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APOLLONIAN PACKINGS AND KAC-MOODY ROOT SYSTEMS

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ABSTRACT. We study Apollonian circle packings using the properties of a certain rank 4 indefinite Kac-Moody root system Φ. We introduce the generating function $Z(\mathbf{s})$ of a packing, an exponential series in four variables with an Apollonian symmetry group, which is a symmetric function for Φ. By exploiting the presence of affine and Lorentzian hyperbolic root subsystems of Φ, with automorphic Weyl denominators, we express $Z(\mathbf{s})$ in terms of Jacobi theta functions and the Siegel modular form Δ_5. We also show that the domain of convergence of $Z(\mathbf{s})$ is the Tits cone of Φ, and discover that this domain inherits the intricate geometric structure of Apollonian packings.

1. INTRODUCTION

The aim of this article is to study Apollonian circle packings from the perspective of Kac-Moody theory, motivating new questions about packings and Kac-Moody root systems. We introduce an indefinite Kac-Moody root system Φ which encodes many of the properties of Apollonian packings, e.g.

- The Cartan matrix of Φ is the matrix of the Descartes quadratic form.
- The Weyl group is the Apollonian group.
- Certain Weyl orbits in the root lattice correspond to Apollonian packings.
- Principal root subsystems of Φ correspond to important subsets of packings: the sets of circles tangent to one or more fixed circles.
- The Tits cone can be constructed geometrically from a packing; see Figure 4 and Theorem 6.2.

We will use the character theory of Φ to approach number-theoretic questions about packings.

Fix a bounded Apollonian packing $\mathcal{P}$. To make the connection between $\mathcal{P}$ and the root system Φ, we define a series that can be considered as a generating function for $\mathcal{P}$, or a symmetric function for Φ. For $s_1, s_2, s_3, s_4 \in \mathbb{C}$, let

$$Z(s_1, s_2, s_3, s_4) = \sum_{(c_1,c_2,c_3,c_4)\in\mathcal{P}} e^{-c_1 s_1 - c_2 s_2 - c_3 s_3 - c_4 s_4}. \tag{1.1}$$

The sum is over all Descartes quadruples of mutually tangent circles appearing in $\mathcal{P}$, with (c_1, c_2, c_3, c_4) being the four curvatures. This series has an infinite group of symmetries W isomorphic to the Apollonian group or the Weyl group of Φ. Its analytic properties–convergence, growth, zeroes and poles–can translate into information about the asymptotic behavior of quadruples in $\mathcal{P}$.

In Section 2, we introduce Apollonian packings and Kac-Moody root systems, for the reader with background in one area but not the other. To further motivate

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this article, we give one example of how the series $Z(\mathbf{s})$ can be used to study the density of curvatures in an integral packing.

Our first question about the series $Z(\mathbf{s})$ is whether it can be expressed in terms of automorphic forms. This question is inspired by Gritsenko and Nikulin's theory of automorphic correction for Lorentzian Kac-Moody root systems [16, 17]. In many cases, one of the simplest symmetric functions for a Kac-Moody root system, the Weyl denominator, can be multiplied by an extra factor to produce an automorphic form. This means roughly that the Weyl denominator has a much larger group of symmetries than just the Weyl group. We do not obtain an automorphic correction for $Z(\mathbf{s})$, but we use known automorphic corrections for root subsystems of Φ to obtain partial results.

In Section 3, we use an affine root subsystem $A_1^{(1)}$ of Φ. By the Jacobi triple product formula, the characters of this root subsystem are Jacobi theta functions, so we find that $Z(\mathbf{s})$ is related to Jacobi theta functions. Proposition 3.1 leads to an expansion of $Z(\mathbf{s})$ as a sum of theta functions. In Section 4, we use the hyperbolic root subsystem $H_{71}^{(3)}$, which is a foundational example for Gritsenko and Nikulin. The Weyl denominator for $H_{71}^{(3)}$ is automorphically corrected to the Siegel automorphic form Δ_5 on Sp(4) [15]. Theorem 4.1 gives an automorphic correction for a series related to $Z(\mathbf{s})$, leading to an expression in terms of Δ_5. These sections are intended to lay the groundwork for further study of $Z(\mathbf{s})$ from an automorphic perspective.

