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GLOBAL DYNAMICS FOR THE STOCHASTIC KDV EQUATION WITH WHITE NOISE AS INITIAL DATA

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Abstract. We study the stochastic Korteweg-de Vries equation (SKdV) with an additive space-time white noise forcing, posed on the one-dimensional torus. In particular, we construct global-in-time solutions to SKdV with spatial white noise initial data. Due to the lack of an invariant measure, Bourgain's invariant measure argument is not applicable to this problem. In order to overcome this difficulty, we implement a variant of Bourgain's argument in the context of an evolution system of measures and construct global-in-time dynamics. Moreover, we show that the white noise measure with variance $1 + t$ is an evolution system of measures for SKdV with the white noise initial data.

Contents

1. Introduction

1.1. **Main result.** The main objective of the present paper is to explain how techniques developed to study invariance of certain measures (in our case, a spatial white noise) under the flow of Hamiltonian partial differential equations (PDEs) can be combined with the analysis of stochastic perturbations of these equations to construct global-in-time solutions in a probabilistic setting.

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In particular, we consider the following Cauchy problem for the stochastic Korteweg-de Vries equation (SKdV) on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = \xi, \\ u|_{t=0} = u_0. \end{cases} \tag{1.1}$$

Here, ξ denotes an additive (Gaussian) space-time white noise forcing whose space-time covariance is (formally) given by

$$\mathbb{E}[\xi(x_1, t_1)\xi(x_2, t_2)] = \delta(x_1 - x_2)\delta(t_1 - t_2) \tag{1.2}$$

for $x_1, x_2 \in \mathbb{T}$ and $t_1, t_2 \in \mathbb{R}_+$ with δ denoting the Dirac delta function. In particular, we study (1.1) with a spatial white noise[1] on $\mathbb{T}$, independent of the forcing ξ, as initial data. More concretely, we take $u_0 = u_0^\omega$ of the form[2]:

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}} g_n(\omega)e^{inx}, \tag{1.3}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is a family of independent standard complex-valued Gaussian random variables conditioned that $g_{-n} = \overline{g_n}$, $n \in \mathbb{Z}$. The main difficulty of this problem comes from the roughness of the noise and the white noise initial data, such that the solution $u(t)$ to (1.1) belongs to $H^s(\mathbb{T}) \setminus H^{-\frac{1}{2}}(\mathbb{T})$, $s < -\frac{1}{2}$, almost surely. Here, $H^s(\mathbb{T})$ denotes the L^2-based Sobolev space defined by the norm:

$$\|u\|_{H^s} = \bigg(\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s}|\widehat{u}(n)|^2\bigg)^{\frac{1}{2}},$$

where $\langle\,\cdot\,\rangle = \sqrt{1 + |\cdot|^2}$.

The well-posedness issue of SKdV with an additive forcing:

$$\partial_t u + \partial_x^3 u + u\partial_x u = \phi\xi, \tag{1.4}$$

where ϕ is a bounded operator on L^2, has been studied both on the real line and on the torus [20–22, 37, 51]. In the periodic setting, de Bouard, Debussche, and Tsutsumi [22] proved local well-posedness of (1.4) on $\mathbb{T}$ when ϕ is a Hilbert-Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$, barely missing the case of an additive space-time white noise. This local well-posedness result in [22] was obtained via a contraction argument, based on the Fourier restriction norm method (namely, utilizing the $X^{s,b}$-spaces) adapted to the Besov space, utilizing the endpoint Besov regularity of the Brownian motion [1, 13, 54]. With an additional assumption that ϕ is Hilbert-Schmidt from $L^2(\mathbb{T})$ to $L^2(\mathbb{T})$, they also proved global well-posedness of (1.4) in $L^2(\mathbb{T})$. In [37], the first author improved this result and proved local well-posedness of (1.4) even when $\phi = \mathrm{Id}$ (thus reducing to (1.1)), thus handling the case of an additive space-time white noise.[3] We point out that the argument in [37] is based on an approximation argument, in particular, not based on a contraction argument. Below, we will describe the approach in [37] more in detail; see Section

[1] As it is customary in the literature, with a slight abuse of notation, we use the term ‘white noise’ to refer to both the distribution-valued random variable u_0^ω in (1.3) and its law $\mu_1 = \mathrm{Law}(u_0^\omega)$, when there is no confusion. Here, $\mathrm{Law}(X)$ denotes the law of a random variable X. For clarity, we may refer to $\mu_1 = \mathrm{Law}(u_0^\omega)$ as the white noise measure.

[2] By convention, we endow $\mathbb{T}$ with the normalized Lebesgue measure $(2\pi)^{-1}dx$.

[3] Note that $\phi = \mathrm{Id}$ is a Hilbert-Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$ but not for $s \geq -\frac{1}{2}$.

3. Our main goal is to construct global-in-time dynamics for (1.1) with the spatial white noise u_0^ω in (1.3) as initial data.

Before proceeding further, let us go over the known well-posedness results for the (deterministic) KdV on $\mathbb{T}$:

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0. \tag{1.5}$$

In [4], Bourgain introduced the so-called Fourier restriction norm method, utilizing the $X^{s,b}$-spaces defined by the norm:

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\|_{\ell^2_n L^2_\tau(\mathbb{Z}\times\mathbb{R})}, \tag{1.6}$$

and proved local well-posedness of (1.5) in $L^2(\mathbb{T})$ via a fixed point argument, immediately yielding global well-posedness in $L^2(\mathbb{T})$ thanks to the conservation of the L^2-norm. Subsequently, Kenig, Ponce, and Vega [31] (also see [14]) improved Bourgain's result and proved local well-posedness of (1.5) in $H^{-\frac{1}{2}}(\mathbb{T})$ by establishing the following bilinear estimate:

$$\|\partial_x(uv)\|_{X^{s,b-1}} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}} \tag{1.7}$$

for $s \geq -\frac{1}{2}$ and $b = \frac{1}{2}$ under the (spatial) *mean-zero* assumption on u and v. In [14], Colliander, Keel, Staffilani, Takaoka, and Tao then proved the corresponding global well-posedness result in $H^{-\frac{1}{2}}(\mathbb{T})$ via the I-method. The KdV equation (1.5) is also known to be one of the simplest completely integrable PDEs, and there are well-posedness results for (1.5), exploiting the completely integrable structure of the equation. In [6], Bourgain proved global well-posedness of (1.5) in the class $\mathcal{M}(\mathbb{T})$ of finite Borel measures λ on $\mathbb{T}$, assuming that its total variation $\|\lambda\|$ is sufficiently small. His proof was based on partially iterating the Duhamel formulation of (1.5) and establishing bilinear and trilinear estimates, assuming an a priori uniform bound of the form:

$$\sup_{t\in\mathbb{R}}\sup_{n\in\mathbb{Z}} |\widehat{u}(n,t)| \leq C \tag{1.8}$$

on the Fourier coefficients of the solution u. Then, he established the global-in-time a priori bound (1.8), using the complete integrability. In [30], Kappeler and Topalov proved global well-posedness of (1.5) in $H^{-1}(\mathbb{T})$ via the inverse spectral method. See also [33].

For SKdV (1.1) with a random perturbation, such an integrable structure is destroyed and thus the approaches based on the complete integrability of KdV are no longer applicable. Nonetheless, in [37], the first author adapted Bourgain's approach [6], based on a partial iteration of the Duhamel formulation (= the mild formulation) of (1.1), and proved local well-posedness of (1.1). In particular, he bypassed the assumption (1.8) by employing the Fourier restriction norm method adapted to the "Fourier-Besov" space $\widehat{b}^s_{p,\infty}(\mathbb{T})$ introduced in [36], defined by the norm:

$$\begin{aligned}\|f\|_{\widehat{b}^s_{p,\infty}} = \|\widehat{f}\|_{b^s_{p,\infty}} &= \sup_{j\in\mathbb{Z}_{\geq 0}} \|\langle n\rangle^s \widehat{f}(n)\|_{\ell^p_{|n|\sim 2^j}} \\ &= \sup_{j\in\mathbb{Z}_{\geq 0}} \bigg(\sum_{|n|\sim 2^j} \langle n\rangle^{sp} |\widehat{f}(n)|^p\bigg)^{\frac{1}{p}},\end{aligned} \tag{1.9}$$

which captures the spatial regularity of the space-time white noise when $sp < -1$; see Proposition 3.4 in [36].[4] Here, $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$, and $\{|n| \sim 2^j\}$ means $\{2^{j-1} < |n| \leq 2^j\}$ when $j \geq 1$ and $\{|n| \leq 1\}$ when $j = 0$. Note that, by taking $p > 2$ (but close to 2), we can take $s > -\frac{1}{2}$, still satisfying $sp < -1$, which is crucial in establishing relevant nonlinear estimates. In Section 3, we go over some aspects of the local well-posedness argument from [37].

We now state our main result, which extends the solution constructed in [37] globally in time in the case of the white noise initial data. We say that u is a solution to (1.1) if it satisfies the following Duhamel formulation (= the mild formulation):

$$u(t) = S(t)u_0 - \frac{1}{2}\int_0^t S(t-t')\partial_x u^2(t')dt + \int_0^t S(t-t')dW(t'), \tag{1.10}$$

where $S(t) = e^{-t\partial_x^3}$ denotes the linear KdV propagator (= the Airy propagator) and W denotes a cylindrical Wiener process on $L^2(\mathbb{T})$:

$$W(t) = \sum_{n\in\mathbb{Z}} \beta_n(t)e^{inx}, \tag{1.11}$$

where $\{\beta_n\}_{n\in\mathbb{Z}}$ is defined by $\beta_n(t) = \langle \xi, \mathbf{1}_{[0,t]} \cdot e_n\rangle_{x,t}$. Here, $\langle \cdot, \cdot \rangle_{x,t}$ denotes the duality pairing on $\mathbb{T} \times \mathbb{R}_+$. As a result, we see that $\{\beta_n\}_{n\in\mathbb{Z}}$ is a family of mutually independent complex-valued Brownian motions conditioned that $\beta_{-n} = \overline{\beta_n}$, $n \in \mathbb{Z}$. In particular, β_0 is a standard real-valued Brownian motion, and we have

$$\mathrm{Var}(\beta_n(t)) = \mathbb{E}\big[\langle \xi, \mathbf{1}_{[0,t]} \cdot e_n\rangle_{x,t}\overline{\langle \xi, \mathbf{1}_{[0,t]} \cdot e_n\rangle_{x,t}}\big] = \|\mathbf{1}_{[0,t]} \cdot e_n\|^2_{L^2_{x,t}} = t \tag{1.12}$$

for any $n \in \mathbb{Z}$. Note that the space-time white noise ξ in (1.1) is a distributional time derivative of the cylindrical Wiener process W in (1.11). The third term on the right-hand side of (1.10) is the so-called stochastic convolution, representing the effect of the stochastic forcing.

In the following, we set

$$s = -\frac{1}{2} + \delta_1 \qquad \text{and} \qquad p = 2 + \delta_2 \tag{1.13}$$

for some small $\delta_1, \delta_2 > 0$ such that $sp < -1$. Given $\alpha \geq 0$,[5] we say that a distribution-valued random variable X on $\mathbb{T}$ (and its law, denoted by μ_α) is a (spatial) white noise on $\mathbb{T}$ with variance α if

$$\mu_\alpha = \mathrm{Law}(X) = \mathrm{Law}(\sqrt{\alpha}\, u_0^\omega), \tag{1.14}$$

where u_0^ω is the white noise (with variance 1) in (1.3).

Theorem 1.1. *The stochastic KdV equation* (1.1) *with an additive space-time white noise forcing is globally well-posed with white noise initial data. More precisely, there exist small* $\delta_1, \delta_2 > 0$ *such that, with probability* 1*, there exists a unique global-in-time solution* u *to* (1.1)*, belonging to the class* $C(\mathbb{R}_+; \widehat{b}^s_{p,\infty}(\mathbb{T}))$ *with* s *and*

[4] In other words, $\phi = \mathrm{Id}$ is a γ-radonifying operator from $L^2(\mathbb{T})$ to $\widehat{b}^s_{p,\infty}(\mathbb{T})$ when $sp < -1$, which is a suitable generalization of the notion of Hilbert-Schmidt operators in the Banach space setting; see [8, 59]. See also [29, Chapter 9].

[5] By convention, we have $X \equiv 0$ when $\alpha = 0$. Namely, $\mu_0 = \delta_0$, where δ_0 is the Dirac delta distribution at the trivial function.

p as in (1.13), with the white noise initial data u_0^ω in (1.3). Moreover, for any $t \geq 0$, we have

$$\text{Law}(u(t)) = \mu_{1+t}. \tag{1.15}$$

Namely, $u(t)$ is a white noise with variance $1+t$.

The proof of Theorem 1.1 is based on a variant of Bourgain's invariant measure argument [5] in the context of an *evolution system of measures* [18, 19], which is a natural generalization of the concept of invariant measures for an autonomous dynamical system. Let us give a somewhat formal definition of an evolution system of measures. Let $\Phi_{t_1,t_2} = \Phi^\omega_{t_1,t_2}$, $t_2 \geq t_1 \geq 0$, be a solution map for a given autonomous (random) dynamical system, sending the data φ at time t_1 to the solution $\Phi_{t_1,t_2}\varphi$ at time t_2. Then, we define the transition semigroup P_{t_1,t_2} by

$$P_{t_1,t_2}F(\varphi) = \mathbb{E}[F(\Phi^\omega_{t_1,t_2}\varphi)] \tag{1.16}$$

for a bounded measurable function F on the phase space $\mathcal{M}$. Then, we say that[6] a family $\{\rho_t\}_{t\in\mathbb{R}_+}$ of probability measures on $\mathcal{M}$ is an evolution system of measures indexed by $\mathbb{R}_+$ if

$$\int_{\mathcal{M}} F(\varphi)\rho_{t_2}(d\varphi) = \int_{\mathcal{M}} P_{t_1,t_2}F(\varphi)\rho_{t_1}(d\varphi) \tag{1.17}$$

for any bounded continuous function F on $\mathcal{M}$ and $t_2 \geq t_1 \geq 0$. Note that (1.17) is equivalent to

$$\rho_{t_2} = P^*_{t_1,t_2}\rho_{t_1}$$

for any $t_2 \geq t_1 \geq 0$. If there exists an invariant measure ρ, then by setting $\rho_t = \rho$, $t \in \mathbb{R}_+$, the family $\{\rho_t\}_{t\in\mathbb{R}_+}$ is obviously an evolution system of measures. It is in this sense that the notion of an evolution system of measures is a generalization of the notion of an invariant measure.

Given $t \in \mathbb{R}_+$, let μ_{1+t} be the white noise of variance $1+t$ defined in (1.14). Then, Corollary 1.2 follows from (1.15) and the flow property

$$\Phi_{t_1,t_3} = \Phi_{t_2,t_3} \circ \Phi_{t_1,t_2} \tag{1.18}$$

for $t_3 \geq t_2 \geq t_1 \geq 0$ of the solution map to SKdV (1.1) constructed in Theorem 1.1.

Corollary 1.2. *Let μ_{1+t} be the white noise measure with variance $1+t$ as in (1.14). Then, the family $\{\mu_{1+t}\}_{t\in\mathbb{R}_+}$ is an evolution system of measures for SKdV (1.1) with the white noise initial data u_0^ω in (1.3).*

Furthermore, we have Corollary 1.3 to Theorem 1.1.

Corollary 1.3.

(i) *Given $\alpha \geq 0$, let $u^\omega_{0,\alpha}$ be a white noise on $\mathbb{T}$ with variance α given by*

$$u^\omega_{0,\alpha}(x) = \sqrt{\alpha}\sum_{n\in\mathbb{Z}} g_n(\omega)e^{inx},$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is as in (1.3). Then, with probability 1, there exists a unique global-in-time solution u to (1.1), with $u|_{t=0} = u^\omega_{0,\alpha}$. Moreover, for any $t \geq 0$, we have

$$\text{Law}(u(t)) = \mu_{\alpha+t}, \tag{1.19}$$

[6]Strictly speaking, an evolution system of measures is the mapping $t \in \mathbb{R}_+ \mapsto \rho_t \in \mathcal{P}(\mathcal{M})$, where $\mathcal{P}(\mathcal{M})$ denotes the family of probability measures on $\mathcal{M}$. However, we simply refer to the family $\{\rho_t\}_{t\in\mathbb{R}_+}$ of measures as an evolution system of measures.

where $\mu_{\alpha+t}$ is as in (1.14)*. Namely, $u(t)$ is a white noise with variance $\alpha + t$.*

(ii) *Let w_0 be a deterministic function in $L^2(\mathbb{T})$ and $\alpha > 0$. Then, with probability* 1*, there exists a unique global-in-time solution u to* (1.1) *with $u|_{t=0} = w_0 + \sqrt{\alpha}\, u^\omega_{0,\alpha}$, where u^ω_0 is the white noise on $\mathbb{T}$ with variance α as in* (1.3)*.*

Part (i) of Corollary 1.3 directly follows from Theorem 1.1 together with the flow property (1.18) and the time translation invariance (in law) of SKdV (1.1). See also Remark 1.5. Part (ii) of Corollary 1.3 follows from Corollary 1.3(i) and the Cameron-Martin theorem [9] by noting that $L^2(\mathbb{T})$ is the Cameron-Martin space of $\mu_\alpha = \mathrm{Law}(\sqrt{\alpha} u^\omega_0)$. See [42] for a further discussion.

Thanks to the time reversibility of the KdV equation, Theorem 1.1 and Corollary 1.3 also hold for negative times (where the variances $1 + t$ in (1.15) and $\alpha + t$ in (1.19) are replaced by $1 + |t|$ and $\alpha + |t|$, respectively. For simplicity of the presentation, however, we only consider positive times in the remaining part of the paper. Moreover, in the following discussion, in considering a stochastic flow on a time interval $[t_1, t_2]$, it is understood that random initial data at time t_1 and a stochastic forcing on $[t_1, t_2]$ are independent (which is justified by (1.2)).

