\leq f_{n \rightarrow n+j} + \sum_{i=0}^t f_{n+j+iN \rightarrow n+j+(i+1)N}(\omega) + f_{n+j+tN \rightarrow n} \\ &\leq S_N |f_1| \circ \sigma^n(\omega) + \sum_{i=0}^t f_N \circ \sigma^{n+j+iN}(\omega) + S_N |f_1| \circ \sigma^{2n-N}(\omega). \end{aligned}$$

The above is valid for all $j \in [0, N)$, so averaging and dividing by nN we obtain

$$\begin{aligned} \frac{1}{n} f_{n \rightarrow 2n} &\leq \frac{1}{n} (S_N |f_1| \circ \sigma^n + S_N |f_1| \circ \sigma^{2n-N}) + \frac{1}{nN} S_N \int f_N \circ \sigma^n \\ &\leq 2\epsilon + \frac{1}{Nn} n(N(C + \epsilon)) \end{aligned}$$

for sufficiently large n . Since ϵ was arbitrary, the result is proven. □

A similar result for $f_{-n \rightarrow n}$ will be crucial to the main result. The proof here uses Lemma 23, which is a simplified version of result 3.9 in [15]:

Lemma 23 (Backward Vitali). *Let $(\Omega, \mathbb{P}, \sigma)$ be an invertible ergodic system on a Lebesgue probability space and $\epsilon > 0$. Suppose that there is a sequence of integer valued functions $j_k \in \mathcal{M}(\Omega \rightarrow \mathbb{N}_0)$ such that $j_k(\omega) \rightarrow \infty$ as $k \rightarrow \infty$. Then there is a measurable $A \subseteq \Omega$ and a measurable $j \in \mathcal{M}(A \rightarrow \mathbb{N}_0)$ such that for every $\omega \in A$ there is a k with $j_\omega = j_k(\omega)$ and writing $I_\omega = \{\sigma^i \omega : i \in \{-j_\omega, \dots, j_\omega\}\}$, the I_ω are disjoint and*

$$\mathbb{P}\left(\bigcup_{\omega \in A} I_\omega\right) > 1 - \epsilon.$$

Theorem 24 (Kingman’s theorem for balanced intervals). *Let $(\Omega, \sigma, \mathbb{P})$ be an ergodic system with invertible base, and let $(f_n)_{n \in \mathbb{N}}$ be subadditive sequence of measurable functions on Ω . Then*

$$\frac{1}{2n} f_{2n}(\sigma^{-n} \omega) \rightarrow C = \lim_{n \rightarrow \infty} \frac{1}{n} \int f_n$$

pointwise almost everywhere.

Proof. Kingman’s theorem immediately yields an upper bound: since

$$\frac{1}{2n} f_{2n}(\sigma^{-n} \omega) \leq \frac{1}{2n} (f_n(\sigma^{-n} \omega) + f_n(\omega)) \rightarrow \frac{1}{2} C + \frac{1}{2} C$$

it is clear that

$$\bar{C}_\omega = \limsup_{n \rightarrow \infty} \frac{1}{2n} f_{2n}(\sigma^{-n} \omega) \leq C.$$

If $C = -\infty$ then there is nothing more to show. Otherwise,

$$\begin{aligned} C_{\sigma^{-1}\omega} &= \liminf_{n \rightarrow \infty} \frac{1}{2^n} f_{2n} \circ \sigma^{-n}(\sigma^{-1}\omega) \\ &\geq \liminf_{n \rightarrow \infty} \left(\frac{1}{2^n} (f_{2(n+1)} \circ \sigma^{-(n+1)}(\omega) - f_2 \circ \sigma^{n-1}(\omega)) \right) \\ &\geq \liminf_{n \rightarrow \infty} \frac{1}{2^n} f_{2n} \circ \sigma^{-n}(\omega) - \limsup f_2 \circ \sigma^{n-1}(\omega) \\ &= \underline{C}_\omega - 0, \end{aligned}$$

whence $C \leq \bar{C}$ is constant. Fix $\epsilon > 0$. It is sufficient to show that $C - \epsilon \leq \Theta(C, \epsilon)$ where $\Theta(\underline{C}, \epsilon) \rightarrow \underline{C}$ as $\epsilon \rightarrow 0$. The set of j such that $f_{-j \rightarrow j}$ is small is infinite, so denote the ordered elements

$$\{j : f_{-j \rightarrow j}(\omega) < 2j(C + \epsilon)\} = \{j_k(\omega) : k \in \mathbb{N}\}.$$

Since f is integrable, it is uniformly integrable and so there exists a $\delta > 0$ such that if $\mathbb{P}(A) < \delta$ then $\int_A f < \epsilon$. Apply Backward Vitali to the j_i 's and the $-j_i$'s with parameter $\epsilon' := \min\{\epsilon, \delta\}$ to obtain an $A \subseteq \Omega$ and a $j : A \rightarrow \mathbb{N}$ according to Theorem 23 such that writing $B = \Omega \setminus \bigcup_{\omega \in A} I_\omega$, we have $\mathbb{P}(B) < \delta$ and so

$$\int_{\Omega \setminus \bigcup_{\omega \in A} I_\omega} f_1^+ < \epsilon.$$

While every $j(\omega) = j_i(\omega)$, for some i so that for every $\omega \in A$,

$$f_{-j(\omega) \rightarrow j(\omega)}(\omega) \leq 2j(C + \epsilon).$$

Write $\Lambda_\pm = \{\sigma^{\pm j_i(\omega)}(\omega) : \omega \in A\}$. In addition we may define measurable maps $r_\pm \in \mathcal{M}(\Omega \rightarrow \mathbb{N})$ by

$$\begin{aligned} r_-(\omega) &= \inf\{t \in \mathbb{N}_0 : \sigma^t \omega \in \Lambda_-\}, \\ r_+(\omega) &= \inf\{t > r_-(\omega) \in \mathbb{N}_0 : \sigma^t \omega \in \Lambda_+\}. \end{aligned}$$

r_- is the earliest nonnegative time such that $\sigma^{r_-(\omega)}\omega$ is the start of a period at growth rate guaranteed close to \underline{C} , r_+ by contrast seeks the first time in the past which was the end of such an interval. Each is almost surely finite since $\mathbb{P}(\Lambda_\pm) = \mathbb{P}(A) > 0$.

The idea of the next step is that $[0, n)$ may be measurably broken up into intervals of slow growth and with small gaps inbetween. Set $b_0 = 0$ and recursively iterate along the orbit for $i > 0$:

$$\begin{aligned} a_i(\omega) &= r_-(\sigma^{b_{i-1}(\omega)}\omega), \\ b_i(\omega) &= r_+(\sigma^{b_{i-1}(\omega)}\omega). \end{aligned}$$

For any $m \in \mathbb{N}_0$ we may set $t_m(\omega) = \max\{t : b_t(\omega) \leq m\}$. Since j and r_\pm are measurable and the $t_m(\omega) \rightarrow \infty$ as $m \rightarrow \infty$, it is possible to pick an N such that for all $n \geq N$, $\mathbb{P}(j \geq N) < \frac{1}{4}\epsilon'$, $\mathbb{P}(t_n = 0) < \frac{1}{4}\epsilon'$ and $\mathbb{P}(r_\pm \geq N) < \frac{1}{4}\epsilon'$. Let $n \geq N$. The orbit of ω is considered over the following intervals of times:

$$\begin{aligned} [0, n] &= [0, a_1(\omega)) \cup [a_1(\omega), b_1(\omega)) \cup \\ &\quad [b_1(\omega), a_2(\omega)) \cup [a_2(\omega), b_2(\omega)) \cup \\ &\quad \dots \\ &\quad [b_{t_n(\omega)-1}(\omega), a_{t_n(\omega)}(\omega)) \cup [a_{t_n(\omega)}(\omega), b_{t_n(\omega)}(\omega)) \cup \\ &\quad [b_{t_n(\omega)}(\omega), n). \end{aligned}$$

Time $a_i \rightarrow b_i$ is guaranteed to have low growth:

$$f_{a_i(\omega) \rightarrow b_i(\omega)}(\omega) \leq (b_i(\omega) - a_i(\omega))(C_\omega + \epsilon),$$

and in addition there is a good chance that the gaps between the a_i 's and the b_i 's won't be too large:

$$\mathbb{P}(a_1(\omega) \geq N) = \mathbb{P}(r_-(\omega) \geq N) < \frac{1}{4}\epsilon'.$$