Our second question about $Z(\mathbf{s})$ is more elementary: where is it defined? In Sections 5 and 6 we describe its domain of absolute convergence, a four-dimensional region which we call the Apollonian cone A. We prove that the real part of the Apollonian cone is the interior of the Tits cone of Φ:

Theorem 5.3. *The domain of absolute convergence of Z is $A = \bigcup_{w\in W} w(C)$.*

Here C is the cone of nonnegative linear combinations of the fundamental weights for Φ with at least three nonzero terms. The Tits cone is $\bigcup_{w\in W} w(\bar{C})$. The domain A is independent of $\mathcal{P}$ and has a rich geometry related to Apollonian packings. In Theorem 6.2, we give a complete geometric description of A in three-dimensional projective space.

Theorem 6.2. *We have $A = J \cup \bigcup_{S\in\mathcal{T}} C_S$ in $\mathbb{RP}^3$.*

Here J is an open ball and $\mathcal{T}$ is an Apollonian packing on the sphere that bounds J. For each circle $S \in \mathcal{T}$, C_S is the unique open cone whose boundary is tangent to the sphere along S. A projective visualization of A is shown in Figure 4. These sections allow us to rediscover Apollonian packings based solely on the Descartes quadratic form. Our argument provides a template to study the geometry of Tits cones for Kac-Moody root systems more generally.

Chen and Labbé have studied the set of limit roots in certain Kac-Moody root systems and related them to sphere packings [9]. Their work involves similar visualizations to ours of the action of Weyl groups on the root space. However, the Apollonian cone constructed here seems to be original, although the method of construction requires no specialized tools.

Having established a connection between Apollonian packings and the Kac-Moody root system Φ, we will suggest some possible generalizations on both sides.

On the Kac-Moody side, one could begin with an indefinite Kac-Moody root system of similar complexity to Φ, and study the geometry of its Weyl group of symmetry and Tits cone. What fractal figures arise this way? In many cases, the Weyl group will be a "thin group," with orbits that are Zariski dense but of infinite covolume in the ambient space [22]. It is an interesting problem to classify Kac-Moody root systems with this property, all of which will be beyond hyperbolic type. Finally, one could generalize the key arithmetic statements about packings—the asymptotic density result of [24], the local-to-global conjecture of [12]—to Weyl orbits for these root systems.

On the packing side, there are many possible generalizations to consider: higher dimensional packings like the sphere packings of Boyd [7] and Maxwell [27], Apollonian superpackings [13], the octahedral packing of Guettler and Mallows [19], and more. Kontorovich and Nakamura introduce a classification of crystallographic sphere packings in all dimensions [23]. They give notions of integrality and super-integrality for general crystallographic sphere packings, and show that the latter yields a finite classification. Stange introduces a collection of packings associated to imaginary quadratic fields and Bianchi groups [30]. In all these cases, the basic unit of the packing is a tuple of circles whose curvatures satisfy one or more quadratic forms. The group of symmetries is generated by reflections which preserve the forms. One could ask which generalizations of Apollonian packings are related to a Kac-Moody root system. The results of Sections 5 and 6 should generalize to many circle and sphere packings besides the Apollonian packing.

2. Background and motivation

2.1. **Background on Apollonian packings.** Figure 1 is a Descartes quadruple of mutually tangent circles in the plane, with curvatures $(c_1, c_2, c_3, c_4) = (-3, 5, 8, 12)$.

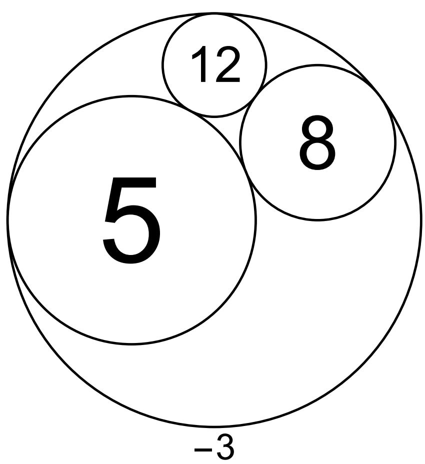

FIGURE 1. Descartes quadruple $\mathbf{c} = -3\alpha_1 + 5\alpha_2 + 8\alpha_3 + 12\alpha_4$