1.2. **Outline of the proof.** Let us now describe some aspects of the proof of Theorem 1.1. Except in the small data regime (including a small perturbation of a known global solution), one usually needs to exploit conservation laws in order to construct global-in-time solutions to nonlinear dispersive PDEs. A remarkable intuition by Bourgain in [5] was to use (formal) invariance of a Gibbs measure as a replacement of a conservation law to construct global-in-time solutions with the Gibbsian initial data. More precisely, he used the rigorous invariance of the truncated Gibbs measures for the associated truncated dynamics and combined it with a PDE approximation argument to construct the desired global-in-time invariant Gibbs dynamics. This argument, known as Bourgain's invariant measure argument, has been applied to many dispersive PDEs with random initial data (and stochastic forcing), in particular over the last fifteen years. See the survey papers [3, 38, 58] for a further discussion on this topic and the references therein. See also [26, 39, 40] for more recent results in the context of stochastic dispersive PDEs. We point out that Bourgain's invariant measure argument has also been applied to globalize solutions to stochastic parabolic PDEs; see, for example, [28, 45, 46].

In the current problem at hand, due to the lack of a damping term, there is no invariant measure for SKdV (1.1), and thus Bourgain's invariant measure argument is not applicable. It is, however, easy to see, at a formal level (as explained below), that SKdV (1.1) with the white noise initial data (1.3) possesses a (formal) evolution system of measures $\{\mu_{1+t}\}_{t\in\mathbb{R}_+}$, where μ_{1+t} is a white noise measure with variance $1 + t$ defined in (1.14). See also Proposition 1.4. Our main strategy is then to *use this (formal) evolution system of measures $\{\mu_{1+t}\}_{t\in\mathbb{R}_+}$ as a replacement of a (formal) invariant measure in Bourgain's invariant measure argument* (and hence as a replacement of a conservation law in the deterministic setting).

Before proceeding further, let us provide a heuristic argument for the claim that $\{\mu_{1+t}\}_{t\in\mathbb{R}_+}$ is an evolution system of measures for SKdV (1.1) with the white noise initial data. First, view the SKdV dynamics (1.1) as a superposition of the deterministic KdV (1.5) and

$$\partial_t u = \xi \tag{1.20}$$

(at the level of infinitesimal generators). On the one hand, the white noise (with any variance) is known to be invariant under the flow of the deterministic KdV (1.5); see [32, 36, 38, 43, 52]. On the other hand, the stochastic flow (1.20) with a white noise initial data (with any variance) increases the variance by the length of the time interval under consideration. Then, the claim follows, at least at a purely formal level, from these observations together with the Lie-Trotter product formula [53, Section VIII.8]:

$$e^{t(A+B)} = \lim_{n\to\infty} \big[e^{\frac{t}{n}A}e^{\frac{t}{n}B}\big]^n \tag{1.21}$$

(which holds, for example, for finite-dimensional matrices A, B). We point out that the Lie-Trotter product formula (1.21) is not directly applicable to our problem, and the core of the proof of Theorem 1.1 consists of justifying this heuristic argument by an approximation argument, which we explain next.

Truncated SKdV dynamics. Given $N \in \mathbb{N}$, let $\mathbf{P}_N$ denotes the Dirichlet projection on (spatial) frequencies $\{|n| \le N\}$. Then, consider the following truncated SKdV equation:

$$\begin{cases} \partial_t u^N + \partial_x^3 u^N + \mathbf{P}_N(\mathbf{P}_N u^N \cdot \partial_x \mathbf{P}_N u^N) = \xi, \\ u^N|_{t=0} = u_0^\omega, \end{cases} \tag{1.22}$$

where u_0^ω is the white noise given in (1.3). Note that the truncation appears only on the nonlinearity, but not on the noise or the initial data. With $\mathbf{P}_N^\perp = \mathrm{Id} - \mathbf{P}_N$, set

$$u_N = \mathbf{P}_N u^N \qquad \text{and} \qquad u_N^\perp = \mathbf{P}_N^\perp u^N.$$

Then, the truncated SKdV dynamics (1.22) decouples into the finite-dimensional nonlinear dynamics for the low frequency part $u_N = \mathbf{P}_N u^N$:

$$\begin{cases} \partial_t u_N + \partial_x^3 u_N + \mathbf{P}_N(u_N \partial_x u_N) = \mathbf{P}_N \xi, \\ u_N|_{t=0} = \mathbf{P}_N u_0^\omega, \end{cases} \tag{1.23}$$

and the linear dynamics for the high frequency part $u_N^\perp = \mathbf{P}_N^\perp u^N$:

$$\begin{cases} \partial_t u_N^\perp + \partial_x^3 u_N^\perp = \mathbf{P}_N^\perp \xi, \\ u_N^\perp|_{t=0} = \mathbf{P}_N^\perp u_0^\omega. \end{cases} \tag{1.24}$$

It is easy to see that both (1.23) and (1.24) are globally well-posed (which implies that (1.22) is globally well-posed); see Section 2. For $t_2 \ge t_1 \ge 0$, we denote by $\Phi_{t_1,t_2}^{N,\mathrm{low}}$ and $\Phi_{t_1,t_2}^{N,\mathrm{high}}$ the solution maps for (1.23) and (1.24) sending data φ at time t_1 to the solutions $\Phi_{t_1,t_2}^{N,\mathrm{low}}\varphi$ and $\Phi_{t_1,t_2}^{N,\mathrm{high}}\varphi$ at time t_2. We let $P_{t_1,t_2}^{N,\mathrm{low}}$ and $P_{t_1,t_2}^{N,\mathrm{high}}$ denote the transition semigroups for (1.23) and (1.24), respectively, defined as in (1.16), where the expectation is taken over the noise restricted to the time interval $[t_1, t_2]$. We also use Φ_{t_1,t_2}^N and P_{t_1,t_2}^N to denote the solution map and the transition semigroup for the truncated SKdV (1.22).

Given $\alpha \ge 0$, let μ_α be the white noise measure (with variance α) as in (1.14). Then, we can write μ_α as

$$\begin{aligned} \mu_\alpha &= \mu_\alpha^{N,\mathrm{low}} \otimes \mu_\alpha^{N,\mathrm{high}} \\ &= (\mathbf{P}_N)_* \mu_\alpha \otimes (\mathbf{P}_N^\perp)_* \mu_\alpha, \end{aligned} \tag{1.25}$$

where $\mu_\alpha^{N,\mathrm{low}} = (\mathbf{P}_N)_*\mu_\alpha$ and $\mu_\alpha^{N,\mathrm{high}} = (\mathbf{P}_N^\perp)_*\mu_\alpha$ the pushforward image measures of μ_α under $\mathbf{P}_N$ and $\mathbf{P}_N^\perp$, respectively. Note that $\mu_\alpha^{N,\mathrm{low}}$ and $\mu_\alpha^{N,\mathrm{high}}$ are nothing but the white noise measures (with variance α) on $E_N = \mathrm{span}\{e^{inx} : |n| \le N\}$ and $E_N^\perp = \mathrm{span}\{e^{inx} : |n| > N\}$, respectively, where the latter span is taken over the space $\mathcal{D}'(\mathbb{T})$ of distributions on $\mathbb{T}$.

The high frequency dynamics (1.23) is linear and it is easy to verify that

$$(P_{t_1,t_2}^{N,\mathrm{high}})^* \mu_{1+t_1}^{N,\mathrm{high}} = \mu_{1+t_2}^{N,\mathrm{high}}. \tag{1.26}$$

By writing it on the Fourier side, we see that the low frequency dynamics (1.23) is nothing but a finite-dimensional system of SDEs, which can be viewed as the superposition of the finite-dimensional KdV dynamics:

$$\partial_t u_N + \partial_x^3 u_N + \mathbf{P}_N(u_N \partial_x u_N) = 0 \tag{1.27}$$

and the linear stochastic dynamics:

$$\partial_t u_N = \mathbf{P}_N \xi. \tag{1.28}$$

While the former (1.27) preserves the white noise $\mu_\alpha^{N,\mathrm{low}}$ (with any variance), the latter (1.28) increases the variance of the white noise initial data by the length of the time interval under consideration. Then, in view of the Lie-Trotter product formula (1.21), we see that

$$(P_{t_1,t_2}^{N,\mathrm{low}})^* \mu_{1+t_1}^{N,\mathrm{low}} = \mu_{1+t_2}^{N,\mathrm{low}}. \tag{1.29}$$

Putting (1.26) and (1.29) together, we then obtain Proposition 1.4.

Proposition 1.4. *Let $N \in \mathbb{N}$. Then, for any $t_2 \ge t_1 \ge 0$, we have*

$$(P_{t_1,t_2}^N)^* \mu_{1+t_1} = \mu_{1+t_2},$$

where P_{t_1,t_2}^N is the transition semigroup for the truncated SKdV (1.22)*. Namely, $\{\mu_{1+t}\}_{t\in\mathbb{R}_+}$ is an evolution system of measures for the truncated SKdV* (1.22)*.*

We present the proof of Proposition 1.4 in Section 2. As for the low frequency part of the claim, instead of decomposing the low frequency dynamics (1.23) into (1.27) and (1.28) and applying the Lie-Trotter product formula (1.21), we verify (1.29) by directly showing that $\mu_{1+t}^{N,\mathrm{low}}$ is the unique solution to the Kolmogorov forward equation (= the Fokker-Planck equation).

Remark 1.5. Let $\alpha \ge 0$. A straightforward modification of the proof of Proposition 1.4 yields

$$(P_{t_1,t_2}^N)^* \mu_\alpha = \mu_{\alpha+(t_2-t_1)},$$

which is the key ingredient for proving Corollary 1.3(i), replacing Proposition 1.4.

Once we obtain Proposition 1.4, we use ideas from Bourgain's invariant measure argument [5] together with the nonlinear analysis in [37], and establish a probabilistic uniform (in N) growth bound on the solutions to the truncated SKdV (1.22). See Proposition 4.1. Finally, Theorem 1.1 follows from a PDE approximation argument and this probabilistic uniform growth bound. See Section 5.

Mean-zero assumption. Recall that the bilinear estimate (1.7) holds only for (spatial) mean-zero functions, namely, the spatial means of $u(t)$ and $v(t)$ are zero for any $t \in \mathbb{R}$. In the case of the deterministic KdV (1.5), if initial data u_0 has nonzero mean α_0, then the following Galilean transformation:

$$u(x,t) \longmapsto u(x+\alpha_0 t, t) - \alpha_0$$

as in [15] together with the conservation of the (spatial) mean under KdV transforms KdV with a nonzero mean into the mean-zero KdV (so that the bilinear estimate (1.7) is applicable). In the case of SKdV with an additive noise, the spatial mean of a solution is no longer conserved. Nonetheless, in [22, 37], a similar transformation was employed to reduce SKdV with an additive noise to the mean-zero case. The transformation in this case depends not only on the mean of the initial condition but also on the Brownian motion β_0 at the zeroth frequency in (1.11). See [22, 37] for details.

For conciseness of the presentation, we impose the following mean-zero assumption in the remaining part of the paper.

- We assume that the white noise initial data u_0^ω in (1.3) and the space-time white noise ξ in (1.1) and (1.22) have spatial mean-zero. This means that the random initial data is now given by

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}_*} g_n(\omega) e^{inx}, \tag{1.30}$$

  where $\mathbb{Z}_* = \mathbb{Z}\setminus\{0\}$, and the stochastic forcing ξ is given by the distributional time derivative of

$$W(t) = \sum_{n\in\mathbb{Z}_*} \beta_n(t) e^{inx}. \tag{1.31}$$

  Namely, we have $\xi = \mathbf{P}_{\neq 0}\xi$, where $\mathbf{P}_{\neq 0}$ is the projection onto the nonzero (spatial) frequencies. This assumption together with the presence of the derivative on the nonlinearity $u\partial_x u = \frac{1}{2}\partial_x u^2$ implies that a solution u to SKdV (1.1) has spatial mean zero as long as it exists.

It is understood that all the functions/distributions have spatial mean zero in the following. The required modifications to handle the general case (i.e. with the white noise u_0^ω in (1.3) and the space-time white noise ξ without the projection $\mathbf{P}_{\neq 0}$) are straightforward and hence we omit details. See [37] for details.

We conclude this introduction by stating several remarks.

Remark 1.6. The usual application of Bourgain's invariant measure argument provides a growth bound[7] on a solution by $\sqrt{\log t}$ for $t \gg 1$, where the implicit constant is random. In the current SKdV problem, we instead obtain a growth bound on a solution by (something slightly faster than) $\sqrt{t \log t}$ for $t \gg 1$, where the extra factor $\sqrt{t}$ comes from the fact that the variance of the white noise at time t grows like $\sim t$. See Remark 5.2.

Remark 1.7.

(i) As mentioned above, by applying the I-method, Colliander, Keel, Staffilani, Takaoka, and Tao [14] proved global well-posedness of the deterministic KdV (1.5)

[7] At least in the setting of [5]. In the singular setting, we have a growth bound by a suitable power of $\log t$. See, for example, Section 5 in [44].

in $H^{-\frac{1}{2}}(\mathbb{T})$. It would be of interest to apply the I-method to study global well-posedness of SKdV (1.1) with general deterministic initial data. In [11], the first author with Cheung and Li adapted the I-method to the stochastic setting and proved global well-posedness, below the energy space, of the stochastic nonlinear Schrödinger equation (SNLS) on $\mathbb{R}^3$ with additive stochastic forcing, white in time and correlated in space. On the one hand, the I-method is suitable for controlling an L^2-based Sobolev norm. On the other hand, the only known local well-posedness result of SKdV (1.1) is in the Fourier-Besov space $\widehat{b}^s_{p,\infty}$ (at this point), and thus there is a nontrivial difficulty in adapting the I-method to this problem.
(ii) In [34], Killip, Vişan, and Zhang exploited the complete integrable structure of the deterministic KdV (1.5) and established a global-in-time a priori bound for solutions to KdV in $H^s(\mathbb{T})$, $s \geq -1$. This a priori bound was given by a sum of suitable rescaled perturbation determinants (each of which is given as an infinite series). It would also be of interest to investigate if their approach can be adapted to the current stochastic setting (and moreover to the Fourier-Besov setting, using the ideas in [49]).

Remark 1.8. Consider the following SNLS on $\mathbb{T}$:

$$i\partial_t u - \partial_x^2 u + |u|^2 u = \xi, \tag{1.32}$$

where ξ is a complex-valued space-time white noise on $\mathbb{T} \times \mathbb{R}_+$, with the complex-valued white noise initial data:

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}} g_n(\omega) e^{inx}, \tag{1.33}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is a family of independent standard complex-valued Gaussian random variables. (Here, we do not impose the condition $g_{-n} = \overline{g_n}$.) Due to the low regularity of the initial data and the forcing, we need to renormalize the nonlinearity in (1.32) to that considered in [12, 25, 27, 50]. In the following discussion, we suppress this renormalization issue.

Let us first consider the (deterministic) nonlinear Schrödinger equation (NLS) on $\mathbb{T}$:

$$i\partial_t u - \partial_x^2 u + |u|^2 u = 0. \tag{1.34}$$

Given $\alpha > 0$, let $\mu_\alpha = \text{Law}(\sqrt{\alpha}\, u_0^\omega)$ with u_0^ω as in (1.33) be the (complex) white noise measure with variance α. Formally, we have

$$d\mu_\alpha = Z_\alpha^{-1} e^{-\frac{1}{2\alpha}\int_\mathbb{T} |u|^2 dx} du.$$

See [38, 43]. Then, in view of the conservation of the L^2-norm under (1.34) and the fact that NLS (1.34) is Hamiltonian, we expect that the white noise measure μ_α is invariant under the NLS dynamics. In [43], the first two authors with Valkó proved formal invariance of the white noise measure under NLS (1.34) in the sense that the white noise measure is a weak limit of invariant measures for NLS (1.34). In the same paper, they also conjectured invariance of the white noise under NLS (1.34). This conjecture remains as a challenging open problem to date, in particular due to the critical nature of the well-posedness issue for (1.34) (and also for (1.32)) with white noise initial data; see [23, 25]. See also [48] for invariance of the white noise measure under the fourth order NLS on $\mathbb{T}$, where $-\partial_x^2$ in (1.34) is replaced by $(-\partial_x^2)^2$.

Let us now turn our attention to SNLS (1.32). As in the SKdV case, by viewing (1.32) as a superposition of the deterministic NLS (1.34) and the stochastic flow $i\partial_t u = \xi$ together with the conjectural invariance of the white noise under NLS (1.34), we arrive at Conjecture 1.

Conjecture 1. *The family* $\{\mu_{1+t}\}_{t\in\mathbb{R}_+}$ *of the white noise measures with variance* $1+t$ *is an evolution system of measures for SNLS* (1.32) *with the white noise initial data* u_0^ω *in* (1.33).

Conjecture 1 is of importance not only from the viewpoint of mathematical analysis but also from the viewpoint of applications due to the importance of SNLS (1.32) (and NLS (1.34)) in nonlinear fiber optics. A straightforward modification of the proof of Proposition 1.4 shows that, for any $N \in \mathbb{N}$, the family $\{\mu_{1+t}\}_{t\in\mathbb{R}_+}$ is an evolution system of measure for the following truncated SNLS:

$$i\partial_t u^N - \partial_x^2 u^N + \mathbf{P}_N(|\mathbf{P}_N u^N|^2 \mathbf{P}_N u^N) = \xi$$

with the white noise initial data u_0^ω in (1.33). The main obstacle for proving Conjecture 1 is the local well-posedness issue as in the case of NLS (1.34) with the white noise initial data.

2. Finite-dimensional approximations and their distributions

In the remaining part of the paper, we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting

- A family $\{g_n\}_{n\in\mathbb{N}}$ of independent standard complex-valued Gaussian random variables:

$$g_n = \operatorname{Re} g_n + i \operatorname{Im} g_n, \qquad n \in \mathbb{N}. \tag{2.1}$$

  Here, $\{\operatorname{Re} g_n, \operatorname{Im} g_n\}_{n\in\mathbb{N}}$ is a family of independent real-valued Gaussian random variables with mean 0 and variance $\frac{1}{2}$. We then set $g_{-n} = \overline{g_n}$, $n \in \mathbb{N}$. The random variables g_n are used to define the spatial white noise u_0^ω on $\mathbb{T}$ in (1.30) which we use as initial data for (1.1) and (1.22).
- A family $\{\beta_n\}_{n\in\mathbb{N}}$ of independent complex-valued Brownian motions, satisfying (1.12):

$$\beta_n(t) = \operatorname{Re} \beta_n(t) + i \operatorname{Im} \beta_n(t), \qquad n \in \mathbb{N},$$

  which is also independent of $\{g_n\}_{n\in\mathbb{N}}$. We then set $\beta_{-n} = \overline{\beta_n}$, $n \in \mathbb{N}$. The Brownian motions β_n serve to define the driving space-time white noise appearing in (1.1) as well as its truncated version (1.22).