Since r_- seeks the endpoints of the periods of guaranteed growth, it holds that either $r_-(\sigma^n\omega) = n - b_{t_n(\omega)}(\omega)$ or $t_n(\omega) = 0$. Therefore,

$$\begin{aligned} \mathbb{P}(n - b_{t_n(\omega)}(\omega) \geq N) &\leq \mathbb{P}(r_-(\sigma^n\omega) \geq N) + \mathbb{P}(t_n(\omega) = 0) \\ &< \mathbb{P}(r_- \geq N) + \frac{1}{4}\epsilon' < \frac{1}{2}\epsilon'. \end{aligned}$$

Set

$$\Lambda = \{a_1 \geq N \text{ or } n - b_{t_n}(\omega) \geq N\},$$

so that it is immediate from above that $\mathbb{P}(\Lambda) < \epsilon'$. Given some measurable g write $S_p g = \sum_{i=0}^{p-1} g \circ \sigma^i$. We then repeatedly apply subadditivity to $f_{0 \rightarrow n}$:

$$\begin{aligned} f_n(\omega) &= f_{0 \rightarrow n}(\omega) \\ &\leq (f_{0 \rightarrow a_1}(\omega) + f_{a_1(\omega) \rightarrow b_1(\omega)} + \cdots + f_{a_{t_n}(\omega) \rightarrow b_{t_n}(\omega)} + f_{b_{t_n}(\omega) \rightarrow n})(\omega) \\ &\leq \sum_{i=1}^{t_n(\omega)} f_{a_i(\omega) \rightarrow b_i(\omega)}(\omega) + \sum_{i=1}^{t_n(\omega)} f_{b_{i-1}(\omega) \rightarrow a_i(\omega)}(\omega) + f_{b_{t_n}(\omega) \rightarrow n}(\omega) \\ &\leq \sum_{i=1}^{t_n(\omega)} (b_i(\omega) - a_i(\omega))(C + \epsilon) + \\ &\quad \sum_{i=a_1(\omega)}^{b_{t_n}(\omega)} 1_B(\sigma^i\omega) f_1 \circ \sigma^i + f_{b_{t_n}(\omega)(\omega) \rightarrow n}(\omega). \end{aligned}$$

In the case where $t_n(\omega) = 0$ the first two terms are empty sums. Outside Λ it is guaranteed that $[0, a_1(\omega)] \subseteq [0, M]$ and $[b_{t_n}(\omega)(\omega), n] \subseteq [n - M, n]$. Therefore

$$\begin{aligned} f_n(\omega) &\leq \sum_{i=0}^{n-1} (C + \epsilon) 1_{B^c}(\sigma^i\omega) + \sum_{i=0}^{n-1} 1_B(\sigma^i\omega) f_1(\sigma^i\omega) + \\ &\quad \sum_{i=0}^M (|f_1|(\sigma^i\omega) + |C| + \epsilon) + \sum_{i=n-M}^{n-1} (|f_1|(\sigma^i\omega) + |C| + \epsilon) + \\ &\quad 1_\Lambda(\omega) S_n |f_1|(\omega) \\ &= (C + \epsilon) S_n 1_{B^c}(\omega) + S_n (f_1 1_B)(\omega) + \\ &\quad S_M |f_1|(\omega) + S_M |f_1 \circ \sigma^{n-M}|(\omega) + 1_\Lambda(\omega) S_n |f_1|(\omega). \end{aligned}$$

It remains to check that each of these terms are small enough to yield the result when we divide through by n and integrate:

$$\begin{aligned} \frac{1}{n} \int (C + \epsilon)S_n 1_{B^c}(\omega) + S_n(1_B f_1)(\omega) &= \int_{\Omega} ((C + \epsilon)1_{B^c} + f_1 1_B) \\ &= (C + \epsilon)\mathbb{P}(B^c) + \int_B |f_1| \\ &\leq \max\{(C + \epsilon)(1 - \epsilon), C + \epsilon\} + \epsilon \\ &\leq (C + \epsilon) + \epsilon(|C| + \epsilon) + \epsilon, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{n} \int (S_M |f_1|(\omega) + S_M |f_1 \circ \sigma^{n-M}|(\omega) + 1_{\Lambda}(\omega)S_n |f_1|(\omega)) \\ &= \frac{1}{n} \int (M |f_1| + M |f_1|) + \frac{1}{n} \int_{\Lambda} S_n |f_1| \\ &\leq \frac{2M\epsilon}{n} + \frac{1}{n} \sum_{i=0}^{n-1} \int_{\sigma^i \Lambda} |f_1| \\ &\leq \frac{2M\epsilon}{n} + \epsilon. \end{aligned}$$

Putting these together we obtain

$$\begin{aligned} (C - \epsilon) &\leq \frac{1}{n} \int_{\Omega} f_n \leq (C + \epsilon) + \epsilon(|C| + \epsilon) + \epsilon + \frac{2M\epsilon}{n} + \epsilon \\ &\rightarrow C + (3 + |C|)\epsilon \text{ as } n \rightarrow \infty. \end{aligned}$$

Letting $\epsilon \rightarrow 0$ we obtain $C \leq \underline{C}$ as required. □

Even if a family f_n isn't subadditive, if slightly weaker inequalities can be written down these results can still be applied. Quantities that can be written as the differences of families of such functions also have such quantities converge.

Lemma 25. *Let sequences of measurable f_n and g_n satisfy the following weak subadditivity condition:*

$$f_{m+n}(\omega) \leq f_m(\sigma^n \omega) + f_n(\omega) + C, g_{m+n}(\omega) \leq g_m(\sigma^n \omega) + g_n(\omega) + D.$$

Then writing $h_n = f_n + t g_n$,

$$\frac{1}{n} h_n \circ \sigma^n, \frac{1}{2n} h_{2n} \circ \sigma^{-n}, \frac{1}{n} h_n \rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \int f_n + t \lim_{n \rightarrow \infty} \frac{1}{n} \int g_n.$$

Proof. The map $f'_n = f_n - nC$ can be seen to be subadditive by subbing $f'_n + nC$ into the inequality f_n satisfies. Then $\frac{1}{n} f'_n$ tends to some L , and thus $\frac{1}{n} f(n)$ to $L + C$. Additivity and scalar multiplicity follow as standard properties of limits. □

The growth statistics of a given cocycle can then be checked to converge.

Definition 26. Given a random linear dynamical system $\mathcal{R} = (\Omega, \sigma, X, \mathcal{L})$, write

$$\lambda_{\omega}(x) = \limsup_n \frac{1}{n} \log \|\mathcal{L}_{\omega}^{(n)} x\| \in [-\infty, \infty).$$

Write m_i for the multiplicity of λ_i , i.e., m_i is defined as the largest $m \in \mathbb{N}$ such that

$$\mu_{m_1 + \dots + m_{i-1} + m} = \lambda_i.$$

Lemma 27. *The quantities λ_i, μ_i and ν are almost everywhere constants. Further, λ_2 exists if and only if $\lambda_1 > \nu$.*

Proof. Consider the following choices of sequences:

$$\begin{aligned} f_n^k(\omega) &= \log \rho_k(\mathcal{L}_\omega^{(n)}), \\ g_n(\omega) &= \log \|\mathcal{L}_\omega^{(n)}\|_c. \end{aligned}$$

In [4] ρ_k is defined (written there as F_k) and it is shown that there exist numbers C_k such that for any operators $T, S \in \mathcal{B}(X)$,

$$\prod_{i=1}^k \rho_i(S \circ T) \leq C_k \prod_{i=1}^k \rho_i(S) \rho_i(T),$$

whence the existence of the desired limits are guaranteed by noting that

$$f_n^k(\omega) = \log \prod_{i=1}^k \rho_i(\mathcal{L}_\omega^{(n)}) - \log \prod_{i=1}^{k-1} \rho_i(\mathcal{L}_\omega^{(n)})$$

converges with an application of 25. The second statement is a consequence of the definitions $\nu = \lim_{n \rightarrow \infty} \mu_n$ and $\lambda_2 = \mu_{\min\{t: \mu_t < \mu_1\}}$. \square

5. DECOMPOSING A COCYCLE

The tools obtained thus far are now used to decompose quasicompact cocycles.