By convention, we take c_i to be negative if this circle is external to the other three; a circle can also degenerate to a straight line with curvature 0. Descartes discovered that the four curvatures satisfy a quadratic equation:

$$2c_1^2 + 2c_2^2 + 2c_3^2 + 2c_4^2 - (c_1 + c_2 + c_3 + c_4)^2 = 0. \tag{2.1}$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the standard basis for $\mathbb{R}^4$, and let $(\ ,\ )$ denote the symmetric bilinear form of signature $(3, 1)$ associated to the Descartes quadratic form, normalized so that $(\alpha_i, \alpha_i) = 2$. A Descartes quadruple can be represented as a vector $\mathbf{c} = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$ satisfying $(\mathbf{c}, \mathbf{c}) = 0$.

The Apollonian group is:

$$W = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 | \sigma_i^2 = 1 \rangle \tag{2.2}$$

with action on $\mathbb{R}^4$ determined by $\sigma_i(\alpha_j) = \alpha_j - (\alpha_j, \alpha_i)\alpha_i$. This preserves the form $(\ ,\ )$. The action of W on a Descartes quadruple $\mathbf{c} = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$ has a beautiful geometric interpretation. If three mutually tangent circles are fixed, then there exist exactly two circles which are tangent to all three. The mapping σ_i corresponds to fixing circles of curvature c_j for $j \neq i$, and swapping out the circle of curvature c_i. This can also be interpreted as a Möbius transformation of the complex plane: an inversion across a circle containing the points of tangency of the three fixed circles.

The orbit of W on an initial Descartes quadruple $\mathbf{c}$ is an Apollonian circle packing, shown in Figure 2.

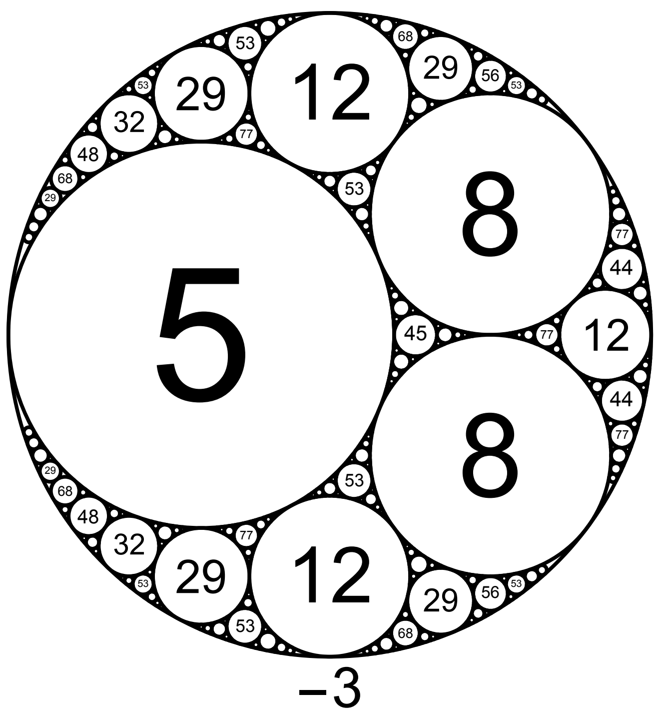

FIGURE 2. Apollonian packing

A quadruple of mutually tangent circles $\mathbf{c}' = c_1'\alpha_1 + c_2'\alpha_2 + c_3'\alpha_3 + c_4'\alpha_4$ appears in this figure if and only if it can be obtained from $\mathbf{c}$ by an element of W. We will denote the multiset of Descartes quadruples in the packing as $\mathcal{P}$ (the multiplicity of quadruples in the packing will be discussed in the proof of Proposition 5.1). Notice that if the initial quadruple $\mathbf{c} \in \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4$, then so is every quadruple in the packing; in this case the packing is called integral. The number-theoretic study of integral Apollonian packings has experienced a renaissance in the last 20 years; see [12, 29]. The central problem in this field is to understand the set of integers which appear as curvatures in a given packing.