We emphasize that we only work with (spatial) mean-zero functions/distributions in the following. Given $\alpha \geq 0$, let $u_{0,\alpha}^\omega$ be a white noise on $\mathbb{T}$ with variance α given by

$$u_{0,\alpha}^\omega(x) = \sqrt{\alpha}\, u_0^\omega(x) = \sqrt{\alpha} \sum_{n\in\mathbb{Z}_*} g_n(\omega) e^{inx}, \tag{2.2}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is as in (2.1), and set

$$\mu_\alpha = \operatorname{Law}(u_{0,\alpha}^\omega) \tag{2.3}$$

to be the (mean-zero) white noise measure with variance α. With this version of μ_α, we set

$$\mu_\alpha^{N,\text{low}} = (\mathbf{P}_N)_* \mu_\alpha \qquad \text{and} \qquad \mu_\alpha^{N,\text{high}} = (\mathbf{P}_N^\perp)_* \mu_\alpha.$$

Then, (1.25) holds in the current setting.

In this section, we study the truncated SKdV (1.22) and present the proof of Proposition 1.4. In view of the discussion in Section 1, it suffices to prove (1.26) and (1.29) for the high and low frequency dynamics, respectively.

We first consider the high frequency dynamics (1.24):

$$\partial_t u_N^\perp + \partial_x^3 u_N^\perp = \mathbf{P}_N^\perp \xi. \tag{2.4}$$

By working on the Fourier side, we see that (2.4) is a system of decoupled linear SDEs for each frequency. In particular, (2.4) is globally well-posed and the solution to (2.4) is given by

$$\widehat{u_N^\perp}(n,t) = e^{itn^3}\widehat{u_N^\perp}(n,0) + \int_0^t e^{i(t-t')n^3} d\beta_n(t'), \qquad |n| > N,$$

for general initial data $u_N^\perp(0) = \mathbf{P}_N^\perp u_N^\perp(0)$. In particular, when the initial data is given by $\mathbf{P}_N^\perp u_0^\omega$ with u_0^ω in (1.30), we have

$$\widehat{u_N^\perp}(n,t) = e^{itn^3} g_n + \int_0^t e^{i(t-t')n^3} d\beta_n(t') =: \mathrm{I}_n + \mathrm{II}_n, \qquad |n| > N.$$

Note that $\mathrm{Law}(\mathrm{I}_n) = \mathrm{Law}(g_n)$ (see Lemma 4.2 in [47]) and $\mathrm{Law}(\mathrm{II}_n) = \mathrm{Law}(\sqrt{t}\, g_n)$. Then, from the independence of I_n and II_n, we conclude that

$$(P_{0,t}^{N,\mathrm{high}})^* \mu_1^{N,\mathrm{high}} = \mu_{1+t}^{N,\mathrm{high}}. \tag{2.5}$$

Therefore, from (2.5) and the flow property of the solution map $\Phi_{t_1,t_2}^{N,\mathrm{high}}$ for (2.4) (analogous to (1.18)), we conclude (1.26).

Let us now turn our attention to the low frequency dynamics (1.23):

$$\partial_t u_N + \partial_x^3 u_N + \mathbf{P}_N(u_N \partial_x u_N) = \mathbf{P}_N \xi. \tag{2.6}$$

Lemma 2.1. *Let $n \in \mathbb{N}$. Given any initial data $u_N(0) = \mathbf{P}_N u_N(0)$ with $\widehat{u}_N(0,0) = 0$, there exists a unique global solution $u_N \in C(\mathbb{R}_+; L^2(\mathbb{T}))$ to (2.6) with $u_N|_{t=0} = u_N(0)$.*

Proof. By writing (2.6) in the Duhamel formulation, we have

$$u_N(t) = S(t)u_N(0) - \frac{1}{2}\int_0^t S(t-t')\partial_x \mathbf{P}_N(u_N^2)(t')dt' + \int_0^t S(t-t')d\mathbf{P}_N W(t'), \tag{2.7}$$

where W is as in (1.31). Note that $u_N(t) = \mathbf{P}_N u_N(t)$ as long as the solution u_N exists. By Bernstein's inequality [57, Appendix A], we have

$$\|\partial_x \mathbf{P}_N u_N^2\|_{C([0,T];L_x^2)} \lesssim N\|u_N\|^2_{C([0,T];L_x^4)} \lesssim N^{\frac{3}{2}}\|u_N\|^2_{C([0,T];L_x^2)}, \tag{2.8}$$

which allows us to control the second term on the right-hand side of (2.7). By the unitarity of $S(t)$ on $L^2(\mathbb{T})$ and the basic property of a Wiener integral, we have

$$\mathbb{E}\bigg[\bigg\|\int_0^t S(t-t')d\mathbf{P}_N W(t')\bigg\|^2_{C([0,T];L_x^2)}\bigg] \lesssim TN.$$

In particular, we have

$$\bigg\|\int_0^t S(t-t')d\mathbf{P}_N W(t')\bigg\|_{C([0,T];L_x^2)} \le C(\omega)T^{\frac{1}{2}}N^{\frac{1}{2}} \tag{2.9}$$

for some almost surely finite random constant $C(\omega) > 0$. Hence, we conclude from a standard contraction argument in $C([0,T];L^2(\mathbb{T}))$ with (2.8) and (2.9) that (2.6) is locally well-posed. Furthermore, the solution exists globally in time as long as its $L^2(\mathbb{T})$-norm remains bounded.

As observed in [22, Theorem 1.5] and [21, Section 3.2], a simple argument, using Ito's formula, Doob's martingale inequality, and the L^2-conservation of the truncated KdV equation (1.27), provides the following bound:

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\|u_N(t)\|_{L^2}^2\Big] &\le \|u_N(0)\|_{L^2}^2 + C(T)\|\mathbf{P}_N\|_{\mathrm{HS}(L^2;L^2)}\\ &\le \|u_N(0)\|_{L^2}^2 + C'(T)N^{\frac{1}{2}}\end{aligned}$$

for any finite $T>0$, where $\|\cdot\|_{\mathrm{HS}(L^2;L^2)}$ denotes the Hilbert-Schmidt norm from $L^2(\mathbb{T})$ to $L^2(\mathbb{T})$. From this a priori bound, we conclude global well-posedness of (2.6). □

In the following, we study the evolution of the distribution of the solution $u_N(t)$ to the low frequency dynamics (2.6). Let $p_n(t) = \operatorname{Re}\widehat{u}_N(n,t)$ and $q_n(t) = \operatorname{Im}\widehat{u}_N(n,t)$ for $1\le |n|\le N$. Since u_N is real-valued, we have

$$p_{-n} = p_n \qquad \text{and} \qquad q_{-n} = -q_n.$$

Then, by writing (2.6) on the Fourier side, we obtain the following finite-dimensional system of SDEs for $(\bar{p},\bar{q}) = (p_1,\dots,p_N,q_1,\dots,q_N)$:

$$\begin{aligned} dp_n &= P_n dt + d(\operatorname{Re}\beta_n),\\ dq_n &= Q_n dt + d(\operatorname{Im}\beta_n)\end{aligned} \tag{2.10}$$

for $n=1,\dots,N$, where P_n and Q_n are defined by

$$\begin{aligned} P_n &:= -n^3 q_n + \sum_{\substack{n=n_1+n_2\\ 1\le|n_1|,|n_2|\le N}} n_2(p_{n_1}q_{n_2} + q_{n_1}p_{n_2}),\\ Q_n &:= n^3 p_n - \sum_{\substack{n=n_1+n_2\\ 1\le|n_1|,|n_2|\le N}} n_2(p_{n_1}p_{n_2} - q_{n_1}q_{n_2}).\end{aligned} \tag{2.11}$$

Define $A(\bar{p},\bar{q}) = A(p_1,\dots,p_N,q_1,\dots,q_N)$ by

$$A(\bar{p},\bar{q}) = (P_1,\dots,P_N,Q_1,\dots,Q_N). \tag{2.12}$$

Then, we have

$$\operatorname{div}_{\bar{p},\bar{q}}A(\bar{p},\bar{q}) = \sum_{n=1}^N(\partial_{p_n}P_n + \partial_{q_n}Q_n) = \sum_{n=1}^N \mathbf{1}_{2n\le N}(nq_{2n} - nq_{2n}) = 0. \tag{2.13}$$

Let $\bar{x} = (x_1,\dots,x_{2N}) = (p_1,\dots,p_N,q_1,\dots,q_N)$. In the following, we briefly go over the derivation of the Kolmogorov forward equation for the evolution of the density of the distribution for

$$\widehat{U}(t) = \big(\operatorname{Re}\widehat{u}_N(1,t),\dots,\operatorname{Re}\widehat{u}_N(n,t),\operatorname{Im}\widehat{u}_N(1,t),\dots,\operatorname{Im}\widehat{u}_N(n,t)\big). \tag{2.14}$$

See, for example, [17,55]. Recalling from (1.12) that $\mathbb{E}[(\operatorname{Re}\beta_n(t))^2]=\mathbb{E}[(\operatorname{Im}\beta_n(t))^2]=\frac{t}{2}$, we see that the Kolmogorov operator $\mathcal{L}$ for (2.10) is given by

$$\mathcal{L} = \frac{1}{4}\Delta_{\bar{x}} + A(\bar{x})\cdot\nabla_{\bar{x}}, \tag{2.15}$$

where $A(\bar{x})$ is given by

$$A(\bar{x}) = (P_1,\dots,P_N,Q_1,\dots,Q_N). \tag{2.16}$$

Lemma 2.2. *Let $f_0(\bar{x})$ be a density of the distribution for $\widehat{U}(0)$. Then, the density $f(\bar{x},t)$ of the distribution for $\widehat{U}(t)$ satisfies the following Kolmogorov forward equation on $\mathbb{R}^{2N}$:*

$$\begin{cases} \partial_t f(\bar{x},t) - \frac{1}{4}\Delta_{\bar{x}} f(\bar{x},t) + A(\bar{x})\cdot\nabla_{\bar{x}} f(\bar{x},t) = 0, \\ f|_{t=0} = f_0, \end{cases} \tag{2.17}$$

where the vector field $A(\bar{x})$ is as in (2.16).

Proof. This is classical, so we only provide a sketch. Consider

$$\begin{cases} (\partial_t - \mathcal{L}) g(\bar{x},t) = 0, \\ g|_{t=0} = g_0, \end{cases} \tag{2.18}$$

where $\mathcal{L}$ is as in (2.15). It is well known that (2.18) has a smooth fundamental solution $p(\bar{x},\bar{y},t)$ for $(\bar{x},\bar{y},t)\in\mathbb{R}^{2N}\times\mathbb{R}^{2N}\times\mathbb{R}_+$, and thus, for initial data $g_0\in C^2(\mathbb{R}^{2N})$ with bounded derivatives, the unique solution to (2.18) is given by

$$g(\bar{x},t) = \int_{\mathbb{R}^{2N}} g_0(\bar{y}) p(\bar{x},\bar{y},t) d\bar{y}. \tag{2.19}$$

Here, $(\bar{x},t)\mapsto p(\bar{x},\bar{y},t)$ satisfies $(\partial_t - \mathcal{L})p(\bar{x},\bar{y},t) = 0$ for each fixed $\bar{y}\in\mathbb{R}^{2N}$. See, for example, [55, Lemma 3.3.3]. Moreover, $g(\bar{x},t)$ has the following probabilistic representation [17, Theorem 9.16]:

$$g(\bar{x},t) = \mathbb{E}_{\bar{y}=\widehat{U}(t)}\big[g_0(\bar{y})\,|\,\bar{x}=\widehat{U}(0)\big], \tag{2.20}$$

where $\widehat{U}(t)$ is as in (2.14) and the expectation on the right-hand side is taken with respect to the vector $\bar{y}=\widehat{U}(t)$ conditioned that $\bar{x}=\widehat{U}(0)$. Hence, it follows from (2.20) and (2.19) that

$$\begin{aligned} \mathbb{E}_{\bar{y}=\widehat{U}(t)}[g_0(\bar{y})] &= \mathbb{E}_{\bar{x}=\widehat{U}(0)}\big[\mathbb{E}_{\bar{y}=\widehat{U}(t)}[g_0(\bar{y})\mid \bar{x}=\widehat{U}(0)]\big] \\ &= \int_{\mathbb{R}^{2N}}\int_{\mathbb{R}^{2N}} g(\bar{y})p(\bar{x},\bar{y},t)d\bar{y} f_0(\bar{x})d\bar{x}. \end{aligned}$$

Therefore, the density $f(\bar{y},t)$ of $\widehat{U}(t)$ is given by

$$f(\bar{y},t) = \int f_0(\bar{x})p(\bar{x},\bar{y},t)d\bar{x}. \tag{2.21}$$

Now, it follows from (2.19), (2.20), and Ito's formula (see, for example, the proof of Proposition 9.9 in [17]), we see

$$\begin{aligned} \int_{\mathbb{R}^{2N}} g_0(\bar{y})\partial_t p(\bar{x},\bar{y},t)d\bar{y} &= \frac{d}{dt}\mathbb{E}_{\bar{y}=\widehat{U}(t)}\big[g_0(\bar{y})\,|\,\bar{x}=\widehat{U}(0)\big] \\ &= \int (\mathcal{L}g_0)(\bar{y})p(\bar{x},\bar{y},t)d\bar{y} \\ &= \int g_0(\bar{y})\,(\mathcal{L}^t_{\bar{y}}p)(\bar{x},\bar{y},t)d\bar{y}, \end{aligned}$$

where $\mathcal{L}^t$ is the formal adjoint of $\mathcal{L}$ given by

$$\mathcal{L}^t_{\bar{y}} = \frac{1}{4}\Delta_{\bar{y}} - A(\bar{y})\cdot\nabla_{\bar{y}}.$$

Note that, in the computation of $\mathcal{L}^t$, we used (2.13): $\mathrm{div}_{\bar{y}} A(\bar{y}) = 0$. Hence, we conclude that $(\bar{y}, t) \mapsto p(\bar{x}, \bar{y}, t)$ satisfies $(\partial_t - \mathcal{L}^t_{\bar{y}})p(\bar{x}, \bar{y}, t) = 0$, and therefore, we conclude from (2.21) that $f(\bar{y}, t)$ satisfies $(\partial_t - \mathcal{L}^t_{\bar{y}}) f(\bar{y}, t) = 0$. □

We are now ready to prove (1.26). Let γ_α be the density for the normal distribution on $\mathbb{R}$ with mean 0 and variance $\frac{\alpha}{2} > 0$:

$$\gamma_\alpha(x) = \frac{1}{\sqrt{\pi\alpha}} e^{-\frac{x^2}{\alpha}}.$$

Then, in the current setting, the density of the distribution for $\mathbf{P}_N u_0^\omega$ with u_0^ω as in (1.30) is given by

$$f_0(\bar{x}) = \prod_{n=1}^{2N} \gamma_1(x_n).$$

Lemma 2.3 shows that the solution $u_N(t)$ to (2.6) with initial data $u^\omega_{0,\alpha} = \sqrt{\alpha} u_0^\omega$ in (2.2) is distributed by the (mean-zero) white noise measure $\mu_{\alpha+t}$ in (2.3), which in particular proves (1.26).

Lemma 2.3. *For any $\alpha > 0$, the function $f_{N,\alpha}$ given by*

$$f_{N,\alpha}(\bar{x}, t) = \prod_{n=1}^{2N} \gamma_{\alpha+t}(x_n) = \frac{1}{(\pi(\alpha+t))^{\frac{N}{2}}} e^{-\frac{|\bar{x}|^2}{\alpha+t}}$$

is the unique solution to (2.17).

Proof. Uniqueness is classical (see [17, Theorem 9.16]). Hence, we only need to check that $f_{N,\alpha}$ is a solution to (2.17).

A direct computation shows

$$\partial_t \gamma_{\alpha+t}(x_n) = \frac{1}{4} \partial^2_{x_n} \gamma_{\alpha+t}(x_n)$$

for $n = 1, \dots, N$. Hence, it suffices to prove

$$A(\bar{x}) \cdot \nabla \bigg(\prod_{n=1}^{2N} \gamma_{\alpha+t}(x_n) \bigg) = 0.$$

Since

$$\partial_{x_n} \gamma_{\alpha+t}(x_n) = -\frac{2x_n}{\alpha+t} \gamma_{\alpha+t}(x_n),$$

it suffices to check $A(\bar{x})\cdot\bar{x}=0$. Recalling $\bar{x}=(x_1,\dots,x_{2N})=(p_1,\dots,p_N,q_1,\dots,q_N)$, it follows from (2.11) and (2.16) that

$$\begin{aligned}A(\bar{x})\cdot\bar{x} &= -\sum_{n=1}^{N} n^3 q_n p_n + \sum_{n=1}^{N} \sum_{\substack{n=n_1+n_2\\1\le|n_1|,|n_2|\le N}} n_2(p_{n_1}q_{n_2}+q_{n_1}p_{n_2})p_n \\ &\quad + \sum_{n=1}^{N} n^3 p_n q_n - \sum_{n=1}^{N} \sum_{\substack{n=n_1+n_2\\1\le|n_1|,|n_2|\le N}} n_2(p_{n_1}p_{n_2}-q_{n_1}q_{n_2})q_n \\ &= -\sum_{n=1}^{N} \operatorname{Re}\big(\mathcal{F}_x(\mathbf{P}_N(u_N\partial_x u_N))(n)\big)\operatorname{Re}\widehat{u}_N(n) \\ &\quad -\sum_{n=1}^{N} \operatorname{Im}\big(\mathcal{F}_x(\mathbf{P}_N(u_N\partial_x u_N))(n)\big)\operatorname{Im}\widehat{u}_N(n),\end{aligned}$$

where $\mathcal{F}_x$ denotes the Fourier transform. In the second step, we used the definition: $p_n(t) = \operatorname{Re}\widehat{u}_N(n,t)$ and $q_n(t) = \operatorname{Im}\widehat{u}_N(n,t)$ together with (2.6) and (2.10). By Parseval's identity with the fact that u_N is real-valued and $u_N = \mathbf{P}_N u_N$, we then have

$$A(\bar{x})\cdot\bar{x} = \frac{1}{2}\int_{\mathbb{T}} \mathbf{P}_N(u_N\partial_x u_N)u_N dx = \frac{1}{6}\int_{\mathbb{T}} \partial_x(u_N^3)dx = 0. \tag{2.22}$$

We point out that, in view of (2.11) and (2.12), $A(\bar{x})\cdot\bar{x} = 0$ in (2.22) is equivalent to the conservation of ℓ^2-norm (= the Euclidean distance in $\mathbb{R}^{2N}$) for the deterministic system:

$$\begin{aligned}\partial_t p_n &= P_n,\\ \partial_t q_n &= Q_n\end{aligned}$$

for $n = 1,\dots,N$ (which in turn is equivalent to the conservation of the L^2-norm for the finite-dimensional KdV (1.27)). This concludes the proof of Lemma 2.3. $\square$

Remark 2.4. A slight modification of the computations in the proof of Lemma 2.3 shows the truncated white noise is invariant under the finite-dimensional KdV dynamics (1.27).