Lemma 28. *Let $\mathcal{R} = (\Omega, \sigma, X, \mathcal{L})$ be a strongly measurable random linear dynamical system with ergodic base on a separable Banach space. Suppose that \mathcal{L} satisfies the quasicompactness condition $\nu < \lambda_1$. Then there is a unique measurable choice of fast space $E : \Omega \rightarrow \mathcal{G}_{m_1} X$ for which the following hold almost surely:*

- *equivariance:* $\mathcal{L}_\omega E(\omega) = E(\sigma\omega)$
- $\lambda(x) = \lambda_1$ for every $x \in E(\omega) \setminus \{0\}$
- $\lim_{n \rightarrow \infty} \frac{1}{n} \log g(\mathcal{L}_\omega^{(n)}, E(\omega)) = \lambda_1$.

Proof. Since X is separable, by Lemma 5 $\mathcal{G}_{m_1} X$ is separable: choose some dense $\{E_i\}_{i \in \mathbb{N}} \subseteq \mathcal{G}_{m_1} X$. Let $\epsilon \in (0, \frac{1}{5}(\lambda_1 - \lambda_2))$. The sets

$$A_i = \{\omega \in \Omega : g(\mathcal{L}_{\sigma^{-n}\omega}^{(2n)}, E_i) > e^{-\epsilon} \rho_k(\mathcal{L}_{\sigma^{-n}\omega}^{(2n)})\}$$

are measurable and cover Ω , since for fixed ω , density of the E_i 's and continuity of $g_{\mathcal{L}_{\sigma^{-n}\omega}^{(2n)}}$ means that we may find an i with $g(\mathcal{L}_{\sigma^{-n}\omega}^{(2n)}, E_i)$ as close to $\rho_k(\mathcal{L}_{\sigma^{-n}\omega}^{(2n)})$ as we please. Then Lemma 9 above provides measurable functions

$$\begin{aligned} \iota^{(n)}(\omega) &= \inf\{i \in \mathbb{N} : g(\mathcal{L}_{\sigma^{-n}\omega}^{(2n)}, E_i) > e^{-\epsilon} \rho_k(\mathcal{L}_{\sigma^{-n}\omega}^{(2n)})\}, \\ \tilde{E}^{(n)}(\omega) &:= E_{\iota^{(n)}(\omega)}, \end{aligned}$$

and the pushforward

$$E^{(n)}(\omega) = \mathcal{L}_{\sigma^{-n}\omega}^{(n)} \tilde{E}^{(n)}(\omega)$$

is also then measurable by Lemma 5.

First, we establish that this sequence is Cauchy, and therefore convergent, to a family of spaces $E \in \mathcal{M}(\Omega \rightarrow \mathcal{G}_{m_1} X)$.

For almost every $\omega \in \Omega$ the fastest m_1 dimensional growth rate

$$\begin{aligned} \frac{1}{n} \log \rho_k(\mathcal{L}_{\sigma^{-n}\omega}^{(n)}) &\rightarrow \lambda_1 \text{ by Kingman's Theorem 20,} \\ \frac{1}{n} \log \rho_k(\mathcal{L}_{\omega}^{(n)}) &\rightarrow \lambda_1 \text{ again by Theorem 20 applied to } \sigma^{-1}, \\ \frac{1}{2n} \log \rho_k(\mathcal{L}_{\sigma^{-n}\omega}^{(2n)}) &\rightarrow \lambda_1 \text{ by the balanced version, Theorem 24,} \end{aligned}$$

and

$$\rho_k(\mathcal{L}_{\sigma^{-n}\omega}^{(n)}) \geq g_{\mathcal{L}_{\sigma^{-n}\omega}^{(n)}}(\tilde{E}^{(n)}(\omega)) > e^{-\epsilon} \rho_k(\mathcal{L}_{\sigma^{-n}\omega}^{(n)}).$$

Thus for each ω in this full measure set we may choose M_ω such that for $n \geq M_\omega$ we can usefully estimate growth under \mathcal{L} in a few cases:

$$\begin{aligned} \|\mathcal{L}_{\sigma^{-n}\omega}^{(n)}\|, \|\mathcal{L}_{\omega}^{(n)}\| &\in (e^{n(\lambda_1-\epsilon)}, e^{n(\lambda_1+\epsilon)}), \\ g(\mathcal{L}_{\sigma^{-n}\omega}^{(2n)}, \tilde{E}^{(n)}(\omega)) &\in (e^{2n(\lambda_1-\epsilon)}, e^{2n(\lambda_1+\epsilon)}) \end{aligned}$$

and

$$\|\mathcal{L}_{\sigma^{-(n+1)}\omega}\| < e^{n\epsilon}.$$

Applying the inequalities in Lemma 7 then,

$$g(\mathcal{L}_{\sigma^{-n}\omega}^{(n)}, \tilde{E}^{(n)}(\omega)) \geq \frac{g(\mathcal{L}_{\sigma^{-n}\omega}^{(2n)}, \tilde{E}^{(n)}(\omega))}{\|\mathcal{L}_{\omega}^{(n)}\|} \geq e^{n(\lambda_1-3\epsilon)}$$

and

$$g(\mathcal{L}_{\omega}^{(n)}, \mathcal{L}_{\sigma^{-n}\omega}^{(n)} \tilde{E}^{(n)}(\omega)) = g(\mathcal{L}_{\omega}^{(n)}, E^{(n)}(\omega)) \geq \frac{g(\mathcal{L}_{\sigma^{-n}\omega}^{(2n)}, \tilde{E}^{(n)}(\omega))}{\|\mathcal{L}_{\sigma^{-n}\omega}^{(n)}\|} \geq e^{n(\lambda_1-3\epsilon)}.$$

In addition, $\mathcal{L}_{\sigma^{-(n+1)}\omega} \tilde{E}^{(n+1)}(\omega)$ is also guaranteed to have fast growth under $\mathcal{L}_{\sigma^{-n}\omega}^{(n)}$:

$$\begin{aligned} g(\mathcal{L}_{\sigma^{-n}\omega}^{(n)}, \mathcal{L}_{\sigma^{-(n+1)}\omega} \tilde{E}^{(n+1)}(\omega)) &\geq \frac{g(\mathcal{L}_{\sigma^{-(n+1)}\omega}^{(n+1)}, \tilde{E}^{(n+1)}(\omega))}{\|\mathcal{L}_{\sigma^{-(n+1)}\omega}\|} \\ &> e^{(n+1)(\lambda_1-3\epsilon)} e^{-n\epsilon} \geq e^{n(\lambda_1-4\epsilon)}. \end{aligned}$$

$E^{(n)}(\omega)$ consists then of the image of vectors that were fast from time $-n$ to 0, and will grow fast from time 0 to n . Then by Lemma 12 with $\Theta = e^{n(\lambda_1-4\epsilon)}$ we have

$$\begin{aligned} d(E^{(n)}(\omega), E^{(n+1)}(\omega)) &= d\left(\mathcal{L}_{\sigma^{-n}\omega}^{(n)} \tilde{E}^{(n)}(\omega), \mathcal{L}_{\sigma^{-n}\omega}^{(n)} (\mathcal{L}_{\sigma^{-(n+1)}\omega} \tilde{E}^{(n+1)}(\omega))\right) \\ &< 4 \frac{e^{n(\lambda_2+\epsilon)}}{e^{n(\lambda_1-4\epsilon)}} < e^{-n(\lambda_1-\lambda_2-5\epsilon)}. \end{aligned}$$

Thus $E^{(n)}(\omega)$ is Cauchy and convergent since $\mathcal{G}_{m_1}X$ is complete, say to $E(\omega)$.