All integral packings are either bounded, with a unique exterior circle of negative curvature, or are strip packings, contained between two parallel lines of curvature zero. If we define the height of a quadruple $\mathrm{ht}(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4) = c_1 + c_2 + c_3 + c_4$, then in each case there is a quadruple in $\mathcal{P}$, called the base quadruple, of minimal height. (This is usually called the root quadruple, but we use the term base quadruple to avoid confusion with root system terminology.) A bounded packing has a unique base quadruple, which appears with multiplicity 1 or 2 in $\mathcal{P}$. A strip packing has $n\alpha_i + n\alpha_j$ for some $i \neq j$ as its unique base quadruple, which appears with infinite multiplicity.

2.2. **Background on Kac-Moody root systems.** We focus here on the symmetric Kac-Moody root systems, and particularly on the root system Φ which will be used throughout this article. For more general definitions, we refer the reader to [21].

A symmetric Kac-Moody root system is defined from a symmetric Cartan matrix, with entries of 2 on the diagonal and nonpositive integers off the diagonal. Our root system Φ has the Cartan matrix

$$\begin{pmatrix} 2 & -2 & -2 & -2 \\ -2 & 2 & -2 & -2 \\ -2 & -2 & 2 & -2 \\ -2 & -2 & -2 & 2 \end{pmatrix} \tag{2.3}$$

which is the matrix of the Descartes quadratic form. The real simple roots are the standard basis vectors α_i. The root space is $\bigoplus_i \mathbb{R}\alpha_i$; the root lattice is $Q = \bigoplus_i \mathbb{Z}\alpha_i$. The Cartan matrix defines a bilinear form on the root space. The Weyl group is the group generated by reflections across the hyperplanes orthogonal to the real simple roots, using this bilinear form. The Weyl group acts on the root space, preserving the root lattice. In our case, by definition, the Weyl group W of Φ is the Apollonian group.

The set of real roots Φ_{re} is the orbit of the real simple roots under W. Each real root α lies in Q and satisfies $(\alpha, \alpha) = 2$. The set of imaginary roots Φ_{im} is more difficult to define, but it arises naturally from the representation theory of Kac-Moody algebras. Each imaginary root α lies in Q and satisfies $(\alpha, \alpha) \leq 0$. The full set of roots $\Phi = \Phi_{\mathrm{re}} \cup \Phi_{\mathrm{im}}$ is a W-invariant subset of Q. Each root is either positive, belonging to the cone Q^+ of nonnegative linear combinations of the real simple roots, or negative, belonging to the cone $Q^- = -Q^+$.

An integral Descartes quadruple can be viewed as a vector $\mathbf{c}$ of length squared $(\mathbf{c}, \mathbf{c}) = 0$ in the root lattice Q of Φ. An Apollonian packing is the W-orbit of $\mathbf{c}$ in Q. Since $(\mathbf{c}, \mathbf{c}) = 0$, one might ask whether integral Descartes quadruples are imaginary roots. The answer is no if $\mathbf{c}$ is in a bounded packing $\mathcal{P}$. In such a packing, quadruples involving the exterior circle do not belong to Q^+ or Q^-, so they cannot be imaginary roots. The orbit $\mathcal{P}$ is bounded below (by the base quadruple) but not contained in the positive cone Q^+; this behavior is not possible for Weyl orbits in affine or hyperbolic types, but it occurs in Φ. Thus $\mathcal{P}$ is not an orbit of imaginary roots. The strip packings do correspond to orbits of imaginary roots; indeed, these are imaginary roots of the affine root subsystems $A_1^{(1)}$ which will be defined below.

The fundamental weights ω_i are a dual basis to the simple roots α_i under $(\ ,\ )$. The weight space is $\bigoplus_i \mathbb{R}\omega_i$; the weight lattice, which contains the root lattice, is $\bigoplus_i \mathbb{Z}\omega_i$. The Weyl group acts on the weight space, preserving the weight lattice. The Weyl vector ρ is the sum of the fundamental weights. In the case of Φ, we have $\omega_i = \frac{1}{8}\left(\alpha_i - \sum_{j\neq i}\alpha_j\right)$ and $\rho = -\frac{1}{4}\left(\sum_i \alpha_i\right)$, but in general the fundamental weights and Weyl vector may not be in the root space. The dominant cone $\bar{C}$ is the cone of nonnegative linear combinations of the fundamental weights, dual to the positive cone. The antidominant cone is $-\bar{C}$. The Tits cone is $\bigcup_{w\in W} w(\bar{C})$.