As a corollary to Proposition 1.4, we obtain the following tail estimate on the size of solutions $u^N(t)$ to (1.22).

Lemma 2.5. *Let $s < 0$ and finite $p > 1$ such that $sp < -1$. Given $\alpha > 0$, let u^N be the solution to* (1.22) *with initial data $u_{0,\alpha}^\omega = \sqrt{\alpha}\,u_0^\omega$ in* (2.2). *Then, we have*

$$\begin{aligned}\mathbb{P}\Big(\|u^N(t)\|_{\widehat{b}^s_{p,\infty}} > \lambda\Big) &= \mathbb{P}\bigg(\sqrt{\frac{\alpha+t}{\alpha}}\|u_0^\omega\|_{\widehat{b}^s_{p,\infty}} > \lambda\bigg)\\ &\le Ce^{-c\frac{\alpha}{\alpha+t}\lambda^2}.\end{aligned} \tag{2.23}$$

for any $t \in \mathbb{R}_+$ and $\lambda > 0$, where the constants $C, c > 0$ are independent of $\alpha > 0$.

The inequality in (2.23) follows from the fact that $(\mu_1, \widehat{b}^s_{p,\infty}(\mathbb{T}), L^2(\mathbb{T}))$ is an abstract Wiener space when $sp < -1$ [36, Proposition 3.4] and Fernique's theorem [24]; see Theorem 3.1 in [35].

3. Review of the local well-posedness argument for SKdV

In this section, we go over the local well-posedness argument in [37] and collect useful estimates.

3.1. **Function spaces.** We first recall the definition of the $X^{s,b}$-spaces adapted to the space $\widehat{b}^s_{p,\infty}(\mathbb{T})$ defined in (1.9). Given $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, define the space $X^{s,b}_{p,q}(\mathbb{T} \times \mathbb{R})$ by the norm:

$$\|u\|_{X^{s,b}_{p,q}} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\|_{b^0_{p,\infty}L^q_\tau}. \tag{3.1}$$

In terms of the interaction representation $v(t) = S(-t)u(t)$, we have

$$\|u\|_{X^{s,b}_{p,q}} = \|\langle \partial_x\rangle^s \langle \partial_t\rangle^b v\|_{(\widehat{b}^0_{p,\infty})_x \mathcal{F}L^{0,q}_t},$$

where $\mathcal{F}L^{b,q}(\mathbb{R})$ denotes the Fourier-Lebesgue space defined by the norm:

$$\|f\|_{\mathcal{F}L^{b,q}} = \|\langle \tau\rangle^b \widehat{f}(\tau)\|_{L^q_\tau}. \tag{3.2}$$

From (1.9), we have $b^0_{p,\infty}(\mathbb{Z}) \supset \ell^p(\mathbb{Z}) \supset \ell^2(\mathbb{Z})$ for $p \ge 2$, and thus we have

$$\|u\|_{X^{s,b}_{p,2}} \le \|u\|_{X^{s,b}} \tag{3.3}$$

for $p \ge 2$, where $X^{s,b}$ is the standard $X^{s,b}$-space defined in (1.6). We also have

$$\|u\|_{X^{-\frac12-\delta,b}} \lesssim \|u\|_{X^{-\frac12+\delta,b}_{p,2}}, \tag{3.4}$$

provided that $\delta > \frac{p-2}{4p}$ (with $p \ge 2$). See [37, eq. (17)]. See also the embedding (5.2).

Given an interval $I \subset \mathbb{R}_+$, we define the restriction space $X^{s,b}_{p,q}(I)$ of $X^{s,b}_{p,q}$ to the interval I by

$$\|u\|_{X^{s,b}_{p,q}(I)} = \inf\big\{\|v\|_{X^{s,b}_{p,q}(\mathbb{T}\times\mathbb{R})} : v|_I = u\big\}. \tag{3.5}$$

When $I = [0,T]$, we also set $X^{s,b,T}_{p,q} = X^{s,b}_{p,q}([0,T])$. When $b > \frac12$, it follows from the Riemann-Lebesgue lemma that

$$X^{s,b}_{p,2}(I) \subset C(I; \widehat{b}^s_{p,\infty}(\mathbb{T})) \tag{3.6}$$

for any $s \in \mathbb{R}$ and $1 \le p \le \infty$.

When $q = 2$, in order to capture the temporal regularity of the stochastic convolution:

$$\Psi(t) = \int_0^t S(t-t')dW(t'), \tag{3.7}$$

where W is as in (1.11), we need to take $b < \frac12$ (see Lemma 3.4), for which the embedding (3.6) fails. When $q = 1$, we have the following embedding:

$$X^{s,0}_{p,1}(I) \subset C(I; \widehat{b}^s_{p,\infty}(\mathbb{T})), \tag{3.8}$$

and thus we use $X^{s,0}_{p,1}(I)$ as an auxiliary function space.

We now recall the basic linear estimate for KdV; given $s \in \mathbb{R}$ and $0 \le b < \frac12$, we have

$$\|S(t)u_0\|_{X^{s,b,T}_{p,2}} \lesssim T^{\frac12-b}\|u_0\|_{\widehat{b}^s_{p,\infty}}, \tag{3.9}$$

for $0 < T \le 1$, where $X^{s,b,T}_{p,2} = X^{s,b}_{p,2}([0,T])$ is the restriction space defined in (3.5). Next, we recall the L^4-Strichartz estimate due to Bourgain [4, Proposition 7.15] (see also [56, Proposition 6.4]):

$$\|u\|_{L^4(\mathbb{T}\times\mathbb{R})} \lesssim \|u\|_{X^{0,\frac{1}{3}}}. \tag{3.10}$$

Lastly, we define the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T})$ in the spatial variable by the norm:

$$\|f\|_{\mathcal{F}L^{s,p}} = \|\langle n\rangle^s \widehat{f}(n)\|_{\ell^p_n}. \tag{3.11}$$

Then, for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, we define the $X^{s,b}$-spaces adapted to the Fourier-Lebesgue spaces by the norm:

$$\|u\|_{Y^{s,b}_{p,q}} = \|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\|_{\ell^p_n L^q_\tau}. \tag{3.12}$$

Trivially, we have

$$\|f\|_{\widehat{b}^s_{p,\infty}} \le \|f\|_{\mathcal{F}L^{s,p}} \qquad \text{and} \qquad \|u\|_{X^{s,b}_{p,q}} \le \|u\|_{Y^{s,b}_{p,q}}. \tag{3.13}$$

Given an interval $I \subset \mathbb{R}_+$, we define the restriction space $Y^{s,b}_{p,q}(I)$ as in (3.5).

3.2. **Partially iterated Duhamel formulation.** In this subsection, we discuss the partially iterated Duhamel formulation used in [37].

First, we consider the deterministic KdV (1.5) considered in [6]. By writing it in the Duhamel formulation, we have

$$u(t) = S(t)u_0 - \frac{1}{2}\mathcal{N}(u,u)(t), \tag{3.14}$$

where $\mathcal{N}(u_1,u_2)$ is given by

$$\mathcal{N}(u_1,u_2)(t) = \int_0^t S(t-t')\partial_x(u_1u_2)(t')dt. \tag{3.15}$$

Note that the Fourier transform $\widehat{u_1u_2}(n,\tau)$ can be written in the convolution form:

$$\widehat{u_1u_2}(n,\tau) = \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)\, d\tau_1.$$

Henceforth, we denote by (n,τ), (n_1,τ_1), and (n_2,τ_2) the space-time frequency variables for the Fourier transforms of $\mathcal{N}(u_1,u_2)$, u_1, and u_2 in (3.15), respectively. In particular, we have

$$n = n_1 + n_2 \qquad \text{and} \qquad \tau = \tau_1 + \tau_2. \tag{3.16}$$

By assuming that the initial data u_0 has spatial mean 0, it follows that $u(t)$ also has mean 0. Furthermore, in view of the derivative on the nonlinearity, we may assume that $n, n_1, n_2 \ne 0$. We also denote the modulations by

$$\sigma_0 = \langle \tau - n^3\rangle \qquad \text{and} \qquad \sigma_j = \langle \tau_j - n_j^3\rangle, \quad j = 1,2. \tag{3.17}$$

Recall the following algebraic relation [4]:

$$n^3 - n_1^3 - n_2^3 = 3nn_1n_2$$

for $n = n_1 + n_2$. Then, under (3.16), we have

$$\text{MAX} := \max(\sigma_0,\sigma_1,\sigma_2) \gtrsim \langle nn_1n_2\rangle. \tag{3.18}$$

In our setting, we need to take $b < \frac{1}{2}$ to capture the temporal regularity of the stochastic convolution Ψ in (3.7). On the other hand, the crucial bilinear estimate

(1.7) holds only for $b = \frac{1}{2}$. In order to overcome this difficulty, we decompose the nonlinearity into three pieces, depending on the sizes of the modulations σ_0, σ_1, and σ_2. Define the sets M_j, $j = 0, 1, 2$, by

$$
\begin{aligned}
M_0 &= \big\{(n, n_1, n_2, \tau, \tau_1, \tau_2) \in \mathbb{Z}_*^3 \times \mathbb{R}^3 : \sigma_0 = \mathrm{MAX}\,\big\}, \\
M_j &= \big\{(n, n_1, n_2, \tau, \tau_1, \tau_2) \in \mathbb{Z}_*^3 \times \mathbb{R}^3 : \sigma_j = \mathrm{MAX} \text{ and } \sigma_j > 1\big\}, \quad j = 1, 2.
\end{aligned} \tag{3.19}
$$

For $j = 0, 1, 2$, let $\mathcal{N}_j(u_1, u_2)$ be the contribution of $\mathcal{N}(u_1, u_2)$ on M_j, and thus we have

$$
\mathcal{N}(u_1, u_2) = \sum_{j=0}^{2} \mathcal{N}_j(u_1, u_2). \tag{3.20}
$$

The standard bilinear estimate (1.7) allows us to estimate $\mathcal{N}_0(u_1, u_2)$ even when $b < \frac{1}{2}$; see [37, eq. (46)]. As for $\mathcal{N}_j(u_1, u_2)$, $j = 1, 2$, however, the bilinear estimate fails for temporal regularity $b < \frac{1}{2}$ (in $X^{s,b}_{p,q}$ for any $s \in \mathbb{R}$ and $1 \le p, q \le \infty$) since, in this case, we do not have a sufficient power for the largest modulation σ_j to control the derivative loss in the nonlinearity. See [31].

This issue was circumvented in [6, 37, 38] by partially iterating the Duhamel formulation (3.14) and writing it as

$$
u(t) = S(t)u_0 - \frac{1}{2}\mathcal{N}_0(u, u)(t) + \frac{1}{4}\mathcal{N}_1(\mathcal{N}(u, u), u) + \frac{1}{4}\mathcal{N}_2(u, \mathcal{N}(u, u)).
$$

Namely, for $j = 1, 2$, we replaced the jth entry in $\mathcal{N}_j(u, u)$ (where the maximum modulation is given by σ_j) by its Duhamel formulation (3.14). It follows from the definition of $\mathcal{N}_j$ (see (3.19)) that there is no contribution from the linear solution $S(t)u_0$ in iterating the Duhamel formulation, since its space-time Fourier transform is supported on $\{\tau = n^3\}$, namely, $S(t)u_0$ has zero modulation, and thus from the definition of $\mathcal{N}_j$, we have $\mathcal{N}_1(S(t)u_0, u) = \mathcal{N}_2(u, S(t)u_0) = 0$.

In the context of SKdV (1.1) and its Duhamel formulation (1.10):

$$
u(t) = S(t)u_0 - \frac{1}{2}\mathcal{N}(u, u) + \Psi, \tag{3.21}
$$

the discussion above leads to

$$
\begin{aligned}
u(t) = S(t)u_0 &- \frac{1}{2}\mathcal{N}_0(u, u)(t) \\
&+ \frac{1}{4}\mathcal{N}_1(\mathcal{N}(u, u), u) - \frac{1}{2}\mathcal{N}_1(\Psi, u) \\
&+ \frac{1}{4}\mathcal{N}_2(u, \mathcal{N}(u, u)) - \frac{1}{2}\mathcal{N}_2(u, \Psi) + \Psi,
\end{aligned} \tag{3.22}
$$

where Ψ is the stochastic convolution in (3.7). In [37], the first author studied this new formulation (3.22) and establish an a priori bound on solutions (with smooth initial data and (spatially) smooth noise), which allowed him to construct a solution (1.10) by an approximation argument.

Lastly, we state an analogous formulation for the truncated SKdV (1.22). By writing (1.22) in the Duhamel formulation, we have

$$
u^N(t) = S(t)u_0 - \frac{1}{2}\mathcal{N}^N(u^N, u^N) + \Psi, \tag{3.23}
$$

where $\mathcal{N}^N(u^N, u^N)$ is given by

$$\begin{aligned}\mathcal{N}^N(u_1, u_2) &= \mathbf{P}_N\mathcal{N}(\mathbf{P}_N u_1, \mathbf{P}_N u_2)\\ &= \int_0^t S(t-t')\partial_x \mathbf{P}_N(\mathbf{P}_N u_1 \mathbf{P}_N u_2)(t')dt.\end{aligned}$$

Then, by partially iterating the Duhamel formulation as above, we rewrite (3.23) as

$$\begin{aligned}u^N(t) &= S(t)u_0 - \frac{1}{2}\mathcal{N}_0^N(u^N, u^N)(t)\\ &\quad + \frac{1}{4}\mathcal{N}_1^N(\mathcal{N}^N(u^N, u^N), u^N) - \frac{1}{2}\mathcal{N}_1^N(\Psi, u^N)\\ &\quad + \frac{1}{4}\mathcal{N}_2^N(u^N, \mathcal{N}^N(u^N, u^N)) - \frac{1}{2}\mathcal{N}_2^N(u^N, \Psi) + \Psi,\end{aligned} \tag{3.24}$$

where $\mathcal{N}_j^N(u_1, u_2)$ is the contribution of $\mathcal{N}^N(u_1, u_2)$ on M_j, $j = 0, 1, 2$.

3.3. **Local well-posedness and an a priori bound.** In this subsection, we collect the useful nonlinear estimates on the iterated formulation (3.22) from [37], and establish an a priori bound for solutions to the truncated SKdV (1.22).

We first recall the following local well-posedness result of SKdV (1.1) from [37].

Theorem 3.1. *Let $s = -\frac{1}{2} + \delta$ and $p = 2 + \delta_0$ for some small $\delta, \delta_0 > 0$ such that $\frac{p-2}{4p} < \delta < \frac{p-2}{2p}$. Given a mean-zero function $u_0 \in \widehat{b}^s_{p,\infty}(\mathbb{T})$, there exist a stopping time $T_\omega > 0$ and a unique solution $u \in C([0, T_\omega]; \widehat{b}^s_{p,\infty}(\mathbb{T})) \cap X^{s,\frac{1}{2}-\delta}_{p,2}([0, T_\omega])$ to (1.1) with $u|_{t=0} = u_0$. Furthermore, denoting by $T_* = T_*(\omega)$ the maximal time of existence, we have the following blowup alternative*:

$$\lim_{t\nearrow T_*} \|u(t)\|_{\widehat{b}^s_{p,\infty}} = \infty \qquad \text{or} \qquad T_* = \infty.$$

Here, the condition $\delta < \frac{p-2}{2p}$ is equivalent to $sp < -1$, while $\delta > \frac{p-2}{4p}$ is used for the embedding (3.4).

In the following, we recall some of the nonlinear estimates from [37]. In the remaining part of the paper, we fix small $\delta, \delta_0 > 0$, satisfying the hypothesis in Theorem 3.1, and set

$$\alpha = \frac{1}{2} - \delta \tag{3.25}$$

as in [6, 37]. The following discussion applies to both the original SKdV (1.1) and its truncated version (1.22). In order to treat them in a uniform manner, we take $N \in \mathbb{Z}_{\geq 0}$ and set $u^\infty = u$, $\mathcal{N}^\infty(u_1, u_2) = \mathcal{N}(u_1, u_2)$, and $\mathcal{N}_j^\infty(u_1, u_2) = \mathcal{N}_j(u_1, u_2)$, $j = 0, 1, 2$. Note that, in the following, all the estimates hold, uniformly in $N \in \mathbb{Z}_{\geq 0}$.