To prove equivariance, observe that for $n \geq \max\{M_\omega, M_{\sigma\omega}\}$, we find $\tilde{E}^{(n+1)}(\sigma\omega)$ is fast under $\mathcal{L}_{\sigma^{-n}\omega}^{(n)}$:

$$g(\mathcal{L}_{\sigma^{-n}\omega}^{(n)}, \tilde{E}^{(n+1)}(\sigma\omega)) \geq \frac{g(\mathcal{L}_{\sigma^{-n}\omega}^{(n+1)}, E^{n+1}(\sigma\omega))}{\|\mathcal{L}_{\omega}^{(n)}\|} \geq e^{(n+1)(\lambda_1-\epsilon)} e^{-n\epsilon} > e^{n(\lambda_1-2\epsilon)}.$$

Then, once again, by closeness of images of fast spaces,

$$d(\mathcal{L}_{\sigma^{-n}\omega}^{(n)} \tilde{E}^{(n+1)}(\sigma\omega), E^{(n)}(\omega)) < e^{-n(\lambda_1-\lambda_2-3\epsilon)},$$

whence

$$\begin{aligned} d(E^{(n+1)}(\sigma\omega), \mathcal{L}_\omega E^{(n)}(\omega)) &= d(\mathcal{L}_{\sigma^{-(n+1)}\omega}^{(n+1)} \tilde{E}^{(n+1)}(\sigma\omega), \mathcal{L}_\omega E^{(n)}(\omega)) \\ &\leq e^{n\epsilon} d(\mathcal{L}_{\sigma^{-(n+1)}\omega}^{(n)} \tilde{E}^{(n+1)}(\sigma\omega), E^{(n)}(\omega)) \\ &< e^{-n(\lambda_1 - \lambda_2 - 4\epsilon)}, \end{aligned}$$

so that $\mathcal{L}_\omega E(\omega) = \lim_{n \rightarrow \infty} \mathcal{L}_\omega E^{(n)}(\omega) = E(\sigma\omega)$.

To check that $E(\omega)$ is fast, choose $x \in \mathbb{S}_{E(\omega)}$. Then since for $n \geq M_\omega$ we have $d(E^{(n)}(\omega), E(\omega)) < e^{-n(\lambda_1 - \lambda_2 - \epsilon)}$, for each such n we may choose an $x_n \in \mathbb{S}_{E^{(n)}(\omega)}$ with $\|x - x_n\| < e^{-n(\lambda_1 - \lambda_2 - \epsilon)}$. Since $x_n \in E^{(n)}(\omega)$ and $g(\mathcal{L}_\omega^{(n)}, E^{(n)}(\omega)) \geq e^{n(\lambda_1 - 3\epsilon)}$ it then follows that,

$$\begin{aligned} \|\mathcal{L}_\omega^{(n)} x\| &\geq \|\mathcal{L}_\omega^{(n)} x_n\| - \|\mathcal{L}_\omega^{(n)}(x - x_n)\| \\ &\geq e^{n(\lambda_1 - 3\epsilon)} - e^{-n(\lambda_1 - \lambda_2 - \epsilon)} \|\mathcal{L}_\omega^{(n)}\| \\ &\implies \|\mathcal{L}_\omega^{(n)} x\| \geq \frac{1}{2} e^{n(\lambda_1 - \epsilon)}. \end{aligned}$$

Thus we may conclude that as well as being equivariant, $E(\omega)$ is fast for all sufficiently large n ; since the choice of x was arbitrary the growth is uniform:

$$g(\mathcal{L}_\omega^{(n)}, E(\omega)) > e^{n(\lambda_1 - \epsilon)}.$$

On the other hand, for n sufficiently large we also have

$$g(\mathcal{L}_\omega^{(n)}, E(\omega)) \leq \rho_k(\mathcal{L}_\omega^{(m)}) < e^{n(\lambda_1 + \epsilon)},$$

whence $\frac{1}{n} \log g(\mathcal{L}_\omega^{(n)}, E(\omega)) \rightarrow \lambda_1$. Finally, we check uniqueness: Suppose that $E(\omega)$ and $E'(\omega)$ are both equivariant and fast, so that for every ω there is some N such that for $n \geq N$,

$$g_{\mathcal{L}_\omega^{(n)}}(E(\omega)), g_{\mathcal{L}_\omega^{(n)}}(E'(\omega)) > e^{n(\lambda_1 - \epsilon)}.$$

Define $\varphi \in \mathcal{M}(\Omega \rightarrow [0, 1])$ by $\varphi(\omega) = d(E(\omega), E'(\omega))$. Applying Lemma 12, for almost every ω we have

$$\begin{aligned} \varphi(\sigma^n \omega) &= d(E(\sigma^n \omega), E'(\sigma^n \omega)) = d(\mathcal{L}_\omega^{(n)} E(\omega), \mathcal{L}_\omega^{(n)} E'(\omega)) \\ &< 4 \frac{\rho_{k+1} \mathcal{L}_\omega^{(n)}}{e^{n(\lambda_1 - \epsilon)}} \rightarrow 0. \end{aligned}$$

φ tends to zero along all orbits. Therefore the sets $\{\varphi(\omega) > \epsilon\}$ all have measure zero, whence φ vanishes almost everywhere and $E = E'$. □

Lemma 29 provides the top fast space for a general quasicompact cocycle:

Lemma 29. *Let $\mathcal{R} = (\Omega, \sigma, X, \mathcal{L})$ be a quasicompact, semi-invertible random linear dynamical system. Then there exists a forward-equivariant decomposition $X = E(\omega) \oplus V(\omega)$, where $V : \Omega \rightarrow \mathcal{G}^k X$ and the corresponding projection is a strongly measurable $\Pi : \Omega \rightarrow \mathcal{B}(X)$. V is a slow-growing space: $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}|_{V(\omega)}\| \rightarrow \lambda_2$ almost surely. Finally, Π_ω is tempered.*

Proof. By Lemma 15 we may choose a measurable family of bases

$$(b_i)_{i=1}^k : \mathcal{G}_{m_1} X \rightarrow \mathbb{S}_X^k.$$

Write $v_i(\omega) = b_i(E(\omega))$ which is itself then measurable. Let $\{(q_{ij})_{i=1}^k\}_{j \in \mathbb{N}}$ be dense in \mathbb{R}^k . Let $T_j \in \mathcal{M}(\Omega \times X \rightarrow X)$ be defined by

$$T_j(\omega, x) = x - \sum_{i=1}^k q_{ij} v_i(\omega),$$

so that for each $\omega \in \Omega$, the collection $\{T(\omega, \cdot)\}$ is dense in translations of X by elements of $E(\omega)$. Let $\epsilon < \frac{1}{2}(\lambda_1 - \lambda_2)$. Define $\iota_n \in \mathcal{M}(X \setminus E(\omega) \rightarrow \mathbb{N})$ for $n \in \mathbb{N}$ by

$$\iota_n(\omega, x) = \inf\{j \in \mathbb{N} : \|\mathcal{L}_\omega^{(n)} T_j x\| \leq e^\epsilon \rho_{k+1}(\mathcal{L}_\omega^{(n)}) \|T_j x\|\}.$$

A sequence of nonlinear maps $\Pi^{(n)} \in \mathcal{M}(X \rightarrow X)$ may then be defined piecewise by

$$\begin{aligned} \Pi_\omega^{(n)}(x) &= \begin{cases} 0, & \text{if } x \in E(\omega), \\ T_{\iota_n(\omega, x)} x & \text{otherwise,} \end{cases} \quad \text{and} \\ P_\omega^{(n)}(x) &= x - \Pi_\omega^{(n)}(x). \end{aligned}$$

To see that ι is finite on $X \setminus E(\omega)$, first note that the set

$$\{T_j x : j \in \mathbb{N}\}$$

is dense in the set $S = E(\omega) + x$. The set

$$U = \{y \in E(\omega) \oplus \text{span}\{x\} : \|\mathcal{L}_\omega^{(n)} y\| < e^{\frac{1}{2}\epsilon} \rho_{k+1}(\mathcal{L}_\omega^{(n)}) \|y\|\}$$

is open and scale invariant - for every $\theta \neq 0$ we have $\theta U = U$, so

$$U' = (E(\omega) + x) \cap U$$

is open and nonempty in S , whence

$$U' \cap \{T_j x\}_j \neq \emptyset,$$

so that an $\iota_n(\omega, x)$ may be found in finite time. To see that ι_n is measurable, note that

$$\iota_n^{-1}\{1, \dots, m\} = \bigcup_{i=1}^m \{(\omega, x) : \|\mathcal{L}_\omega^{(n)} T_i x\| \leq e^\epsilon \rho_{k+1}(\mathcal{L}_\omega^{(n)}) \|\mathcal{L}_\omega^{(n)} T_i x\|\}$$