Using the notation of this section, we may take $\mathbf{c} \in \mathcal{P}$ a Descartes quadruple, $\mathbf{s} = s_1\omega_1 + s_2\omega_2 + s_3\omega_3 + s_4\omega_4$, and write the generating function

$$Z(\mathbf{s}) = \sum_{w\in W} e^{-(w\mathbf{c},\mathbf{s})} = \sum_{w\in W} e^{-(\mathbf{c},w\mathbf{s})}. \tag{2.4}$$

This series is invariant under $\mathbf{s} \mapsto w\mathbf{s}$ for $w \in W$.

The Weyl denominator identity is

$$\sum_{w\in W} (-1)^{\ell(w)} e^{(w(\rho)-\rho,\mathbf{s})} = \prod_{\alpha\in\Phi^+} (1 - e^{-(\alpha,\mathbf{s})})^{\mathrm{mult}(\alpha)}, \tag{2.5}$$

where $\ell(w)$ denotes the length of a reduced word for w, Φ^+ is the set of positive real and imaginary roots, and $\mathrm{mult}(\alpha)$ is a positive integer, called the multiplicity, of each root. This formula can serve as an artificial definition of the set of imaginary roots. The multiplicity of each real root is 1; describing the multiplicities of the imaginary roots is a difficult problem. The Weyl denominator (2.5) is a convergent series for $\mathbf{s}$ in the interior of the Tits cone.

The idea of automorphic correction is to augment the set of imaginary roots for Φ so that the Weyl denominator (2.5) becomes an automorphic form. In its sum expression, the corrected denominator function contains infinitely many different Weyl orbits, not just one. In its product expression, the imaginary roots and their multiplicities can be parametrized explicitly. Weyl denominators of affine Kac-Moody root systems are theta functions, as explained in detail in Chapter 13 of [21]. Beginning with examples due to Feingold and Frenkel [11] and Borcherds [3], there have been automorphic corrections for Weyl denominators in certain indefinite Kac-Moody root systems. Gritsenko and Nikulin develop the theory of automorphic correction in [14–17].

We will need some terminology from the classification of Kac-Moody root systems. A symmetric root system Φ is finite if the Cartan matrix is positive definite, affine if it is positive semidefinite with a nontrivial kernel, and indefinite otherwise. A principal root subsystem of Φ is obtained by deleting rows and columns $i_1, \ldots,$ i_k from the Cartan matrix. Φ is called hyperbolic if it is indefinite, but each proper principal root subsystem is finite or affine. The Cartan matrix of a hyperbolic root system must have exactly one negative eigenvalue. Then the projectivized timelike cone of vectors $\mathbf{s}$ satisfying $(\mathbf{s},\mathbf{s}) < 0$ is a hyperbolic space. The condition on principal root subsystems implies that the fundamental weights ω_i are all inside or on the boundary of this space. The dominant cone $\bar{C}$ is a hyperbolic simplex of finite volume, and its orbit under the Weyl group tiles the space, so the Tits cone is the timelike cone.

Our root system Φ is not finite, affine, or hyperbolic. In Maxwell's terminology, Φ has level 2, because each proper principal root subsystem is finite, affine, or hyperbolic [27]. The rank 2 principal root subsystems have affine type $A_1^{(1)}$. The rank 3 principal root subsystems have hyperbolic type $H_{71}^{(3)}$. (The notation $H_{71}^{(3)}$ is taken from Carbone et al. [8]; this root system is called $A_{1,II}$ in [14] and $\Pi_{3,1}$ in [1].) We will describe the dominant cone for Φ and its orbit under the Weyl group in Section 6.