Given $T > 0$, we define the random quantity $L_\omega(T)$ by

$$L_\omega(T) = \|\mathbf{1}_{[0,T]}\Psi\|_{X^{-\frac{1}{2}-\frac{1}{2}\delta, \frac{1}{2}-\delta}} + \|\mathbf{1}_{[0,T]}\Psi\|_{Y_{2,4}^{-\frac{1}{2}-\frac{1}{2}\delta, \frac{11}{16}+\delta}}, \tag{3.26}$$

which is a pathwise[8] upper bound for the $X^{-\alpha, 1-\alpha, T}$-norms of $\mathcal{N}_1(\Psi, u)$ in (3.22) and $\mathcal{N}_1^N(\Psi, u^N)$ in (3.24). See Appendix B. We point out that, while the analysis in Appendix B (see (B.2) and (B.4)) yields the spatial regularity $-\frac{1}{2} - \delta$, we use a slightly worse spatial regularity for the definition of $L_\omega(T)$ in (3.26) (so that

[8]In [37, "Estimate on (ii)" on pp. 296-297], an expectation was taken on the $X^{-\alpha,1-\alpha,T}$-norms of $\mathcal{N}_1(\Psi, u)$. However, we in fact need a pathwise bound, which is established in Appendix B.

the estimate (5.8) holds, allowing us to gain a decay in N). Note that, in contrast to [37], we defined $L_\omega(T)$ on the "long" interval $[0, T]$. This will be useful in Sections 4 and 5 when we iterate the local-in-time argument on many small subintervals of $[0, T]$ but with a fixed driving space-time white noise. From (3.26) and Remark 3.5, we have

$$\|L_\omega(T)\|_{L^r(\Omega)} \lesssim \sqrt{r}\, T^{\frac{3}{2}} \tag{3.27}$$

for any $T > 0$ and $1 \le r < \infty$, provided that $\delta > 0$ is sufficiently small such that

$$\Big(\frac{11}{16} + \delta - 1\Big) 4 < -1.$$

With this notation, the main nonlinear estimate [37, eq. (73)] (see also Appendix B) reads (with some small $\theta > 0$)

$$\begin{aligned} \|u^N\|_{X^{-\alpha,\alpha,T_1}_{p,2}} \le C_1 \|u_0^N\|_{\widehat{b}^{-\alpha}_{p,\infty}} + \frac{1}{2} C_2 T_1^\theta \|u^N\|^2_{X^{-\alpha,\alpha,T_1}_{p,2}} + 2C_3 T_1^\theta \|u^N\|^3_{X^{-\alpha,\alpha,T_1}_{p,2}} \\ + 2C_3 T_1^\theta L_\omega(T) \|u^N\|_{X^{-\alpha,\alpha,T_1}_{p,2}} + C_4 \|\Psi\|_{X^{-\alpha,\alpha,T_1}_{p,2}} \end{aligned} \tag{3.28}$$

for any $T > 0$ and $0 < T_1 \le \min(1, T)$, provided that

$$C_3 T_1^\theta R \le \frac{1}{2} \qquad \text{and} \qquad \|u^N\|_{X^{-\alpha,\alpha,T_1}_{p,2}} \le R. \tag{3.29}$$

Here, in importing (3.28) from [37], we use the fact that $\mathbf{P}_N$ is bounded on relevant function spaces, uniformly in $N \in \mathbb{N}$. The nonlinear estimate (3.28) together with an analogous difference estimate [37, eq. (74)] allows us to construct a solution $u \in X^{-\alpha,\alpha,T_1}_{p,2}$ to (1.1) as a limit of smooth solutions.

Next, we estimate the $\widehat{b}^{-\alpha}_{p,\infty}$-norm of a solution $u^N(t)$ to (3.23). Since $\alpha < \frac{1}{2}$, the embedding (3.6) does not hold and thus (3.28) is not directly applicable. However, some terms can be estimated in a stronger norm. Indeed, from (3.6), (3.3), and [37, eq. (47) and (72)], we have, under the condition (3.29),

$$\begin{aligned} &\|\mathcal{N}_1^N(u^N, u^N) + \mathcal{N}_2^N(u^N, u^N)\|_{C([0,T_1];\widehat{b}^{-\alpha}_{p,\infty})} \\ &\quad \lesssim \|\mathcal{N}_1^N(u^N, u^N) + \mathcal{N}_2^N(u^N, u^N)\|_{X^{-\alpha,1-\alpha,T_1}} \\ &\quad \le 2C_3 \Big(T_1^\theta \|u^N\|^3_{X^{-\alpha,\alpha,T_1}_{p,2}} + T_1^\theta L_\omega(T) \|u^N\|_{X^{-\alpha,\alpha,T_1}_{p,2}} \Big) \end{aligned} \tag{3.30}$$

for any $T > 0$ and $0 < T_1 \le \min(1, T)$, where the first inequality follows from (3.6) since $b = 1 - \alpha = \frac{1}{2} + \delta > \frac{1}{2}$. As for $\mathcal{N}_0^N$, we write it as

$$\mathcal{N}_0^N(u^N, u^N) = \mathcal{N}_3^N(u^N, u^N) + \mathcal{N}_4^N(u^N, u^N),$$

where $\mathcal{N}_3^N$ denotes the contribution of $\mathcal{N}_0^N$ on $\{\max(\sigma_1, \sigma_2) \gtrsim \langle n n_1 n_2 \rangle^{\frac{1}{100}}\}$. Then, from (3.6), (3.8), and [37, (a) and (b) on p. 302)], we have

$$\begin{aligned} \|\mathcal{N}_0^N(u^N, u^N)\|_{C([0,T_1];\widehat{b}^{-\alpha}_{p,\infty})} &\lesssim \|\mathcal{N}_3^N(u^N, u^N)\|_{X^{-\alpha,1-\alpha,T_1}_{p,2}} \\ &\quad + \|\mathcal{N}_4^N(u^N, u^N)\|_{X^{-\alpha,0,T_1}_{p,1}} \\ &\lesssim T_1^\theta \|u^N\|^2_{X^{-\alpha,\alpha,T_1}_{p,2}}. \end{aligned} \tag{3.31}$$

Hence, putting (3.23), (3.30), and (3.31) together, we obtain

$$\begin{aligned}\|u^N\|_{C([0,T_1];\widehat{b}^{-\alpha}_{p,\infty})} \le \|u_0\|_{\widehat{b}^{-\alpha}_{p,\infty}} &+ C_5 T_1^\theta \|u^N\|^2_{X^{-\alpha,\alpha,T_1}_{p,2}} \\ &+ C_6 T_1^\theta \|u^N\|^3_{X^{-\alpha,\alpha,T_1}_{p,2}} + C_7 T_1^\theta L_\omega(T)\|u^N\|_{X^{-\alpha,\alpha,T_1}_{p,2}} \\ &+ \|\Psi\|_{C([0,T_1];\widehat{b}^{-\alpha}_{p,\infty})}\end{aligned} \tag{3.32}$$

for any $T > 0$ and $0 < T_1 \le \min(1, T)$, provided that (3.29) holds.

We now state an a priori bound on a solution u^N to the truncated SKdV (1.22).

Lemma 3.2. *Let $N \in \mathbb{N}$. Then, there exist absolute constants $\gamma > 0$ and $C_*, c_* > 0$ such that, given any $T > 0$, we have*

$$\|u^N\|_{X^{-\alpha,\alpha}_{p,2}(I)} \le C_*\Big(\|u^N(t_0)\|_{\widehat{b}^{-\alpha}_{p,\infty}} + \|\Psi\|_{X^{-\alpha,\alpha}_{p,2}([t_0,t_0+1])} + 1\Big) =: R_\omega(t_0) \tag{3.33}$$

for any time interval $I = [t_0, t_0 + T_1] \subset [0, T]$ of length $T_1 \le 1$ and any solution u^N to the truncated SKdV (1.22), provided that

$$T_1 \le c_*(R_\omega(R_\omega + 1) + L_\omega(T))^{-\gamma}. \tag{3.34}$$

Here, the constants $\gamma > 0$ and $C_*, c_* > 0$ are independent of $N \in \mathbb{N}$.

Proof. Under (3.29), it follows from (3.28) that

$$\|u^N\|_{X^{-\alpha,\alpha}_{p,2}(I)} \le 2C_1\|u^N(t_0)\|_{\widehat{b}^{-\alpha}_{p,\infty}} + 2C_4\|\Psi\|_{X^{-\alpha,\alpha}_{p,2}(I)}, \tag{3.35}$$

provided that

$$T_1^\theta\bigg(\frac{1}{2}C_2 R + 2C_3 R^2 + 2C_3 L_\omega(T)\bigg) \le \frac{1}{2}.$$

Then, under (3.34) with $\gamma = \theta^{-1}$, the bound (3.33) follows from (3.35) and a continuity argument. ☐

Remark 3.3. In order use a continuity argument in the proof of Lemma 3.2 presented above, we need the continuity of the $X^{-\alpha,\alpha}_{p,2}([t_0, t_1])$-norm with respect to the right endpoint t_1. While it may be possible to check this directly (see, for example, [2, Appendix A] and [27, Lemma 8.1]), let us use the following equivalence:

$$\|u\|_{X^{-\alpha,\alpha}_{p,2}([t_0,t_1])} \sim \|\mathbf{1}_{[t_0,t_1]}u\|_{X^{-\alpha,\alpha}_{p,2}}, \tag{3.36}$$

where the norm on the left-hand side is defined in (3.5), and study the latter norm. Recall that the equivalence (3.36) holds since the temporal regularity $b = \alpha = \frac{1}{2} - \delta$ is below $\frac{1}{2}$ (see [16, eq. (3.5)]). Such equivalence also holds for the general $X^{s,b}_{p,q}([t_0, t_1])$ for $0 \le b < \frac{q-1}{q}$; see [10].

Given small $h > 0$, from the triangle inequality, we have

$$\|\mathbf{1}_{[t_0,t_1+h]}u\|_{X^{-\alpha,\alpha}_{p,2}} - \|\mathbf{1}_{[t_0,t_1]}u\|_{X^{-\alpha,\alpha}_{p,2}} \le \|\mathbf{1}_{[t_1,t_1+h]}u\|_{X^{-\alpha,\alpha}_{p,2}}, \tag{3.37}$$

and thus it suffices to show that the right-hand side of (3.37) tends to 0 as $h \to 0$. In view of the definition (3.1), such a claim follows once we prove

$$\lim_{h\to 0}\|\mathbf{1}_{[t_1,t_1+h]}f\|_{H^\alpha} = 0 \tag{3.38}$$

for a function $f \in H^\alpha(\mathbb{R})$. Obviously, we have $\lim_{h\to 0} \|\mathbf{1}_{[t_1,t_1+h]} f\|_{L^2} = 0$. Using the physical side characterization of the homogeneous Sobolev norm, we have

$$\begin{aligned}\|\mathbf{1}_{[t_1,t_1+h]} f\|^2_{\dot{H}^\alpha} &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\mathbf{1}_{[t_1,t_1+h]}(t)f(t) - \mathbf{1}_{[t_1,t_1+h]}(\tau)f(\tau)|^2}{|t-\tau|^{1+2\alpha}} dt d\tau \\ &= \mathrm{I}(h) + \mathrm{II}(h) + \mathrm{III}(h),\end{aligned}$$

where I, II, and III are defined by

$$\begin{aligned}\mathrm{I}(h) &= \int_{[t_1,t_1+h]}\int_{[t_1,t_1+h]} \frac{|f(t) - f(\tau)|^2}{|t-\tau|^{1+2\alpha}} dt d\tau, \\ \mathrm{II}(h) &= \int_{[t_1,t_1+h]}\int_{[t_1,t_1+h]^c} \frac{|f(t)|^2}{|t-\tau|^{1+2\alpha}} d\tau dt, \\ \mathrm{III}(h) &= \int_{[t_1,t_1+h]}\int_{[t_1,t_1+h]^c} \frac{|f(\tau)|^2}{|t-\tau|^{1+2\alpha}} dt d\tau.\end{aligned}$$

By the dominated convergence theorem with the fact that $f \in H^\alpha(\mathbb{R})$, we see that $\lim_{h\to\infty} \mathrm{I}(h) = 0$. As for $\mathrm{II}(h)$, integration in τ yields

$$\begin{aligned}\mathrm{II}(h) &\sim \int_{[t_1,t_1-h]} \frac{|f(t)|^2}{|t-t_1-h|^{2\alpha}} dt + \int_{[t_1,t_1+h]} \frac{|f(t)|^2}{|t-t_1|^{2\alpha}} dt \\ &\lesssim \|f\|_{\dot{H}^\alpha(\mathbb{R})},\end{aligned}$$

where the second step follows from Hardy's inequality [57, Lemma A.2] since $0 \le \alpha < \frac{1}{2}$. Noting that $\mathrm{III}(h) = \mathrm{II}(h)$, we see that the term $\mathrm{III}(h)$ also satisfies the bound above. Also, the case $h < 0$ follows from an analogous consideration. Putting everything together, we conclude (3.38). See also Lemma 4.4 in [7].

We conclude this section by stating a lemma on growth of the stochastic convolution Ψ in (3.7) over long time intervals. We point out that analogous regularity results were obtained in [37, Propositions 4.1 and 4.5] but they are only for short times.

Lemma 3.4. *Let $s < 0$ and $1 \le p, q < \infty$ such that $sp < -1$.*
(i) *Let $(b-1)q < -1$. Given any $1 \le r < \infty$ and $T \ge 1$, we have*

$$\Big\| \|\Psi\|_{X^{s,b}_{p,q}(I)} \Big\|_{L^r(\Omega)} \le \Big\| \|\Psi\|_{Y^{s,b}_{p,q}(I)} \Big\|_{L^r(\Omega)} \lesssim \sqrt{rT} \tag{3.39}$$

for any interval $I \subset [0,T]$ with length $|I| \le 1$, where the implicit constant is independent of r and T.
(ii) *Given any $1 \le r < \infty$ and $T \ge 1$, we have*

$$\Big\| \|\Psi\|_{C([0,T];\widehat{b}^s_{p,\infty})} \Big\|_{L^r(\Omega)} \lesssim \sqrt{rT\log T}, \tag{3.40}$$

where the implicit constant is independent of r and T.

We present the proof of Lemma 3.4 in Appendix A.

Remark 3.5. We point out that the bound (3.39) holds only for intervals I of short lengths. Indeed, a slight modification of the proof yields the following estimate for $I = [0,T]$:

$$\Big\| \|\Psi\|_{X^{s,b,T}_{p,q}} \Big\|_{L^r(\Omega)} \le \Big\| \|\Psi\|_{Y^{s,b,T}_{p,q}} \Big\|_{L^r(\Omega)} \lesssim \sqrt{r}\, T^{\frac{3}{2}}, \tag{3.41}$$

where the right-hand side is much worse than those in (3.39) and (3.40). See Remark A.1.

4. Probabilistic uniform growth bound

Given $N \in \mathbb{N}$, let u^N be the global solution to the truncated SKdV (1.22) with the mean-zero white noise initial data u_0^ω in (1.30). Our main goal in this section is to establish the following probabilistic growth bound on the solution u^N to (1.1) whose proof is based on a variant of Bourgain's invariant measure argument in the current setting of an evolution system of measures (Proposition 1.4).

Proposition 4.1. *Let α and p be as in* (3.25) *and Theorem* 3.1*, respectively. Given any $T \gg 1$ and $0 < \varepsilon \ll 1$, there exists a set $\Omega_{T,\varepsilon}(N)$ such that $\mathbb{P}(\Omega_{T,\varepsilon}(N)^c) < \varepsilon$ and*

$$\sup_{t\in[0,T]} \|u^N(t)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \le C\sqrt{\log\frac{1}{\varepsilon}}\sqrt{T\log T} \tag{4.1}$$

on $\Omega_{T,\varepsilon}(N)$, where the constant $C > 0$ is independent of $N \in \mathbb{N}$, $T \gg 1$, and $\varepsilon \ll 1$.

Proof. Fix small $T_1 > 0$ (to be chosen later), and let $I_j = [jT_1, (j+1)T_1] \cap [0, T]$, $j \in \mathbb{Z}_{\ge 0}$. Recall from Proposition 1.4 that the solution $u^N(jT_1)$ at time $t = jT_1$ is distributed by the white noise measure μ_{1+jT_1} with variance $1 + jT_1$, where μ_{1+jT_1} is as in (2.3). Then, given $K_1 \gg 1$, set $\Omega_1 = \Omega_1(T, \varepsilon, N) \subset \Omega$ by

$$\Omega_1 = \bigcap_{j=0}^{[T/T_1]} \Big\{\|u^N(jT_1)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \le K_1\Big\}. \tag{4.2}$$

Then, it follows from Lemma 2.5 and choosing

$$K_1 = r_1\sqrt{T\log\frac{T}{\varepsilon}} \tag{4.3}$$

for some $r_1 \gg 1$(to be chosen later) that

$$\mathbb{P}(\Omega_1^c) \lesssim \sum_{j=0}^{[T/T_1]} e^{-\frac{c}{1+jT_1}K_1^2} \sim \frac{T}{T_1}e^{-\frac{c'}{T}K_1^2} = T_1^{-1}T^{1-c'r_1^2}\varepsilon^{c'r_1^2}. \tag{4.4}$$

Next, define $\Omega_2 = \Omega_2(T, \varepsilon) \subset \Omega$ by

$$\Omega_2 = \bigcap_{j=0}^{[T/T_1]} \Big\{\|\Psi\|_{X^{-\alpha,\alpha}_{p,2}([jT_1, jT_1+1])} \le K_1\Big\}, \tag{4.5}$$

where K_1 is as in (4.3). Then, by Lemma 3.4(i) and Chebyshev's inequality, we have

$$\mathbb{P}(\Omega_2^c) \lesssim \sum_{j=0}^{[T/T_1]} e^{-\frac{c}{1+jT_1}K_1^2} \sim T_1^{-1}T^{1-c'r_1^2}\varepsilon^{c'r_1^2} \tag{4.6}$$

just as in (4.4). Lastly, define $\Omega_3 = \Omega_3(T, \varepsilon) \subset \Omega$ by

$$\Omega_3 = \big\{L_\omega(T) \le K_2\big\}, \tag{4.7}$$

where $L_\omega(T)$ is as in (3.26) and

$$K_2 = r_2\sqrt{T^3\log\frac{1}{\varepsilon}}. \tag{4.8}$$

Then, by choosing $r_2 > 0$ sufficiently large, it follows from (3.27) and Chebyshev's inequality that

$$\mathbb{P}(\Omega_3^c) \le Ce^{-\frac{c}{T^3}K_2^2} < \frac{\varepsilon}{4}. \tag{4.9}$$

Let R_ω be as in (3.33). Then, on $\Omega_1 \cap \Omega_2 \cap \Omega_3$, we have

$$R_\omega(jT_1) \le C_*(2K_1 + 1) \sim K_1 \qquad \text{and} \qquad L_\omega(T) \le K_2 \tag{4.10}$$

for $j = 0, 1, \dots, \big[\frac{T}{T_1}\big]$. In view of (3.29) and (3.34) in Lemma 3.2 with (4.10), we now choose $T_1 > 0$ by setting

$$T_1 \sim \min\Big\{K_1^{-\frac{1}{\theta}},\, (K_1^2 + K_2)^{-\gamma}\Big\}. \tag{4.11}$$

Then, by choosing $r_1 > 0$ sufficiently large, it follows from (4.4) and (4.6) with (4.3), (4.8), and (4.11) that

$$\mathbb{P}(\Omega_k^c) < \frac{\varepsilon}{4} \tag{4.12}$$

for $k = 1, 2$. Furthermore, from Lemma 3.2 and (4.10) with (4.3), we obtain

$$\|u^N\|_{X_{p,2}^{-\alpha,\alpha}(I_j)} \le C_*(2K_1 + 1) \sim K_1 \sim \sqrt{T \log \frac{T}{\varepsilon}} \tag{4.13}$$

for $j = 0, 1, \dots, \big[\frac{T}{T_1}\big]$.