Let $x \in \mathbb{S}_X$. Immediately from the definition, $P_\omega^{(n)} x \in E(\omega)$. By convergence of the $\frac{1}{n} \log \rho_k$ s, for all n greater than or equal to some N_ω ,

$$\begin{aligned} \|\mathcal{L}_\omega^{(n+1)} \Pi_\omega^{(n)}(x)\| &\leq e^{(n+1)(\lambda_2 + \epsilon)} \|\Pi_\omega^{(n)} x\|, \\ \|\mathcal{L}_\omega^{(n+1)} \Pi_\omega^{(n+1)}(x)\| &\leq e^{(n+1)(\lambda_2 + \epsilon)} \|\Pi_\omega^{(n)} x\|, \\ \|\mathcal{L}_{\sigma^n \omega}\| &< e^{n\epsilon} \quad \text{and} \\ g_{\mathcal{L}_\omega^{(n)}}(E(\omega)) &\geq e^{n(\lambda_1 - \epsilon)}. \end{aligned}$$

In addition there is an easy bound independent of $x \in \mathbb{S}_X$ on $\Pi_\omega^{(n)}$:

$$\begin{aligned} \|\Pi_\omega^{(n)} x\| &\leq \|x\| + \|P_\omega^{(n)} x\| \\ &\leq 1 + e^{-n(\lambda_1 - \epsilon)} \|\mathcal{L}_\omega^{(n)} P_\omega^{(N)} x\| \\ &\leq 1 + e^{-n(\lambda_1 - \epsilon)} (\|\mathcal{L}_\omega^{(n)} x\| + \|\mathcal{L}_\omega^{(n)} \Pi_\omega^{(n)} x\|) \\ &\leq 1 + e^{-n(\lambda_1 - \epsilon)} (e^{n(\lambda_1 + \epsilon)} + e^{n(\lambda_2 + \epsilon)} \|\Pi_\omega^{(n)} x\|), \end{aligned}$$

which rearranged yields

$$\|\Pi_\omega^{(n)}x\| \leq \frac{1 + e^{2n\epsilon}}{1 - e^{-n(\lambda_1 - \lambda_2 - 2\epsilon)}} \leq e^{3n\epsilon}.$$

Consider differences between successive approximate slow components:

$$\begin{aligned} & \|\Pi_\omega^{(n+1)}(x) - \Pi_\omega^{(n)}(x)\| \\ &= \|P_\omega^{(n+1)}(x) - P_\omega^{(n)}(x)\| \\ &\leq \frac{\|\mathcal{L}_\omega^{(n+1)}(P_\omega^{(n+1)}x - P_\omega^{(n)}(x))\|}{g_{\mathcal{L}_\omega^{(n)}}(E(\omega))} \\ &= \frac{\|\mathcal{L}_\omega^{(n+1)}(\Pi_\omega^{(n+1)}x - \Pi_\omega^{(n)}(x))\|}{g_{\mathcal{L}_\omega^{(n)}}(E(\omega))} \\ &\leq e^{-(n+1)(\lambda_1 - \epsilon)} (\|\mathcal{L}_\omega^{(n+1)}\Pi_\omega^{(n)}(x)\| + \|\mathcal{L}_\omega^{(n+1)}\Pi_\omega^{(n+1)}(x)\|) \\ &\leq e^{-(n+1)(\lambda_1 - \epsilon)} (\|\mathcal{L}_{\sigma^n\omega}\| \|\mathcal{L}_\omega^{(n)}\Pi_\omega^{(n)}(x)\| + e^{n(\lambda_2 + \epsilon)} \|\Pi_\omega^{(n+1)}(x)\|) \\ &\leq e^{-(n+1)(\lambda_1 - \lambda_2 - 3\epsilon)} (e^{3n\epsilon} + e^{3(n+1)\epsilon}) \\ &\leq e^{-(n+1)(\lambda_1 - \lambda_2 - 7\epsilon)}, \end{aligned}$$

whence gaps between subsequent points decay exponentially, so that $(\Pi_\omega^{(n)}(x))_{n \in \mathbb{N}}$ forms a Cauchy and thus convergent sequence. Set

$$\Pi_\omega(x) = \lim_{n \rightarrow \infty} \Pi_\omega^{(n)}(x).$$

Note that $\Pi_\omega \mathbb{S}_X$ is then bounded, since

$$\begin{aligned} \|\Pi_\omega x\| &= \lim_{m \rightarrow \infty} \|\Pi_\omega^{(m)}x\| \\ &\leq \|\Pi_\omega^{(N)}x\| + \lim_{m \rightarrow \infty} \|\Pi_\omega^{(N)}x - \Pi_\omega^{(m)}(x)\| \\ &\leq e^{3N\epsilon} + \frac{e^{-N(\lambda_1 - \lambda_2 - 7\epsilon)}}{1 - e^{-(\lambda_1 - \lambda_2 - 7\epsilon)}}, \end{aligned}$$

the final line being independent of choice of x . Not only then is $\Pi_\omega^{(n)}x$ convergent, but we have the estimate

$$\|\Pi_\omega^{(n)}x - \Pi_\omega x\| \leq \sum_{i=1}^{\infty} \|\Pi_\omega^{(n+i)}x - \Pi_\omega^{(n+i-1)}x\| < \frac{e^{-n(\lambda_1 - \lambda_2 - \epsilon)}}{1 - e^{-\lambda_1 - \lambda_2 - \epsilon}}.$$

Π_ω is linear: to see this let $b, c \in X$ and $t \in \mathbb{R}$, and set

$$d := \Pi_\omega^{(n)}(b + tc) - \Pi_\omega^{(n)}b - t\Pi_\omega^{(n)}c.$$

Certainly $d \in E(\omega)$, since

$$\begin{aligned} d &= \Pi_\omega^{(n)}(b + tc) - (b + tc) + b - \Pi_\omega^{(n)}b + tc - t\Pi_\omega^{(n)}c \\ &= -P_\omega^{(n)}(b + tc) + P_\omega^{(n)}b + tP_\omega^{(n)}c \in E(\omega). \end{aligned}$$

Then applying the estimate for $\Pi_\omega^{(n)}$

$$\begin{aligned} \|d\| &\leq e^{-n(\lambda_1-\epsilon)} \|\mathcal{L}_\omega^{(n)} d\| \\ &\leq e^{-n(\lambda_1-\epsilon)} (\|\mathcal{L}_\omega^{(n)} \Pi_\omega^{(n)}(b+tc)\| + \|\mathcal{L}_\omega^{(n)} \Pi_\omega^{(n)}(b)\| + \|t\mathcal{L}_\omega^{(n)} \Pi_\omega^{(n)} c\|) \\ &< e^{-n(\lambda_1-\lambda_2-2\epsilon)} (\|\Pi_\omega^{(n)}(b+tc)\| + \|\Pi_\omega^{(n)} b\| + \|t\Pi_\omega^{(n)} c\|) \\ &< e^{-n(\lambda_1-\lambda_2-2\epsilon)} (\|\Pi_\omega(b+tc)\| + \|\Pi_\omega b\| + \|t\Pi_\omega c\| + \\ &(\|b+tc\| + \|b\| + \|tc\|) \frac{e^{-n(\lambda_1-\lambda_2-\epsilon)}}{1-e^{-\lambda_1-\lambda_2-\epsilon}}) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus $d = 0$ and $\Pi_\omega \in \mathcal{B}(X)$. By construction $\Pi_\omega E(\omega) = 0$. On the other hand, $P_\omega X = E(\omega)$ since for any $x \in X$ and $n \in \mathbb{N}$ we have $x - \Pi_\omega^{(n)} x \in E(\omega)$. Π_ω is then idempotent, since for any $x \in X$

$$\Pi_\omega^2 x - \Pi_\omega x = \Pi_\omega \circ P_\omega x \in \Pi_\omega E(\omega) = \{0\},$$

and is thus a projection. Set $V(\omega) = \Pi_\omega X$ so that $X = V(\omega) \oplus E(\omega)$. For all $n \geq N_\omega$, and any $x \in V_\omega(x)$, because of the exponential rate of convergence there is a sequence of approximants

$$\|x_n - x\| < C_\omega e^{-n(\lambda_1-\lambda_2-7\epsilon)}$$

with

$$\|\mathcal{L}_\omega^{(n)} x_n\| \in [0, e^{n(\lambda_2+\epsilon)}].$$

Therefore,

$$\|\mathcal{L}_\omega^{(n)} x\| < \|\mathcal{L}_{\sigma^{-n}\omega}^{(n)}\| \|x - x_n\| + \|\mathcal{L}_\omega^{(n)} x_n\| < e^{n(\lambda_2+8\epsilon)},$$

whence

$$\|\mathcal{L}_\omega^{(n)}|_{V(\omega)}\| < e^{n(\lambda_2+8\epsilon)}$$

for $n \geq N_\omega$. By the definition of λ_2 , it is possible to choose $Y_n \in \mathcal{G}_{m_1+1}(\omega)$ with $\frac{1}{n} g_{\mathcal{L}_\omega^{(n)}}(Y_n) \rightarrow \lambda_2$, which means by dimension counting that there is always some $x_n \in \mathbb{S}_{V(\omega) \cap Y_n}$ with $\frac{1}{n} \|\mathcal{L}_\omega^{(n)} x_n\| \rightarrow \lambda_2$ and so

$$\frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}|_{V(\omega)}\| \rightarrow \lambda_2.$$