2.3. **Motivation.** We will sketch one application of the series $Z(\mathbf{s})$, which also illustrates why automorphic correction would be valuable. There has been great interest in the Apollonian "L-function"

$$L(u) = \sum_{c \in \mathcal{P}^*} c^{-u}. \tag{2.6}$$

Here $\mathcal{P}^*$ is the collection of curvatures of circles in $\mathcal{P}$, again counted with multiplicity. This series is known to converge for $\Re(u) > \delta$, where $\delta \approx 1.30568$ is the Hausdorff dimension of the residual set of any packing [6]. Meromorphic continuation to the left of δ would yield an asymptotic for the growth of circles in $\mathcal{P}$. Important work of Kontorovich and Oh [24], Vinogradov [31] and Lee and Oh [26] has shown that

$$|\{c \in \mathcal{P}^* | c < X\}| = rX^\delta + O(X^{\delta - \frac{2(\delta - s_1)}{63}}), \tag{2.7}$$

where r is a constant depending on the packing, and s_1 is a constant independent of the packing. Their approach is based on equidistribution of horocycles on a hyperbolic 3-manifold and does not explicitly involve $L(u)$. Meromorphic continuation of $L(u)$ would yield a new proof.

In fact, $L(u)$ can be obtained from $Z(\mathbf{s})$ by an integral transform. First, for $t > 0$, we take

$$\begin{aligned} Z_1(t) = \tfrac{1}{2\pi i} \int_{(\frac{1}{2})} & Z(st\omega_1 + st\omega_2 + st\omega_3 + (1-s)t\omega_4) \\ & + Z(st\omega_1 + st\omega_2 + (1-s)t\omega_3 + st\omega_4) \\ & + Z(st\omega_1 + (1-s)t\omega_2 + st\omega_3 + st\omega_4) \\ & + Z((1-s)t\omega_1 + st\omega_2 + st\omega_3 + st\omega_4) \, \tfrac{ds}{s}, \end{aligned} \tag{2.8}$$

where the integral on the vertical line $\Re(s) = \frac{1}{2}$ is taken in the principal value sense.

An individual summand in $Z(st\omega_1 + st\omega_2 + st\omega_3 + (1-s)t\omega_4)$ has the form $e^{(-c_1-c_2-c_3+c_4)st - c_4 t}$. The integral in s will be 0 if $c_1 + c_2 + c_3 > c_4$, $\frac{e^{-c_4 t}}{2}$ if $c_1 + c_2 + c_3 = c_4$, and $e^{-c_4 t}$ if $c_1 + c_2 + c_3 < c_4$. The results of the integration for the three other terms of the integrand are entirely parallel. For any Descartes quadruple $(c_1, c_2, c_3, c_4) \in \mathcal{P}$ other than the base quadruple, there is a unique Apollonian group reflection σ_i which reduces c_i, yielding a Descartes quadruple of larger circles in $\mathcal{P}$. For this i, we have $c_i > \sum_{j \neq i} c_j$, and c_i is the maximal circle in the quadruple. It follows that the integral of

$$e^{(-c_1-c_2-c_3+c_4)st-c_4t}+e^{(-c_1-c_2+c_3-c_4)st-c_3t}$$
$$+e^{(-c_1+c_2-c_3-c_4)st-c_2t}+e^{(c_1-c_2-c_3-c_4)st-c_1t}$$

will be simply $e^{-\max(c_i)t}$. The base quadruple does not contribute to the integral at all. Note that packings with symmetry type D_2, i.e. with the base quadruple being a multiple of $(-1,2,2,3)$, contain two copies of the base quadruple. In this case each copy of the base quadruple (c_1,c_2,c_3,c_4) will contribute $\frac{e^{-\max(c_i)t}}{2}$, so it is as if one copy of the base quadruple is removed.

Because each $c\in\mathcal{P}^*$ is the maximum of a unique Descartes quadruple other than the base quadruple, we have shown that

$$Z_1(t)+e^{-c_1t}+e^{-c_2t}+e^{-c_3t}+e^{-c_4t}=\sum_{c\in\mathcal{P}^*}e^{-ct}, \tag{2.9}$$

where c_1,c_2,c_3,c_4 are the four curvatures of the base quadruple. Finally, a Mellin transform in t yields

$$\int_0^\infty t^u(Z_1(t)+e^{-c_1t}+e^{-c_2t}+e^{-c_3t}+e^{-c_4t})\tfrac{dt}{t}=\Gamma(u)L(u). \tag{2.10}$$

We will see from Proposition 5.1 that $Z(\mathbf{s})$ converges absolutely in the domain of the two integrations. The first integral converges conditionally in the principal value sense. The integrand in the second integral has a potential singularity at $t=0$. The integral converges as $t\to\infty$ because of the rapid decay of $Z_1(t)$, but it may diverge as $t\to 0$ for sufficiently small $\Re(u)$.