Now, define $\Omega_4 = \Omega_4(T, \varepsilon) \subset \Omega$ by

$$\Omega_4 = \bigg\{\|\Psi\|_{C([0,T];\widehat{b}^s_{p,\infty})} \le r_3 \sqrt{\log \frac{1}{\varepsilon}} \sqrt{T \log T}\bigg\}. \tag{4.14}$$

Then, from Lemma 3.4(ii) and Chebyshev's inequality, we have

$$\mathbb{P}(\Omega_4^c) < \frac{\varepsilon}{4} \tag{4.15}$$

by choosing $r_3 > 0$ sufficiently large. In view of (3.32) with (4.10) and (4.13), we further impose that

$$T_1^\theta\big(C_5 C_*(2K_1 + 1) + C_6 C_*^2(2K_1 + 1)^2 + C_7 K_2\big) \le 1. \tag{4.16}$$

Note that (4.16) yields $T_1 \lesssim (K_1^2 + K_2)^{-\frac{1}{\theta}}$, which is essentially implied by (4.11) (by possibly making r_1 larger) and thus the bound (4.12) still holds.

Finally, set $\Omega_{T,\varepsilon}(N) = \Omega_1 \cap \dots \cap \Omega_4$. Then, from (4.9), (4.12), and (4.15), we have

$$\mathbb{P}(\Omega_{T,\varepsilon}(N)^c) < \varepsilon.$$

Furthermore, on $\Omega_{T,\varepsilon}(N)$, we conclude from (3.32) with (4.2), (4.3), (4.13), (4.14), and (4.16) that

$$\|u^N\|_{C(I_j;\widehat{b}^{-\alpha}_{p,\infty})} \lesssim \sqrt{\log \frac{1}{\varepsilon}} \sqrt{T \log T},$$

uniformly in $j = 0, 1, \dots, \big[\frac{T}{T_1}\big]$, which implies (4.1). □

5. Approximation argument

In this section, we present the proof of Theorem 1.1. We first establish the following 'almost' almost sure global well-posedness of SKdV (1.1) via an approximation argument.

Given $N \in \mathbb{N}$, let u^N be the global solution to the truncated SKdV (1.22) with the mean-zero white noise initial data u_0^ω in (1.30), and let u be the solution to SKdV (1.1) with the mean-zero white noise initial data u_0^ω in (1.30), whose local existence is guaranteed by Theorem 3.1.

Proposition 5.1. *Let $\alpha = \frac{1}{2} - \delta$ and $p = 2 + \delta_0$ for some small $\delta, \delta_0 > 0$ such that $\frac{p-2}{3p} < \delta < \frac{p-2}{2p}$. Given any $T \gg 1$ and $0 < \varepsilon \ll 1$, there exist a set $\Omega_{T,\varepsilon}$ and $N_* = N_*(T,\varepsilon) \in \mathbb{N}$ such that $\mathbb{P}(\Omega_{T,\varepsilon}^c) < \varepsilon$ and, on $\Omega_{T,\varepsilon}$, we have*

$$\sup_{t\in[0,T]} \|u(t) - u^{N_*}(t)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \leq C(T,\varepsilon) N_*^{-\frac{\delta}{2}}. \tag{5.1}$$

In particular, on $\Omega_{T,\varepsilon}$, the solution u to SKdV (1.1) with the mean-zero white noise initial data u_0^ω in (1.30) exists on the time interval $[0,T]$.

As compared to Theorem 3.1, we need an extra restriction $\delta > \frac{p-2}{3p}$ in order to obtain a decay in N. See (5.2) and (5.12).

Proof. We first record the following embedding, which requires the additional condition $\delta > \frac{p-2}{3p}$. Let $p > 2$. By Hölder's inequality, we have

$$\begin{aligned}\|f\|_{H^{-\frac{1}{2}-\frac{1}{2}\delta}} &\leq \bigg(\sum_{j=0}^\infty 2^{-2\varepsilon j} \|\langle n\rangle^{-\frac{1}{2}-\frac{1}{2}\delta+\varepsilon} \widehat{f}(n)\|^2_{\ell^2_{|n|\sim 2^j}}\bigg)^{\frac{1}{2}} \\ &\leq \|\langle n\rangle^{-\frac{3}{2}\delta+\varepsilon}\|_{\ell_n^{\frac{2p}{p-2}}} \sup_{j\in\mathbb{Z}_{\geq 0}} \|\langle n\rangle^{-\frac{1}{2}+\delta}\widehat{f}(n)\|_{\ell^p_{|n|\sim 2^j}} \lesssim \|f\|_{\widehat{b}^{-\frac{1}{2}+\delta}_{p,\infty}},\end{aligned}$$

provided that $\delta > \frac{p-2}{3p}$ (by taking $\varepsilon > 0$ sufficiently small). Hence, we have

$$\|u\|_{X^{-\frac{1}{2}-\frac{1}{2}\delta,b}} \lesssim \|u\|_{X_{p,2}^{-\frac{1}{2}+\delta,b}} \tag{5.2}$$

for any $s, b \in \mathbb{R}$, provided that $\delta > \frac{p-2}{3p}$. Instead of (3.4), we use (5.2) in the following.

Step 1. In the following, we first study the difference of the Duhamel formulations (3.21) and (3.23) for SKdV (1.1) and the truncated SKdV (1.22), respectively, on *short* time intervals. Our first main goal is to estimate the difference

$$\|\mathcal{N}(u,u) - \mathcal{N}^N(u^N,u^N)\|_{X_{p,2}^{-\alpha,\alpha,T_1}}$$

for small $T_1 > 0$, where $\alpha = \frac{1}{2} - \delta$ as in (3.25). From the discussion in Section 3.2, we have

$$\begin{aligned}\mathcal{N}(u,u) - \mathcal{N}^N(u^N,u^N) &= \sum_{j=0}^{2} \Big(\mathcal{N}_j(u,u) - \mathcal{N}_j^N(u^N,u^N)\Big) \\ &= \mathcal{N}_0(u,u) - \mathcal{N}_0^N(u^N,u^N) \\ &\quad - \frac{1}{2}\Big(\mathcal{N}_1(\mathcal{N}(u,u),u) - \mathcal{N}_1^N(\mathcal{N}^N(u^N,u^N),u^N)\Big) \\ &\quad + \mathcal{N}_1(\Psi,u) - \mathcal{N}_1^N(\Psi,u^N) \\ &\quad - \frac{1}{2}\Big(\mathcal{N}_2(u,\mathcal{N}(u,u)) - \mathcal{N}_2^N(u^N,\mathcal{N}^N(u^N,u^N))\Big) \\ &\quad + \mathcal{N}_2(u,\Psi) - \mathcal{N}_2^N(u^N,\Psi).\end{aligned}$$

From the definitions of $\mathcal{N}_1(u,u)$ and $\mathcal{N}_1^N(u^N,u^N)$, we have

$$\begin{aligned}&\mathcal{N}_1(\mathcal{N}(u,u),u) - \mathcal{N}_1^N(\mathcal{N}^N(u^N,u^N),u^N) \\ &\quad = \mathcal{N}_1(\mathcal{N}(u,u),u) - \mathbf{P}_N\mathcal{N}_1(\mathbf{P}_N\mathcal{N}(\mathbf{P}_N u^N,\mathbf{P}_N u^N),\mathbf{P}_N u^N) \\ &\quad = \mathcal{N}_1\big(\mathcal{N}(u,u) - \mathbf{P}_N\mathcal{N}(\mathbf{P}_N u^N,\mathbf{P}_N u^N),u\big) \\ &\qquad + \mathcal{N}_1(\mathbf{P}_N\mathcal{N}(\mathbf{P}_N u^N,\mathbf{P}_N u^N),u-u^N) \\ &\qquad + \mathcal{N}_1(\mathbf{P}_N\mathcal{N}(\mathbf{P}_N u^N,\mathbf{P}_N u^N),\mathbf{P}_N^\perp u^N) \\ &\qquad + \mathbf{P}_N^\perp\mathcal{N}_1(\mathbf{P}_N\mathcal{N}(\mathbf{P}_N u^N,\mathbf{P}_N u^N),\mathbf{P}_N u^N) \\ &\quad =: A_1 + A_2 + A_3 + A_4\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\mathcal{N}_1(\Psi,u) - \mathcal{N}_1^N(\Psi,u^N) &= \mathcal{N}_1(\Psi,u) - \mathbf{P}_N\mathcal{N}_1(\mathbf{P}_N\Psi,\mathbf{P}_N u^N) \\ &= \mathcal{N}_1(\mathbf{P}_N^\perp\Psi,u) + \mathcal{N}_1(\mathbf{P}_N\Psi,u-u^N) \\ &\quad + \mathcal{N}_1(\mathbf{P}_N\Psi,\mathbf{P}_N^\perp u^N) + \mathbf{P}_N^\perp\mathcal{N}_1(\mathbf{P}_N\Psi,\mathbf{P}_N u^N) \\ &=: B_1 + B_2 + B_3 + B_4.\end{aligned} \tag{5.4}$$

Similar expressions hold for the differences

$$\mathcal{N}_2(u,\mathcal{N}(u,u)) - \mathcal{N}_2^N(u^N,\mathcal{N}^N(u^N,u^N)) \tag{5.5}$$

and

$$\mathcal{N}_2(u,\Psi) - \mathcal{N}_2^N(u^N,\Psi). \tag{5.6}$$

Let us first estimate (5.4). Given $N \in \mathbb{N}$, define $\widetilde{L}_{\omega,N}^\perp(T)$ by

$$\widetilde{L}_{\omega,N}^\perp(T) = \|\mathbf{1}_{[0,T]}\mathbf{P}_N^\perp\Psi\|_{X^{-\frac{1}{2}-\delta,\frac{1}{2}-\delta}} + \|\mathbf{1}_{[0,T]}\mathbf{P}_N^\perp\Psi\|_{Y_{2,4}^{-\frac{1}{2}-\delta,\frac{11}{16}+\delta}}. \tag{5.7}$$

See (B.5). Then, from (3.26) and (5.7), we have

$$\widetilde{L}_{\omega,N}^\perp(T) \lesssim N^{-\frac{\delta}{2}} L_\omega(T). \tag{5.8}$$

From the estimates in Appendix B, (5.8), and (5.2) (see also (5.12)), we have

$$
\begin{aligned}
&\|B_1 + B_2 + B_3\|_{X^{-\alpha,1-\alpha,T_1}} \\
&\quad \lesssim T_1^\theta \widetilde{L}^\perp_{\omega,N}(T)\|u\|_{X^{-(1-\alpha),\alpha,T_1}} \\
&\qquad + T_1^\theta L_\omega(T)\Big(\|u-u^N\|_{X^{-(1-\alpha),\alpha,T_1}} + \|\mathbf{P}_N^\perp u^N\|_{X^{-(1-\alpha),\alpha,T_1}}\Big) \\
&\quad \lesssim T_1^\theta L_\omega(T)\|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}} \\
&\qquad + N^{-\frac{\delta}{2}} T_1^\theta L_\omega(T)\Big(\|u\|_{X_{p,2}^{-\alpha,\alpha,T_1}} + \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\Big)
\end{aligned}
\tag{5.9}
$$

and

$$
\begin{aligned}
\|B_4\|_{X^{-\alpha,1-\alpha,T_1}} &\lesssim N^{-\frac{\delta}{2}}\|B_4\|_{X^{-\alpha+\frac{\delta}{2},1-\alpha,T_1}} \lesssim N^{-\frac{\delta}{2}} T_1^\theta L_\omega(T)\|u^N\|_{X^{-\frac{1}{2}-\frac{\delta}{2},\alpha,T_1}} \\
&\lesssim N^{-\frac{\delta}{2}} T_1^\theta L_\omega(T)\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}
\end{aligned}
\tag{5.10}
$$

for some small $\theta > 0$. Therefore, from (5.9), (5.10), and the symmetry between $\mathcal{N}_1$ and $\mathcal{N}_2$, we have

$$
\begin{aligned}
\|(5.4) + (5.6)\|_{X^{-\alpha,1-\alpha,T_1}} &\lesssim T_1^\theta L_\omega(T)\|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}} \\
&\quad + N^{-\frac{\delta}{2}} T_1^\theta L_\omega(T)\Big(\|u\|_{X_{p,2}^{-\alpha,\alpha,T_1}} + \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\Big).
\end{aligned}
\tag{5.11}
$$

Next, we estimate the terms in (5.3). The main nonlinear analysis comes from [6, (2.27)-(2.59) pp. 125-130] and [37, "Estimate on (i)" on pp. 295-296]. Here, the latter replaces [6, Estimation of (2.62) on p. 131], where the a priori assumption (1.8) was used. In [6], the nonlinear analysis [6, (2.27)-(2.59) pp. 125-130] was estimated by the $X^{-(1-\alpha),\alpha}$-norm of u. In particular, in estimating the terms with $\mathbf{P}_N^\perp u^N$ in (5.3) (namely, the first and third terms on the right-hand side of (5.3)), we can apply (5.2) to gain a negative power of N as follows:

$$
\begin{aligned}
\|\mathbf{P}_N^\perp u^N\|_{X^{-(1-\alpha),\alpha,T_1}} &\lesssim N^{-\frac{\delta}{2}}\|\mathbf{P}_N^\perp u^N\|_{X^{-\frac{1}{2}-\frac{1}{2}\delta,\alpha,T_1}} \\
&\lesssim N^{-\frac{\delta}{2}}\|u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}.
\end{aligned}
\tag{5.12}
$$

As for [37, "Estimate on (i)" on pp. 295-296] on R_α in [37, (58)], we used $\langle n\rangle^{-1-\alpha} \le \langle n\rangle^{-3\alpha}$. This can be replaced by $\langle n\rangle^{-1-\alpha+\frac{\delta}{2}} \le \langle n\rangle^{-3\alpha}$, which allows us to gain $N^{-\frac{\delta}{2}}$ from $\mathbf{P}_N^\perp$.

From the discussion above, a straightforward modification of the estimates in [6, (2.27)-(2.59) pp. 125-130] and [37, "Estimate on (i)" on pp. 295-296] yields

$$
\begin{aligned}
&\|A_1\|_{X^{-\alpha,1-\alpha,T_1}} \\
&\quad \lesssim T_1^\theta \|\mathcal{N}_1(u,u) - \mathbf{P}_N\mathcal{N}_1(\mathbf{P}_N u^N, \mathbf{P}_N u^N)\|_{X^{-\alpha,1-\alpha,T_1}}\|u\|_{X^{-(1-\alpha),\alpha,T_1}} \\
&\qquad + T_1^\theta\Big(\|u\|^2_{X^{-(1-\alpha),\alpha,T_1}} + \|u^N\|^2_{X^{-(1-\alpha),\alpha,T_1}}\Big)\|u-u^N\|_{X^{-(1-\alpha),\alpha,T_1}} \\
&\qquad + N^{-\frac{\delta}{2}} T_1^\theta \|u^N\|^3_{X^{-(1-\alpha),\alpha,T_1}} \\
&\quad \lesssim T_1^\theta \|\mathcal{N}_1(u,u) - \mathcal{N}_1^N(u^N,u^N)\|_{X^{-\alpha,1-\alpha,T_1}}\|u\|_{X_{p,2}^{-\alpha,\alpha,T_1}} \\
&\qquad + T_1^\theta\Big(\|u\|^2_{X_{p,2}^{-\alpha,\alpha,T_1}} + \|u^N\|^2_{X_{p,2}^{-\alpha,\alpha,T_1}}\Big)\|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}} \\
&\qquad + N^{-\frac{\delta}{2}} T_1^\theta \|u^N\|^3_{X_{p,2}^{-\alpha,\alpha,T_1}}.
\end{aligned}
\tag{5.13}
$$

Here, the first term on the right-hand side comes from [6, (II.1) on pp. 126-127], while the second and third terms on the right-hand side come from estimating the other cases trilinearly, using

$$\begin{aligned}\mathcal{N}(u,u) - \mathbf{P}_N\mathcal{N}(\mathbf{P}_N u^N, \mathbf{P}_N u^N) &= \mathcal{N}(u,u) - \mathcal{N}(u^N, u^N)\\ &\quad + \mathcal{N}(u^N, \mathbf{P}_N^\perp u^N) + \mathcal{N}(\mathbf{P}_N^\perp u^N, \mathbf{P}_N u^N)\\ &\quad + \mathbf{P}_N^\perp \mathcal{N}(\mathbf{P}_N u^N, \mathbf{P}_N u^N).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|A_2\|_{X^{-\alpha,1-\alpha,T_1}} &\lesssim T_1^\theta \|\mathcal{N}_1^N(u^N,u^N)\|_{X^{-\alpha,1-\alpha,T_1}} \|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\\ &\quad + T_1^\theta \|u^N\|^2_{X_{p,2}^{-\alpha,\alpha,T_1}} \|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\|A_3\|_{X^{-\alpha,1-\alpha,T_1}} &\lesssim N^{-\frac{\delta}{2}} T_1^\theta \|\mathcal{N}_1^N(u^N,u^N)\|_{X^{-\alpha,1-\alpha,T_1}} \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\\ &\quad + N^{-\frac{\delta}{2}} T_1^\theta \|u^N\|^3_{X_{p,2}^{-\alpha,\alpha,T_1}}.\end{aligned} \tag{5.15}$$

In handling the term A_4 in (5.3) with $\mathbf{P}_N^\perp$ outside the nonlinearity we simply use

$$\langle n\rangle^{\frac{\delta}{2}} \lesssim \langle n_1\rangle^{\frac{\delta}{2}}\langle n_2\rangle^{\frac{\delta}{2}} \qquad \text{and} \qquad \langle n\rangle^{\frac{\delta}{2}} \lesssim \langle n_2\rangle^{\frac{\delta}{2}}\langle n_3\rangle^{\frac{\delta}{2}}\langle n_4\rangle^{\frac{\delta}{2}},$$

where n_3 and n_4 are the spatial frequencies of the first and second factors of $\mathbf{P}_N\mathcal{N}(\mathbf{P}_N u^N, \mathbf{P}_N u^N)$ in A_4; see also (5.10). Thus, we have

$$\begin{aligned}\|A_4\|_{X^{-\alpha,1-\alpha,T_1}} &\lesssim N^{-\frac{\delta}{2}} \|A_4\|_{X^{-\alpha+\frac{\delta}{2},1-\alpha,T_1}}\\ &\lesssim N^{-\frac{\delta}{2}} T_1^\theta \|\mathcal{N}_1^N(u^N,u^N)\|_{X^{-\alpha,1-\alpha,T_1}} \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\\ &\quad + N^{-\frac{\delta}{2}} T_1^\theta \|u^N\|^3_{X_{p,2}^{-\alpha,\alpha,T_1}}.\end{aligned} \tag{5.16}$$