Further, the map $\omega \mapsto \Pi_\omega$ is strongly measurable, since for each $x \in X$ the map $\omega \mapsto \Pi_\omega x$ is the limit of a sequence of measurable functions.

To see that $V(\omega)$ is equivariant it is sufficient to show that $P_{\sigma\omega} \circ \mathcal{L}_\omega x = 0$ for all $x \in V(\omega)$. If this were not the case, then $\mathcal{L}_\omega x$ would have a nonzero component in $E(\sigma\omega)$. From this it would follow that $\lambda_\omega(x) = \lambda_1$ which would contradict the fact that $\|\mathcal{L}_\omega^{(n)}|_{V(\omega)}\| < e^{n(\lambda_2+8\epsilon)}$ for sufficiently large n .

As for temperedness of the projections: Since Π_ω is bounded pointwise we may choose an $A \subseteq \Omega$ of positive measure on which $\|\Pi_\omega\|$ is at most some $M > 0$. Then define a new cocycle \mathcal{L}' by

$$\mathcal{L}'_\omega = \begin{cases} \mathcal{L}_\omega \circ \Pi_\omega & \text{if } \omega \in A, \\ \mathcal{L} & \text{otherwise.} \end{cases}$$

Then \mathcal{L}'_ω is forward-integrable since

$$\int_\Omega \log^+ \|\mathcal{L}'_\omega\| \leq \int_\Omega \log^+ \|\mathcal{L}\| + \int_A \log^+ M < \infty.$$

Since A has positive measure, there is almost surely an n such that $\mathcal{L}'_\omega^{(n)}E(\omega) = 0$, and so

$$\mathcal{L}'_\omega^{(n)}X \subseteq V(\sigma^n\omega).$$

We may then conclude that for each $x \in X$, $\lambda'_\omega(x) \leq \lambda_2$ and $\lambda'_1 \leq \lambda_2$. Applying Lemma 22 to the subadditive families $g_n = \log \|\mathcal{L}^{(n)}_\omega\|$ and $g'_n = \log \|\mathcal{L}'^{(n)}_\omega\|$ there is an N_1 such that for $n \geq N_1$,

$$\|\mathcal{L}'_{n \rightarrow 2n}\| < e^{n(\lambda_2 + \epsilon)} \text{ and } \|\mathcal{L}_{n \rightarrow 2n}\| > e^{n(\lambda_1 - \epsilon)}.$$

Clearly $\mathcal{L}'_{n \rightarrow 2n} \neq \mathcal{L}_{n \rightarrow 2n}$ since the latter has a greater norm. Therefore there must be some j such that

$$\mathcal{L}'_{n \rightarrow 2n} = \mathcal{L}'_{n \rightarrow j} \circ \Pi_{\sigma^j\omega} \circ \mathcal{L}'_{j \rightarrow 2n} = \mathcal{L}_{n \rightarrow 2n} \circ \Pi_{\sigma^n\omega}.$$

In addition, there exists an N_2 such that for $n \geq N_2$,

$$g_{\mathcal{L}'_{\sigma^n\omega}}(E(\sigma^n\omega)) \in (e^{n(\lambda_1 - \epsilon)}, e^{n(\lambda_1 + \epsilon)}).$$

As a final condition, there exists some $N_3 \in \mathbb{N}$ such that for all

$$n \geq N_3, g_{\mathcal{L}'_\omega^{(n)}}(E(\omega)) \in (e^{n(\lambda_1 - \epsilon)}, e^{n(\lambda_1 + \epsilon)}).$$

Let $x \in \mathbb{S}_X$. Putting these together, for all $n \geq \max\{N_1, N_2, N_3, \frac{1}{\epsilon}\}$,

$$\begin{aligned} \|P_{\sigma^n\omega}x\| &\leq e^{-n(\lambda_1 - \epsilon)} \|\mathcal{L}'_{\sigma^n\omega} P_{\sigma^n\omega}x\| \\ &\leq e^{-n(\lambda_1 - \epsilon)} (\|\mathcal{L}'_{\sigma^n\omega}x\| + \|\mathcal{L}'_{\sigma^n\omega}\Pi_{\sigma^n\omega}x\|) \\ &\leq e^{-n(\lambda_1 - \epsilon)} (e^{n(\lambda_1 + \epsilon)}\|x\| + \|\mathcal{L}'_{n \rightarrow 2n\omega}x\|) \\ &\leq e^{-n(\lambda_1 - \epsilon)} (e^{n(\lambda_1 + \epsilon)} + \|\mathcal{L}'_{n \rightarrow 2n}\|)\|x\| \\ &\leq e^{2n\epsilon} + e^{-n(\lambda_1 - \lambda_2 - 2\epsilon)} \leq e^{3n\epsilon}. \end{aligned}$$

ϵ was arbitrary and the norms of Π and P differ by at most 1 so

$$\frac{1}{n} \log \|P_{\sigma^n\omega}\|, \frac{1}{n} \log \|\Pi_{\sigma^n\omega}\| \rightarrow 0$$

as required. □

Corollary 30. $V(\omega) = \tilde{V}(\omega) = \{x \in X : \lambda_\omega(x) \leq \lambda_2\}$.

Proof. $\frac{1}{n} \log \|\mathcal{L}^{(n)}_\omega|_V(\omega)\| \rightarrow 0$ establishes the fact that $V(\omega) \subseteq \tilde{V}(\omega)$. Conversely, any $x \in \tilde{V}(\omega) \setminus V(\omega)$ would have $P_\omega(x) \neq 0$ so that $\lambda_\omega(x) = \lambda_1$, contradicting the definition of \tilde{V} . □

6. PROOF OF MAIN RESULT

Finally, we may conclude with the main result, a well-behaved decomposition of the space acted on by a random linear dynamical system:

Proof of Theorem 3. The decomposition is obtained inductively. At each stage it is shown that if λ_{i+1} exists, there exists an equivariant decomposition

$$X = E_{\leq i}(\omega) \oplus V_{i+1}(\omega)$$

with $E_{\leq i} \in \mathcal{SM}(\omega \rightarrow \mathcal{G}_{M_i}X)$ and bounded projections

$$\begin{aligned} \Pi_{i+1\omega} : X &\rightarrow V_{i+1}(\omega) \text{ and} \\ P_{i\omega} : X &\rightarrow E_{\leq i}(\omega). \end{aligned}$$

The existence of the top fast space has already been established - here denote this $E_{<2}(\omega) \oplus V_2(\omega)$ with measurable projections $\Pi_{2\omega}$ and $P_{2\omega}$.

Suppose that the statement is true up to $i = l - 1$. If $\lambda_l = \nu$ then we are done, so suppose otherwise that there exists $\lambda_{l+1} \geq \nu$. The projection $\Pi_{l\omega}$ is pointwise bounded, so that there exists some $M > 0$ such that $A = \{\|\Pi_{l\omega}\| < M\}$ has positive measure.

$$\mathcal{L}'_\omega = \begin{cases} \mathcal{L}_\omega \circ \Pi_{l\omega}, \omega \in A, \\ \mathcal{L}_\omega \text{ otherwise.} \end{cases}$$

As before \mathcal{L}' is forward integrable. Write $\lambda', \mu'_i, \lambda'_i, \nu', E_l \oplus V', M'_i, P'$ and Π' for the asymptotic growth rates, Lyapunov exponents, decomposition, fast space multiplicities, fast projection and slow projection with \mathcal{L}' . There almost surely exists an $N \in \mathbb{N}$ such that for each $n \geq N$, the following hold:

- $\mathcal{L}'_\omega^{(n)} = \mathcal{L}_\omega^{(n)} \circ \Pi_{l\omega} = \Pi_{l\sigma^n\omega} \circ \mathcal{L}_\omega^{(n)}$.
- $\|\Pi_{\sigma^n\omega}\| < e^{n\epsilon}$
- $g_{\mathcal{L}'_\omega^{(n)}}(E(\omega)) > \|\mathcal{L}_\omega^{(n)}|_{V_{l+1}(\omega)}\|$.

Therefore applying Lemmas 16 and 17 to $\mathcal{L}'_\omega^{(n)} = \mathcal{L}_\omega^{(n)} \circ \Pi_{l\omega}$, for each k the following holds:

$$\rho_{k+M_l}(\mathcal{L}'_\omega^{(n)}) \leq \rho_k(\mathcal{L}'_\omega^{(n)}) \leq 4\|\Pi_{\sigma^n\omega}\| \|\Pi_{l\omega}\| \rho_{k+M_l}(\mathcal{L}_\omega^{(n)}) \leq 4e^{n\epsilon} \|\Pi_{l\omega}\| \rho_{k+M_l}(\mathcal{L}_\omega^{(n)}),$$

whence $\mu_{M_l+k} = \mu'_k$ and $\nu = \nu'$. Further then, $m'_k = m_{l+k}$ and $\lambda + k' = \lambda_{k+l}$. Set $V_{l+1}(\omega) = V_l(\omega) \cap V'(\omega)$ and $\Pi_{l+1\omega} = \Pi_{l\omega} \circ \Pi'_\omega$. Write $E_{<l}(\omega) = E_{<l}(\omega) + E_l(\omega)$. The equality $\mathcal{L}_\omega = \mathcal{L}'_\omega$ holds on $V_l(\omega)$, whence $\lambda_\omega = \lambda'_\omega$ on $V_l(\omega)$. Since A has positive measure, there is almost surely an N such that for all $n \geq N$, $\mathcal{L}'_\omega^{(n)} E_{<l}(\omega) = 0$, $\mathcal{L}'_{\sigma^{-n}\omega} X \subseteq V_l(\omega)$, and $\mathcal{L}'_\omega^{(n)} X \subseteq V_l(\sigma^n\omega)$. We may then conclude that for each $x \in X$, $\lambda'_\omega(x) \leq \lambda_l$ and $\lambda'_1 \leq \lambda_l$. For each $i < l$ it follows that $E_i(\omega) \subseteq V'(\omega)$. As for $E_l(\omega)$, by equivariance $E_l(\omega) = \mathcal{L}'_{\sigma^{-n}\omega} E(\sigma^{-n}\omega) \subseteq V_l(\omega)$. Thus $X = E_{<l}(\omega) \oplus E_l(\omega) \oplus (V_l(\omega) \cap V'(\omega)) = E_{<l}(\omega) \oplus E_l(\omega) \oplus V_{l+1}(\omega)$.

Π_{l+1} is then also tempered:

$$0 \leq \frac{1}{n} \log \|\Pi_{l+1\sigma^n\omega}\| \leq \frac{1}{n} \log \|\Pi_{l\sigma^n\omega}\| + \frac{1}{n} \log \|\Pi'_{\sigma^n\omega}\| \rightarrow 0.$$

Let $\epsilon > 0$. There exists an N such that for $n \geq N$, $\|P'_{\sigma^n\omega}\|, \|\Pi'_{\sigma^n\omega}\| < e^{n\epsilon}$ and so for all $x \in X$ we have $\max\{\|P'_{\sigma^n\omega} \mathcal{L}_\omega^{(n)} x\|, \|\Pi'_{\sigma^n\omega} \mathcal{L}_\omega^{(n)} x\|\} \leq \|\mathcal{L}_\omega^{(n)} x\|$.

Rearranging this last inequality and letting $x \in E_{<l}(\omega)$,

$$\begin{aligned} \|\mathcal{L}_\omega^{(n)} x\| &\geq e^{-n\epsilon} \max\{\|P'_{\sigma^n\omega} \mathcal{L}_\omega^{(n)} x\|, \|\Pi'_{\sigma^n\omega} \mathcal{L}_\omega^{(n)} x\|\} \\ &\geq e^{-n\epsilon} \max\{\|\mathcal{L}_\omega^{(n)} P'_\omega x\|, \|\mathcal{L}_\omega^{(n)} \Pi'_\omega x\|\} \\ &\geq e^{-n\epsilon} \max\{e^{n(\lambda_l - \epsilon)} \|P'_\omega x\|, e^{n(\lambda'_1 - \epsilon)} \|\Pi'_\omega x\|\} \\ &\geq e^{-n\epsilon} e^{n(\lambda_l - \epsilon)} \max\{\|\Pi'_\omega x\|, \|P'_\omega x\|\} \geq \frac{1}{2} e^{n(\lambda_l - 2\epsilon)}. \end{aligned}$$

For $n \geq \max\{N, \frac{1}{\epsilon} \log 2\}$ it follows that $g_{\mathcal{L}_\omega^{(n)}}(E_{<l}(\omega)) \geq e^{n(\lambda_l - 3\epsilon)}$. The characterization

$$V_{l+1}(\omega) = \{x \in X : \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)} x\| \leq \lambda_{l+1}\}$$

holds. □

APPENDIX: EQUIVALENCE OF GROWTH STATISTICS

The Gelfand numbers may be defined by $s_k(T) = \inf_{V \in \mathcal{G}^{k-1}X} \|T|_V\|$. Throughout this section we assume $T \in \mathcal{B}(X)$.

Lemma 31. $\rho_k \leq s_k$.

Proof. Let $\epsilon > 0$. Choose $V \in \mathcal{G}_k X$ with $g_T(V) \geq \rho_k(T) - \epsilon$. Then since any $W \in \mathcal{G}^{k-1}X$ intersects V nontrivially, suppose x is a unit vector in the intersection. Then $\rho_k(T) \leq \|Tx\| \leq \|T|_W\|$. W was arbitrary so $\rho_k(T) \leq s_k(T)$. \square

A better bound than the following may be found in the work of Pietsch on s -numbers [13]:

Lemma 32. For all k , $s_k \leq 4^{k-1}((k-1)!)^{\frac{1}{2}} \rho_k$.

Proof. For $k = 1$ each quantity is just the norm of T , so equality holds. Write $c_i = 4^{1-i}((i-1)!)^{-\frac{1}{2}}$. Suppose that the proposition holds for each $l < k$ with $k \geq 2$. Choose $U \in \mathcal{G}_{k-1}X$ with $g_T(U) \geq (1-\epsilon)\rho_{k-1}(T)$, so that in particular

$$g_T(U) \geq (1-\epsilon)c_{k-1}s_{k-1}(T) \geq (1-\epsilon)c_{k-1}s_k(T).$$

Choose a complement $TU \oplus V = X$ with $\Pi = \Pi_{TU \parallel V}$ such that $\|\Pi\| \leq (k-1)^{\frac{1}{2}} + \epsilon$. Since $T^{-1}V \in \mathcal{G}^{k-1}X$, we may choose $x \in \mathbb{S}_{T^{-1}V}$ with $\|Tx\| \geq s_k(T) - \epsilon$. Set $W = U \oplus \text{span}\{x\} \in \mathcal{G}_k X$ and let $a = b + tx \in \mathbb{S}_W$. Applying T ,

$$\begin{aligned} \|Ta\| &\geq \max\left\{\frac{\|Tb\|}{\|\Pi\|}, \frac{\|tTx\|}{\|1-\Pi\|}\right\} \\ &\geq (1 + \sqrt{k-1} + \epsilon)^{-1} \max\{(1-\epsilon)c_{k-1}s_k(T)\|b\|, s_k(T)(1-\epsilon)|t|\} \\ &\geq (2\sqrt{k-1} + \epsilon)^{-1} (1-\epsilon)c_{k-1} \max\{\|b\|, |t|\} \\ &\geq \frac{(1-\epsilon)c_{k-1}}{4(\sqrt{k-1} + \epsilon)} s_k(T). \end{aligned}$$

a was arbitrary so the final line is a lower bound for $g_T(W)$. ϵ was also arbitrary, so that $\rho_k(T) \geq \frac{1}{4c_{k-1}\sqrt{k-1}}s_k(T) = c_k s_k(T)$. \square