Here, the first term on the right-hand side of (5.16) comes from [6, (II.1) on pp. 126-127], where we used the fact that $\langle n_1\rangle^{2\alpha-1} = \langle n_1\rangle^{-2\delta}$. (In [6], in view of $2\alpha-1<0$, this factor $\langle n_1\rangle^{2\alpha-1}$ was simply thrown away; see [6, (2.37)].) Hence, from (5.13), (5.14), (5.15), (5.16), and the symmetry between $\mathcal{N}_1$ and $\mathcal{N}_2$, we obtain

$$\begin{aligned}&\|(5.3)+(5.5)\|_{X^{-\alpha,1-\alpha,T_1}}\\ &\quad\lesssim T_1^\theta \|\mathcal{N}_1(u,u) - \mathcal{N}_1^N(u^N,u^N)\|_{X^{-\alpha,1-\alpha,T_1}} \|u\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\\ &\qquad + T_1^\theta \Big(\|u\|^2_{X_{p,2}^{-\alpha,\alpha,T_1}} + \|u^N\|^2_{X_{p,2}^{-\alpha,\alpha,T_1}}\Big)\|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\\ &\qquad + T_1^\theta \|\mathcal{N}_1^N(u^N,u^N)\|_{X^{-\alpha,1-\alpha,T_1}} \|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\\ &\qquad + N^{-\frac{\delta}{2}} T_1^\theta \|\mathcal{N}_1^N(u^N,u^N)\|_{X^{-\alpha,1-\alpha,T_1}} \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\\ &\qquad + N^{-\frac{\delta}{2}} T_1^\theta \|u^N\|^3_{X_{p,2}^{-\alpha,\alpha,T_1}}.\end{aligned} \tag{5.17}$$

Given $R \geq 1$, by choosing $T_1 = T_1(R) > 0$ sufficiently small such that the condition (3.29) is satisfied. Then, by possibly making $T_1 = T_1(R) > 0$ small, it follows from (5.11) and (5.17) with (3.30) that

$$\begin{aligned}\sum_{j=1}^{2} &\|\mathcal{N}_j(u,u) - \mathcal{N}_j^N(u^N,u^N)\|_{X^{-\alpha,1-\alpha,T_1}} \\ &\lesssim T_1^\theta\Big(\|u\|^2_{X_{p,2}^{-\alpha,\alpha,T_1}} + R^3 + L_\omega(T)R\Big)\|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}} \\ &\quad + N^{-\frac{\delta}{2}}T_1^\theta\Big(R^4 + L_\omega(T)R^2\Big)\end{aligned} \tag{5.18}$$

under an extra assumption u:

$$\|u\|_{X_{p,2}^{-\alpha,\alpha,T_1}} \le 2R. \tag{5.19}$$

As mentioned in Section 3, the temporal regularity on the left-hand side of (5.18) is $b = 1-\alpha = \frac{1}{2}+\delta > \frac{1}{2}$, which is used in (5.23).

The following estimate follows from a slight modification of the bilinear estimate (1.7) (see [6, (I.1) and (I.2) on pp. 122-125] and (3.4)):

$$\begin{aligned}\|\mathcal{N}_0&(u,u) - \mathcal{N}_0^N(u^N,u^N)\|_{X^{-\alpha,\alpha,T_1}} \\ &\lesssim T_1^\theta\Big(\|u\|_{X^{-(1-\alpha),\alpha,T_1}} + \|u^N\|_{X^{-(1-\alpha),\alpha,T_1}}\Big)\|u-u^N\|_{X^{-(1-\alpha),\alpha,T_1}} \\ &\lesssim T_1^\theta\Big(\|u\|_{X_{p,2}^{-\alpha,\alpha,T_1}} + \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\Big)\|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}.\end{aligned} \tag{5.20}$$

As for the difference of the linear solutions, it follows from (3.5) (with $T_1 \le 1$) and (3.9) that

$$\begin{aligned}\|S(t)u(0) - S(t)u^N(0)\|_{X_{p,2}^{-\alpha,\alpha,T_1}} &\le \|S(t)u(0) - S(t)u^N(0)\|_{X_{p,2}^{-\alpha,\alpha,1}} \\ &\lesssim \|u(0) - u^N(0)\|_{\widehat{b}^{-\alpha}_{p,\infty}}.\end{aligned}$$

Therefore, putting (3.21), (3.20), (3.23), (5.18), and (5.20) together we obtain

$$\begin{aligned}\|u &- u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}} \\ &\le D_0\|u(0)-u^N(0)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \\ &\quad + D_1T_1^\theta\Big(\|u\|^2_{X_{p,2}^{-\alpha,\alpha,T_1}} + R^3 + L_\omega(T)R\Big)\|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}} \\ &\quad + D_2N^{-\frac{\delta}{2}}T_1^\theta\Big(R^4 + L_\omega(T)R^2\Big)\end{aligned} \tag{5.21}$$

under the assumptions (3.29) and (5.19). Here, we took general initial data $u(0)$ and $u^N(0)$ so that we can apply the estimate (5.21) to a general time interval of length T_1.

Next, let us bound the difference of u and u^N in the $C([0,T_1];\widehat{b}^{-\alpha}_{p,\infty}(\mathbb{T}))$-norm. A bilinear version of (3.31) yields

$$\begin{aligned}\|\mathcal{N}_0&(u,u) - \mathcal{N}_0^N(u^N,u^N)\|_{C([0,T_1];\widehat{b}^{-\alpha}_{p,\infty})} \\ &\lesssim T_1^\theta\Big(\|u\|_{X_{p,2}^{-\alpha,\alpha,T_1}} + \|u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}\Big)\|u-u^N\|_{X_{p,2}^{-\alpha,\alpha,T_1}}.\end{aligned} \tag{5.22}$$

Hence, from (5.18) and (5.22), we have[9]

$$\begin{aligned}
&\|u-u^N\|_{C([0,T_1];\widehat{b}^{-\alpha}_{p,\infty})} \\
&\quad \le \|u(0)-u^N(0)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \\
&\qquad + D_1T_1^\theta\Big(\|u\|^2_{X^{-\alpha,\alpha,T_1}_{p,2}} + R^3 + L_\omega(T)R\Big)\|u-u^N\|_{X^{-\alpha,\alpha,T_1}_{p,2}} \\
&\qquad + D_2N^{-\frac{\delta}{2}}T_1^\theta\Big(R^4 + L_\omega(T)R^2\Big)
\end{aligned} \tag{5.23}$$

under the assumptions (3.29) and (5.19). We point out that the estimates (5.21) and (5.23) hold true on a general time interval of length T_1.

Step 2. Fix $T \gg 1$ and $0 < \varepsilon \ll 1$. We now establish the difference estimate (5.1) on the time interval $[0,T]$ by iterating the local-in-time estimates (5.21) and (5.23) with the probabilistic input from Proposition 4.1.

Given $N \in \mathbb{N}$, let $\Omega_{T,\varepsilon}(N) = \Omega_1 \cap \dots \cap \Omega_4$ be as in Proposition 4.1, where Ω_k, $k = 1,\dots,4$, are as in (4.2), (4.5), (4.7), and (4.14), respectively. In particular, if necessary, we have made T_1 smaller such that (4.11) is satisfied. In the following, it is understood that we work on $\Omega_{T,\varepsilon}(N)$ and that all the estimates are restricted to $\Omega_{T,\varepsilon}(N)$, where the value of N may increase in each step.

For now, assume that

$$\|u\|_{X^{-\alpha,\alpha}_{p,2}(I_j)} \le \|u^N\|_{X^{-\alpha,\alpha}_{p,2}(I_j)} + 1 \lesssim K_1 \tag{5.24}$$

for $I_j = [jT_1,(j+1)T_1] \cap [0,T]$, $j = 0,1,\dots,\big[\frac{T}{T_1}\big]$, where the second inequality follows from (4.13). Note that, with $R = C_*(2K_1+1)$, (3.29) (on the interval I_j) and (5.24) (see also (4.10) and (4.13)) implies (5.19) (on the interval I_j). Then, in view of (5.21) and (5.23) with (4.10) (see also (3.33) in Lemma 3.2), we further impose that $T_1 > 0$ be sufficiently small such that

$$\begin{aligned}
&T_1^\theta\Big(K_1^3 + K_1K_2\Big) \ll 1,\\
&T_1^\theta\Big(K_1^4 + K_1^2K_2\Big) \ll 1.
\end{aligned} \tag{5.25}$$

In the following, we work iteratively on each interval I_j and verify (5.24).

Let us now consider the first time interval $I_0 = [0,T_1]$. By the local well-posedness theory (see (3.28)), there exists small $T_0 > 0$ such that

$$\|u\|_{X^{-\alpha,\alpha,T_0}_{p,2}} \lesssim K_1. \tag{5.26}$$

Then, from (5.21) (but with T_0 replacing T_1 and with $u(0) = u^N(0)$) with (5.25) and (5.26), we have

$$\|u-u^N\|_{X^{-\alpha,\alpha,T_0}_{p,2}} \le \frac{1}{2}\|u-u^N\|_{X^{-\alpha,\alpha,T_0}_{p,2}} + N^{-\frac{\delta}{2}}.$$

Hence, we have

$$\|u-u^N\|_{X^{-\alpha,\alpha,T_0}_{p,2}} \le 2N^{-\frac{\delta}{2}}.$$

[9]In general, the constants D_1 and D_2 in (5.21) and (5.23) are different, but we simply take the worse ones.

Therefore, by a standard continuity argument (see also Remark 3.3), we conclude that there exists $N_0 \in \mathbb{N}$ such that (5.24) holds on the entire time interval $I_0 = [0, T_1]$ for any $N \geq N_0$. As a result, we obtain

$$\|u - u^N\|_{X^{-\alpha,\alpha}_{p,2}(I_0)} \leq 2N^{-\frac{\delta}{2}} \tag{5.27}$$

for any $N \geq N_0$. By applying (5.25) and (5.27) (with (4.10)) to (5.23), we then obtain

$$\|u - u^N\|_{C(I_0;\widehat{b}^{-\alpha}_{p,\infty})} \leq 2N^{-\frac{\delta}{2}} \tag{5.28}$$

for any $N \geq N_0$.

On the second interval $I_1 = [T_1, 2T_1] \cap [0, T]$, we repeat an analogous analysis. From (5.28), we have

$$\|u(T_1) - u^N(T_1)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \leq 2N^{-\frac{\delta}{2}}$$

for any $N \geq N_0$. By the local theory, there exists small $T_0 > 0$ such that

$$\|u\|_{X^{-\alpha,\alpha}_{p,2}([T_1,T_1+T_0])} \lesssim K_1.$$

Then, from (5.21) (but on $[T_1, T_1 + T_0]$) with (5.25) and (5.26), we have

$$\|u - u^N\|_{X^{-\alpha,\alpha}_{p,2}([T_1,T_1+T_0])} \leq 2N^{-\frac{\delta}{2}} + \frac{1}{2}\|u - u^N\|_{X^{-\alpha,\alpha}_{p,2}([T_1,T_1+T_0])} + N^{-\frac{\delta}{2}},$$

which yields

$$\|u - u^N\|_{X^{-\alpha,\alpha}_{p,2}([T_1,T_1+T_0])} \leq 6N^{-\frac{\delta}{2}}.$$

Therefore, it follows from a standard continuity argument that there exists $N_1 \in \mathbb{N}$ such that (5.24) holds on the entire time interval I_1 for any $N \geq N_1$. As a result, we obtain

$$\|u - u^N\|_{X^{-\alpha,\alpha}_{p,2}(I_1)} \leq 6N^{-\frac{\delta}{2}} \tag{5.29}$$

for any $N \geq N_1$. Hence, from (5.23), (5.24) (5.25), and (5.29), we obtain

$$\|u - u^N\|_{C(I_1;\widehat{b}^{-\alpha}_{p,\infty})} \leq 2N^{-\frac{\delta}{2}} + \frac{1}{2} \cdot 6N^{-\frac{\delta}{2}} + N^{-\frac{\delta}{2}} = 6N^{-\frac{\delta}{2}},$$

for any $N \geq N_1$.

Proceeding iteratively, we conclude that, on the jth interval $I_j = [jT_1, (j+1)T_1] \cap [0, T]$, $j = 0, 1, \dots, \left[\frac{T}{T_1}\right]$, there exists $N_j \in \mathbb{N}$ such that

$$\begin{aligned} \|u - u^N\|_{X^{-\alpha,\alpha}_{p,2}(I_j)} &\leq \bigg(\sum_{k=0}^{j} 2^{k+1}\bigg) N^{-\frac{\delta}{2}}, \\ \|u - u^N\|_{C(I_j;\widehat{b}^{-\alpha}_{p,\infty})} &\leq \bigg(\sum_{k=0}^{j} 2^{k+1}\bigg) N^{-\frac{\delta}{2}} \end{aligned} \tag{5.30}$$

for any $N \geq N_j$. Note that T_1 depends only on T and ε; see (4.11) and (5.25) with (4.3) and (4.8). See also (3.28) with $R \lesssim K_1$ as in (4.10). Therefore, by setting

$$N_* = N_*(T, \varepsilon) = N_{[T/T_1]} \qquad \text{and} \qquad \Omega_{T,\varepsilon} = \Omega_{T,\varepsilon}(N_*(T, \varepsilon)),$$

where the latter is as in Proposition 4.1, we conclude from (5.30) that, on $\Omega_{T,\varepsilon}$, we have

$$\|u - u^{N_*}\|_{C([0,T];\widehat{b}^{-\alpha}_{p,\infty})} \leq C(T, \varepsilon) N_*^{-\frac{\delta}{2}}.$$

This concludes the proof of Proposition 5.1.

□

We now present the proof of Theorem 1.1. We first note that the claimed almost sure global well-posedness of SKdV (1.1) with the white noise initial data immediately follows from the 'almost' almost sure global well-posedness result established in Proposition 5.1; see [2,16]. Indeed, define $\Sigma \subset \Omega$ by

$$\Sigma = \bigcup_{k=1}^{\infty} \bigcap_{j=1}^{\infty} \Omega_{2^j, \frac{1}{k2^j}}, \tag{5.31}$$

where $\Omega_{T,\varepsilon}$ is as in Proposition 5.1. Then, we have

$$\mathbb{P}(\Sigma^c) \leq \inf_{k\in\mathbb{N}} \sum_{j=1}^{\infty} \mathbb{P}(\Omega^c_{2^j, \frac{1}{k2^j}}) = \inf_{k\in\mathbb{N}} \frac{1}{k} = 0.$$

Moreover, if $\omega \in \Sigma$, then there exists $k \in \mathbb{N}$ such that $\omega \in \Omega_{2^j, \frac{1}{k2^j}}$ for any $j \in \mathbb{N}$, which implies that the corresponding solution $u = u(\omega)$ to SKdV (1.1) exists globally in time.

It remains to prove (1.15). It follows from the proof of Proposition 5.1 that, on $\Omega_{T,\varepsilon} = \Omega_{T,\varepsilon}(N_*(T,\varepsilon))$, we have

$$\sup_{t\in[0,T]} \|u^N(t) - u^{N_*}(t)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \leq C(T,\varepsilon) N_*^{-\frac{\delta}{2}}, \tag{5.32}$$

for any $N \geq N_*$. Define $\widetilde{\Omega}_1(N) = \widetilde{\Omega}_1(T,\varepsilon,N) \subset \Omega$ by

$$\widetilde{\Omega}_1(N) = \bigcap_{j=0}^{[T/T_1]} \Big\{ \|u^N(jT_1)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \leq 2K_1 \Big\}. \tag{5.33}$$

Namely, we replaced K_1 in (4.2) by $2K_1$. By taking N_* sufficiently large, it follows from (5.32) that $\Omega_{T,\varepsilon} \subset \widetilde{\Omega}_1(N)$ for any $N \geq N_*$. Hence, by setting $\widetilde{\Omega}_{T,\varepsilon}(N) = \widetilde{\Omega}_1 \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$, where Ω_2, Ω_3, and Ω_4 are (4.5), (4.7), and (4.14), respectively, we have

$$\Omega_{T,\varepsilon} \subset \widetilde{\Omega}_{T,\varepsilon}(N) \tag{5.34}$$

for any $N \geq N_*$. Now, by repeating Step 2 in the proof of Proposition 5.1,[10] we conclude that there exists $N_{**} = N_{**}(T,\varepsilon) \in \mathbb{N}$ such that, on $\Omega_{T,\varepsilon}$, we have

$$\sup_{t\in[0,T]} \|u(t) - u^N(t)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \leq C(T,\varepsilon) N^{-\frac{\delta}{2}}, \tag{5.35}$$

for any $N \geq N_{**}$. This in particular implies that, for each $\omega \in \Omega_{T,\varepsilon}$, the solution $u = u(\omega)$ to SKdV (1.1) is the limit of $u^N = u^N(\omega)$ in $C([0,T]; \widehat{b}^{-\alpha}_{p,\infty}(\mathbb{T}))$. Hence, given $t \in \mathbb{R}_+$, it follows from the discussion above that, for each $\omega \in \Sigma$,

$$\|u^N(t;\omega) - u(t;\omega)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \longrightarrow 0$$

[10] In (5.33), we replaced K_1 by $2K_1$, which worsens constants in the argument. We can, however, implement the proof of Proposition 5.1 to incorporate these worse constants from the beginning.

as $N \to \infty$. This in particular implies convergence in law of $u^N(t)$ to $u(t)$. Recalling that $\text{Law}(u^N(t)) = \mu_{1+t}$ for any $N \in \mathbb{N}$, we then conclude that

$$\text{Law}(u(t)) = \mu_{1+t}.$$

This concludes the proof of Theorem 1.1.