Lemma 33. The Gelfand numbers satisfy

$$s_k(T) \geq k^{-\frac{1}{2}} \|T\|_c.$$

Proof. Let $V \in \mathcal{G}^{k-1}X$. We may by the theorem of Kadets in [18] choose a complement $V \oplus W = X$ with $\|\Pi_{V \parallel W}\| \leq \sqrt{k-1} + \epsilon$. Then $T \circ \Pi_{W \parallel V}$ is finite rank, so $\|T\|_c \leq \|T - T \circ \Pi_{W \parallel V}\| = \|T \circ \Pi_{V \parallel W}\| \leq \|T|_V\| \|\Pi_{V \parallel W}\| \leq \|T|_V\|(\sqrt{k-1} + \epsilon)$. Taking the inf over such V and letting $\epsilon \rightarrow 0$ we obtain the bound. \square

Lemma 34. The index defined at the start of this article agrees with the usual index of compactness

$$\lim_{n \rightarrow \infty} \mu_n = \nu = \kappa = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{L}_\omega^{(n)}\|_c.$$

Proof. Since ρ dominates the compactness seminorm up to a multiplicative constant, $\kappa \leq \mu_k$ for every $k \in \mathbb{N}$, so certainly $\kappa \leq \nu$. It remains to verify that

$\kappa \geq \nu$. There exists a $\delta > 0$ such that for all $\mathbb{P}(\Lambda) < \delta$, $\int_{\Lambda} \log \|\mathcal{L}\| < \frac{\epsilon}{2}$. Choose N sufficiently large

$$\mathbb{P}(\|\mathcal{L}_{\omega}^{(N)}\|_c \geq e^{N(\kappa+\epsilon)}) < \frac{1}{2}\delta.$$

Choose r sufficiently large

$$\mathbb{P}(G) = \mathbb{P}(\mathcal{L}_{\omega}^{(N)}\mathbb{B}_X \text{ may be covered by at most } e^{rN} e^{N(\kappa+\frac{1}{2}\epsilon)\text{-balls}}) > 1 - \delta.$$

Set $f_n(\omega) = \log \inf\{t > 0 : \mathcal{L}_{\omega}^{(n)}\mathbb{B}_X \text{ is covered by } e^{rn} t\text{-balls}\}$. In this case

$$\begin{aligned} f_{m+n}(\omega) &\leq \log \inf\{ab : \mathcal{L}_{\omega}^{(n)}\mathbb{B}_X \text{ is covered by } e^{rn} a\text{-balls, and } \mathcal{L}_{\sigma^n\omega}^{(m)}\mathbb{B}_X \text{ by } e^{rm} b\text{-balls}\} \\ &= f_n(\omega) + f_m(\sigma^n\omega) \end{aligned}$$

so that the family is subadditive, $\int_{\Omega} f_1 \leq \int_{\Omega} \log \|\mathcal{L}_{\omega}\| < \infty$ and

$$\frac{1}{N} \int_{\Omega} f_N \leq \kappa + \frac{1}{2}\epsilon + \frac{1}{N} \int_{G^c} \log \|\mathcal{L}_{\omega}^{(N)}\| < \kappa + \frac{1}{2}\epsilon + \frac{1}{2}\epsilon.$$

Thus we may apply Kingman again to obtain that $\frac{1}{n}f_n(\omega) \rightarrow C < \kappa + \epsilon$. Almost surely, for sufficiently large n we may guarantee the following:

- $\|\mathcal{L}_{\omega}^{(n)}\|_c < e^{n(\kappa+\epsilon)}$
- There is a $V \in \mathcal{G}_k X$ with $g_{\mathcal{L}_{\omega}^{(n)}}(V) > e^{n(\mu_k - \epsilon)}$.
- $\mathcal{L}_{\omega}^{(n)}\mathbb{B}_X$ is covered by at most $e^{rn} e^{n(\kappa+\epsilon)}$ -balls
- $e^{n\epsilon} > (2k)^k$.

Choose a basis of unit vectors x_i for V with $d(x_i, \text{span}\{x_j\}_{j < i}) = 1$. Write

$$\Lambda = \left\{ \sum_{i=1}^k a_i x_i : a_i \in \{0, \pm \frac{2e^{n(\kappa+\epsilon)}}{\|\mathcal{L}_{\omega}^{(n)} x_i\|}, \pm \frac{4e^{n(\kappa+\epsilon)}}{\|\mathcal{L}_{\omega}^{(n)} x_i\|}, \dots\}, |a_i| < \frac{1}{k} \right\} \subseteq \mathbb{B}_X.$$

If two members $a = \sum_i a_i x_i$ and $b = \sum_i b_i x_i$ are distinct then there is a maximal $j \leq k$ with $a_j \neq b_j$. Then

$$\|a - b\| \geq |a_j - b_j| d(\mathcal{L}_{\omega}^{(n)} x_j, \mathcal{L}_{\omega}^{(n)} \text{span}\{x_i : i \leq j\}) > e^{n(\kappa+\epsilon)} > \|\mathcal{L}_{\omega}^{(n)}\|_c.$$

The points in $\mathcal{L}_{\omega}^{(n)}\Lambda$ are then of distance at least $2e^{n(\kappa+\epsilon)} > \|\mathcal{L}_{\omega}^{(n)}\|_c$ apart, and there are at least

$$|\Lambda| = \prod_{i=1}^k \frac{1}{2} \lfloor \frac{\|\mathcal{L}_{\omega}^{(n)} x_i\|}{e^{n(\kappa+\epsilon)k}} \rfloor \geq (2k)^{-k} \prod_{i=1}^k e^{n(\mu_k - \epsilon) - n(\kappa+\epsilon)} = e^{nk(\mu_k - \kappa - 3\epsilon)}$$

of them. Each member of a cover of $\mathcal{L}_{\omega}^{(n)}\mathbb{B}_X$ by $e^{n(\kappa+\epsilon)}$ -balls contains at most one element of Λ , so the cover has cardinality at least $e^{nk(\mu_k - \kappa - 3\epsilon)}$. On the other hand, this quantity is bounded by e^{rn} :

$$e^{rn} \geq e^{nk(\mu_k - \kappa - 4\epsilon)}.$$

Taking log and rearranging we obtain

$$k \leq \frac{r}{\mu_k - \kappa - 4\epsilon}.$$

In the case that $\mu_k > \kappa + 5\epsilon$, it must hold that $k < \frac{r}{\epsilon}$. This bound on k shows the number of μ 's greater than $\kappa + 5\epsilon$ is finite, whence $\mu_k \downarrow \kappa = \nu$ as $k \rightarrow \infty$. \square

Example 35. Let $\Omega = \{\pm 1\}^{\mathbb{Z}}$ and $\sigma(\omega)_n = \omega_{n+1}$. Let $X = \ell^1(\mathbb{N}_0)$. Let $\mathcal{L}_\omega(x) = (\sum_{i=1}^{\infty} \omega_i x_i, 0, 0, \dots)$. It is then easy to verify that $E_\omega = \text{span}\{(1, 0, \dots)\}$ with $\lambda_1 = 0$. Let $\omega, \omega' \in \Omega$ so that $\omega \neq \pm\omega'$. Then there are $i \neq j$ with $\omega_i = \omega'_i$ and $\omega_j = -\omega'_j$. Then the $x \in \ell^1(\mathbb{N}_0)$ which has $x_i = 1, x_j = -1$ and $x_l = 0$ otherwise will be in exactly one of V_ω or $V_{\omega'}$ but has distance at least 1 from the other. The family $\{V_\omega : \omega \in \Omega\} \subseteq \mathcal{G}^1 X$ is then a discrete uncountable set. Taking open balls around each V_ω , any subset $S \subseteq \Omega$ with $S = -S$ is a preimage of some measurable subset of $\mathcal{G}^1 X$, whence V isn't measurable.

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