Remark 5.2. Let $\omega \in \Omega_{T,\varepsilon}$. Then, from (5.34), (5.35), and Proposition 4.1, we have

$$\sup_{t\in[0,T]} \|u(t)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \le C\sqrt{\log\frac{1}{\varepsilon}}\sqrt{T\log T}. \tag{5.36}$$

Fix $k \in \mathbb{N}$, and suppose that $\omega \in \bigcap_{j=1}^{\infty} \Omega_{2^j, \frac{1}{k2^j}}$. Then, from (5.36), we obtain

$$\|u(t)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \le C\sqrt{\log k}\sqrt{1+t}\log(1+t)$$

for any $t \in \mathbb{R}_+$. Namely, we have

$$\|u(t)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \le C(\omega)\sqrt{1+t}\log(1+t) \tag{5.37}$$

for any $t \in \mathbb{R}_+$ and $\omega \in \Sigma$. Note that the growth bound (5.37) is not optimal, and we can improve it by modifying the definition (5.31) of Σ. For example, by redefining Σ by

$$\Sigma = \bigcup_{k=1}^{\infty}\bigcap_{j=1}^{\infty} \Omega_{2^j, \frac{1}{kj^2}}$$

and repeating the argument, we obtain the following growth bound:

$$\|u(t)\|_{\widehat{b}^{-\alpha}_{p,\infty}} \le C(\omega)\sqrt{1+t}\sqrt{\log(1+t)}\sqrt{\log\log(1+t)}.$$

In this way, we can obtain a growth bound which is only slightly faster than $\sqrt{t\log t}$, $t \gg 1$ (but the random constant $C(\omega)$ gets worse).

Appendix A. Growth bound on the stochastic convolution for large times

In this appendix, we present the proof of Lemma 3.4.

Proof of Lemma 3.4. Fix $s < 0$ and $1 \le p, q < \infty$ such that $sp < -1$, and $(b-1)q < -1$. We also fix $1 \le r < \infty$ and $T \ge 1$. Without loss of generality, we assume

$$r \ge \max(p, q). \tag{A.1}$$

Before proceeding further, we first recall the following bound for a Gaussian random variable g:

$$\|g\|_{L^r(\Omega)} \lesssim \sqrt{r}\|g\|_{L^2(\Omega)}. \tag{A.2}$$

(i) Let $I = [t_0, t_1] \subset [0, T]$ be an interval of length $|I| \le 1$. The first inequality in (3.39) follows from (3.13), and thus we focus on proving the second inequality in (3.39).

Recall that

$$\|u\|_{Y^{s,b}_{p,q}} = \|S(-t)u(t)\|_{\mathcal{F}L^{s,p}_x \mathcal{F}L^{b,q}_t}, \tag{A.3}$$

where $\mathcal{F}L_t^{b,q}$ and $\mathcal{F}L_x^{s,p}$ are the Fourier-Lebesgue spaces defined in (3.2) and (3.11), respectively. Let $\Phi(t) = S(-t)\Psi(t)$ be the interaction representation of Ψ. From (3.7) with (1.11), we have

$$\widehat{\mathbf{1}_I\Phi}(n,t) = \mathbf{1}_I(t)\int_0^t e^{-it'n^3} d\beta_n(t').$$

By taking the temporal Fourier transform, we then have

$$\begin{aligned} \widehat{\mathbf{1}_I\Phi}(n,\tau) &= \int_{t_0}^{t_1} e^{-it\tau}\int_0^t e^{-it'n^3} d\beta_n(t')dt \\ &= \int_0^{t_1} e^{-it'n^3}\int_{\max(t_0,t')}^{t_1} e^{-it\tau} dt d\beta_n(t'). \end{aligned} \tag{A.4}$$

The inner integral can be estimated as

$$\bigg|\int_{\max(t_0,t')}^{t_1} e^{-it\tau}dt\bigg| \lesssim \min\bigg(1, \frac{1}{|\tau|}\bigg) \lesssim \frac{1}{\langle\tau\rangle}. \tag{A.5}$$

From (3.5) (for the $Y^{s,b}_{p,q}$-space) and (A.3), we have

$$\|\Psi\|_{Y^{s,b}_{p,q}(I)} \le \|\mathbf{1}_I\Psi\|_{Y^{s,b}_{p,q}(I)} = \|\langle n\rangle^s\langle\tau\rangle^b\widehat{\mathbf{1}_I\Phi}(n,\tau)\|_{\ell^p_nL^q_\tau}. \tag{A.6}$$

Then, by (A.6), Minkowski's integral inequality, and (A.2) followed by the Ito isometry with (A.4), (A.5) and $t_1 \le T$, we have

$$\begin{aligned} \Big\|\|\Psi\|_{Y^{s,b}_{p,q}(I)}\Big\|_{L^r(\Omega)} &= \Big\|\|\langle n\rangle^s\langle\tau\rangle^b\widehat{\mathbf{1}_I\Phi}(n,\tau)\|_{\ell^p_nL^q_\tau}\Big\|_{L^r(\Omega)} \\ &\le \Big\|\|\langle n\rangle^s\langle\tau\rangle^b\widehat{\mathbf{1}_I\Phi}(n,\tau)\|_{L^r(\Omega)}\Big\|_{\ell^p_nL^q_\tau} \\ &\lesssim \sqrt{r}\Big\|\|\langle n\rangle^s\langle\tau\rangle^b\widehat{\mathbf{1}_I\Phi}(n,\tau)\|_{L^2(\Omega)}\Big\|_{\ell^p_nL^q_\tau} \\ &\lesssim \sqrt{rT}\|\langle n\rangle^s\langle\tau\rangle^{b-1}\|_{\ell^p_nL^q_\tau} \\ &\lesssim \sqrt{rT}, \end{aligned}$$

since $sp < -1$ and $(b-1)q < -1$. This proves (3.39).

(ii) It follows from [37, Proposition 4.5] that the stochastic convolution is continuous in time with values in $\widehat{b}^s_{p,\infty}(\mathbb{T})$ when $sp < -1$, at least locally in time. In the following, we estimate its growth in a direct manner by following the argument in [41, Lemma 3.4].

Without loss of generality, assume that $T \in 2^{\mathbb{N}}$. For an integer $k \in \mathbb{Z} \cap [-\log_2 T, \infty)$, let $\{t_{\ell,k} : \ell = 0, 1, \dots, 2^kT\}$ be $2^kT + 1$ equally spaced points on $[0,T]$, i.e. $t_{0,k} = 0$ and $t_{\ell,k} - t_{\ell-1,k} = 2^{-k}$ for $\ell = 1, \dots, 2^kT$. Let $\Phi(t) = S(-t)\Psi(t)$ be the interaction representation of Ψ. Then, given $t \in [0,T]$, it follows from the continuity (in time) of Ψ and $\Psi(0) = 0$ that

$$\Phi(t) = \sum_{k=-\log_2 T}^{\infty} \big(\Phi(t_{\ell_k,k}) - \Phi(t_{\ell_{k-1},k-1})\big) \tag{A.7}$$

for some $\ell_k = \ell_k(t) \in \{0, \dots, 2^k T\}$. Then, from (3.13), (A.7), and Minkowski's integral inequality with (A.1), we have

$$\begin{aligned}
&\Big\| \|\Psi\|_{C[0,T];\widehat{b}^s_{p,\infty})} \Big\|_{L^r(\Omega)} \\
&\quad \le \Big\| \|\Phi(t)\|_{C([0,T];\mathcal{F}L^{s,p})} \Big\|_{L^r(\Omega)} \\
&\quad \le \sum_{k=-\log_2 T}^{\infty} \Big\| \max_{0\le \ell_k \le 2^k T} \|\Phi(t_{\ell_k,k}) - \Phi(t_{l'_{k-1},k-1})\|_{\mathcal{F}L^{s,p}} \Big\|_{L^r(\Omega)},
\end{aligned} \tag{A.8}$$

where $t_{l'_{k-1},k-1}$ is one of the $2^{(k-1)}T+1$ equally spaced points such that

$$|t_{\ell_k,k} - t_{l'_{k-1},k-1}| \le 2^{-k}. \tag{A.9}$$

For $k \in \mathbb{Z} \cap [-\log_2 T, \infty)$, let

$$q_k = \max(\log 2^k T, p, r) \sim \log(2^k T) + r.$$

Then, noting that $(2^k T + 1)^{\frac{1}{q_k}} \lesssim 1$, it follows from (A.8) that

$$\begin{aligned}
&\Big\| \|\Phi\|_{C[0,T];\widehat{b}^s_{p,\infty})} \Big\|_{L^r(\Omega)} \\
&\quad \le \sum_{k=-\log_2 T}^{\infty} \Bigg\| \bigg(\sum_{\ell_k=0}^{2^k T} \|\Phi(t_{\ell_k,k}) - \Phi(t_{l'_{k-1},k-1})\|^{q_k}_{\mathcal{F}L^{s,p}} \bigg)^{\frac{1}{q_k}} \Bigg\|_{L^{q_k}(\Omega)} \\
&\quad = \sum_{k=-\log_2 T}^{\infty} \bigg(\sum_{\ell_k=0}^{2^k T} \Big\| \|\Phi(t_{\ell_k,k}) - \Phi(t_{l'_{k-1},k-1})\|_{\mathcal{F}L^{s,p}} \Big\|^{q_k}_{L^{q_k}(\Omega)} \bigg)^{\frac{1}{q_k}} \\
&\quad \lesssim \sum_{k=-\log_2 T}^{\infty} \max_{0\le \ell_k \le 2^k T} \Big\| \|\Phi(t_{\ell_k,k}) - \Phi(t_{l'_{k-1},k-1})\|_{\mathcal{F}L^{s,p}} \Big\|_{L^{q_k}(\Omega)}.
\end{aligned} \tag{A.10}$$

From (3.11), Minkowski's integral inequality, and (A.2), we have

$$\begin{aligned}
&\Big\| \|\Phi(t_{\ell_k,k}) - \Phi(t_{l'_{k-1},k-1})\|_{\mathcal{F}L^{s,p}} \Big\|_{L^{q_k}(\Omega)} \\
&\quad = \Big\| \big\| \langle n \rangle^s \big(\widehat{\Phi}(n, t_{\ell_k,k}) - \widehat{\Phi}(n, t_{l'_{k-1},k-1}) \big) \big\|_{\ell^p_n} \Big\|_{L^{q_k}(\Omega)} \\
&\quad \lesssim \sqrt{q_k} \Big\| \langle n \rangle^s \|\widehat{\Phi}(n, t_{\ell_k,k}) - \widehat{\Phi}(n, t_{l'_{k-1},k-1})\|_{L^2(\Omega)} \Big\|_{\ell^p_n} \\
&\quad = \sqrt{q_k} \Bigg\| \langle n \rangle^s \bigg\| \int_{t_{l'_{k-1},k-1}}^{t_{\ell_k,k}} e^{-it'n^3} d\beta_n(t') \bigg\|_{L^2(\Omega)} \Bigg\|_{\ell^p_n} \\
&\quad \lesssim \sqrt{\frac{q_k}{2^k}},
\end{aligned} \tag{A.11}$$

where the last step follows from (A.9) and $sp < -1$. Hence, from (A.10) and (A.11), we obtain (A.8) that

$$\begin{aligned}
\Big\| \|\Psi\|_{C[0,T];\widehat{b}^s_{p,\infty})} \Big\|_{L^r(\Omega)} &\lesssim \sqrt{r} \sum_{k=-\log T}^{\infty} \frac{\log 2^k + \log_2 T}{2^{\frac{1}{2}k}} \\
&\lesssim \sqrt{r}\sqrt{T \log T}.
\end{aligned}$$

This proves (3.40). □

Remark A.1. Let us consider the bound (3.39) when $I = [0, T]$, as discussed in Remark 3.5. In this case, (A.4) becomes

$$\widehat{\mathbf{1}_{[0,T]}\Phi}(n,\tau) = \int_0^T e^{-it'n^3} \int_{t'}^T e^{-it\tau} dt d\beta_n(t').$$

In particular, the inner integral is estimated as

$$\left| \int_{t'}^T e^{-it\tau} dt \right| \lesssim \frac{T}{\langle \tau \rangle}. \tag{A.12}$$

Then, by repeating the computation above with (A.12), we obtain (3.41).

Appendix B. Pathwise bound on the iterated term with the stochastic convolution

In this appendix, we establish a pathwise bound on the $X^{-\alpha,1-\alpha,T}$-norm of $\mathcal{N}_1(\Psi, u)$ appearing in (3.22). This was essentially carried out in [37, "Estimate on (ii)" on pp. 296-297] but was done with an expectation. In the following, based on the analysis in [37], we instead present straightforward pathwise analysis. By duality, it suffices to estimate

$$\begin{aligned} &\sum_{\substack{n, n_1 \in \mathbb{Z} \\ n = n_1 + n_2}} \int_{\tau = \tau_1 + \tau_2} d\tau d\tau_1 \mathbf{1}_{\sigma_1 = \text{MAX}} \frac{\langle n \rangle^{1-\alpha} d(n,\tau)}{\sigma_0^\alpha} \\ &\qquad \times |\widehat{\mathbf{1}_{[0,T]}\Psi}(n_1, \tau_1)| \frac{\langle n_2 \rangle^{1-\alpha} |c(n_2, \tau_2)|}{\sigma_2^\alpha}, \end{aligned} \tag{B.1}$$

where σ_j, $j = 0, 1, 2$, is as in (3.17), $d = d(n, \tau)$ with $\|d\|_{\ell^2_n L^2_\tau} = 1$, and $c(n,\tau) = \langle n \rangle^{-(1-\alpha)} \langle \tau - n^3 \rangle^\alpha \widehat{u}(n, \tau)$ such that $\|c\|_{\ell^2_n L^2_\tau} = \|u\|_{X^{-(1-\alpha),\alpha}}$.

Case 1 ($\max(\sigma_0, \sigma_2) \gtrsim \langle n n_1 n_2 \rangle^{\frac{1}{100}}$). Without loss of generality, assume $\sigma_0 \gtrsim \langle n n_1 n_2 \rangle^{\frac{1}{100}}$. Then, by (3.18) and the $L^4_{x,t}, L^2_{x,t}, L^4_{x,t}$-Hölder's inequality followed by the L^4-Strichartz estimate (3.10), we have

$$\begin{aligned} \text{(B.1)} \lesssim &\sum_{\substack{n, n_1 \in \mathbb{Z} \\ n = n_1 + n_2}} \int_{\tau = \tau_1 + \tau_2} d\tau d\tau_1 \frac{d(n,\tau)}{\sigma_0^{\alpha - 200\delta}} \\ &\times \langle n_1 \rangle^{-\frac{1}{2} - \delta} \sigma_1^{\frac{1}{2} - \delta} |\widehat{\mathbf{1}_{[0,T]}\Psi}(n_1, \tau_1)| \frac{|c(n_2, \tau_2)|}{\sigma_2^\alpha} \\ \lesssim &\|\mathbf{1}_{[0,T]}\Psi\|_{X^{-\frac{1}{2}-\delta, \frac{1}{2}-\delta}} \|u\|_{X^{-(1-\alpha),\alpha,T}} \end{aligned} \tag{B.2}$$

by taking $\delta > 0$ sufficiently small.

Case 2 ($\max(\sigma_0, \sigma_2) \ll \langle n n_1 n_2 \rangle^{\frac{1}{100}}$). Define the set $\Omega(n)$ by

$$\begin{aligned} \Omega(n) = \big\{ \sigma \in \mathbb{R} : \sigma = -3 n n_1 n_2 + o(\langle n n_1 n_2 \rangle^{\frac{1}{100}}) \\ \text{for some } n_1, n_2 \in \mathbb{Z}_* \text{ with } n = n_1 + n_2 \big\}. \end{aligned}$$

Then, we have

$$\int \langle \tau - n^3 \rangle^{-\frac{3}{4}} \mathbf{1}_{\Omega(n)}(\tau - n^3) d\tau \lesssim 1. \tag{B.3}$$

See [37, Lemma 5.3]. By (3.18), the $L^4_{x,t}, L^2_{x,t}, L^4_{x,t}$-Hölder's inequality, the L^4-Strichartz estimate (3.10), and Hölder's inequality (in τ) with (B.3), we have

$$\begin{aligned}
\text{(B.1)} \lesssim & \sum_{\substack{n,n_1\in\mathbb{Z}\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} d\tau d\tau_1 \frac{d(n,\tau)}{\sigma_0^\alpha} \\
& \times \mathbf{1}_{\Omega(n_1)}(\tau_1 - n_1^3)\langle n_1\rangle^{-\frac12-\delta}\sigma_1^{\frac12+\delta}|\widehat{\mathbf{1}_{[0,T]}\Psi}(n_1,\tau_1)|\frac{|c(n_2,\tau_2)|}{\sigma_2^\alpha} \qquad \text{(B.4)} \\
\lesssim & \|\mathbf{1}_{\Omega(n_1)}(\tau_1 - n_1^3)\langle n_1\rangle^{-\frac12-\delta}\sigma_1^{\frac12+\delta}\widehat{\mathbf{1}_{[0,T]}\Psi}(n_1,\tau_1)\|_{\ell^2_{n_1}L^2_{\tau_1}}\|u\|_{X^{-(1-\alpha),\alpha}} \\
\lesssim & \|\mathbf{1}_{[0,T]}\Psi\|_{Y_{2,4}^{-\frac12-\delta,\frac{11}{16}+\delta}}\|u\|_{X^{-(1-\alpha),\alpha,T}},
\end{aligned}$$

where the $Y^{s,b}_{p,q}$-norm is defined in (3.12).

Given $N \in \mathbb{N}$, a similar computation yields

$$\begin{aligned}
&\|\mathcal{N}_1(\mathbf{P}_N^\perp\Psi, u)\|_{X^{-\alpha,1-\alpha,T}} \\
&\lesssim \Big(\|\mathbf{1}_{[0,T]}\mathbf{P}_N^\perp\Psi\|_{X^{-\frac12-\delta,\frac12-\delta}} + \|\mathbf{1}_{[0,T]}\mathbf{P}_N^\perp\Psi\|_{Y_{2,4}^{-\frac12-\delta,\frac{11}{16}+\delta}}\Big)\|u\|_{X^{-(1-\alpha),\alpha,T}}, \qquad \text{(B.5)}
\end{aligned}$$

which motivates the definition of $\widetilde{L}^\perp_{\omega,N}(T)$ in (5.7).

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