One would hope to meromorphically continue $L(u)$ following the procedure of Riemann's second proof of the meromorphic continuation and functional equation for the zeta function. This requires finding a symmetry for $Z_1(t)$ in $t\mapsto\frac{1}{t}$. Such a symmetry does not arise from the group of functional equations W for $Z(\mathbf{s})$, but it might come from additional automorphic behavior. In particular, both the theta functions of Section 3 and the Siegel automorphic form Δ_5 of Section 4 possess such symmetries. This is one reason to search for an automorphic correction of $Z(\mathbf{s})$.

Gritsenko and Nikulin give conditions on generalized Kac-Moody root systems which are good candidates for automorphic correction in [18]. They call these root systems "Lorentzian." A Lorentzian root system is hyperbolic and has Weyl vector ρ in the root space. Our root system Φ satisfies the second condition but not the first. Automorphic correction may still be possible for Φ but it is more difficult than the automorphic corrections which are currently known. This indicates the difficulty of generalizing Riemann's proof as discussed above. As another indication, note that the constant δ would appear in the calculation as a pole of $L(u)$. Since little is known about this constant, it is not clear how it would arise. A more tractable problem is to make use of other Lorentzian root subsystems of Φ (besides the principal ones). This could yield new information about the density of curvatures appearing in different subsets of an Apollonian packing.

3. Expansion of $Z(\mathbf{s})$ in theta functions

The Kac-Moody root system Φ has a principal rank 2 root subsystem $A_1^{(1)}$ with the Cartan matrix $\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix}$. The Weyl group of this root subsystem is an infinite dihedral group. Sums over Weyl orbits are theta functions—this is equivalent to the fact that the set of circles tangent to two fixed circles in a packing have curvatures parametrized by a quadratic polynomial. Theta functions satisfy a group $\mathrm{GL}(2,\mathbb{Z})$ of symmetries, in which the Weyl group elements act as upper-triangular matrices. Their appearance here is preliminary evidence that $Z(\mathbf{s})$ may have automorphic properties beyond its Apollonian group of symmetries. In this section we will briefly explain the connection between $Z(\mathbf{s})$ and theta functions.

Let W_2 denote the subgroup $\langle\sigma_3,\sigma_4\rangle \subset W$, which is the Weyl group of an $A_1^{(1)}$ root subsystem. Fix a pair of tangent circles in $\mathcal{P}$, assuming without loss of generality that their curvatures are c_1, c_2 in a Descartes quadruple $\mathbf{c} = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$. The set of all Descartes quadruples including these two circles is an orbit of W_2 in $\mathcal{P}$. Define

$$Z_2^+(\mathbf{s}) = \sum_{w\in W_2} e^{-(w\mathbf{c},\mathbf{s})}, \qquad Z_2^-(\mathbf{s}) = \sum_{w\in W_2} (-1)^{\ell(w)} e^{-(w\mathbf{c},\mathbf{s})}. \tag{3.1}$$

Proposition 3.1 relates these to Jacobi theta functions, which we denote as

$$\theta_{00}(z,\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n z+\pi i n^2\tau}, \qquad \theta_{01}(z,\tau) = \sum_{n\in\mathbb{Z}} (-1)^n e^{2\pi i n z+\pi i n^2\tau}. \tag{3.2}$$

Proposition 3.1. *We have:*

(3.3)
$$\begin{aligned} Z_2^{\pm}(\mathbf{s}) = e^{-(\mathbf{c},\mathbf{s})} \Bigg(&\left(\tfrac{\theta_{00}+\theta_{01}}{2}\right)\left(\tfrac{(c_1+c_2+c_3-c_4)s_3-(c_1+c_2-c_3+c_4)s_4}{2\pi i}, \tfrac{-(c_1+c_2)(s_3+s_4)}{\pi i}\right) \\ &\pm e^{(c_3-c_4)(s_3-s_4)} \left(\tfrac{\theta_{00}-\theta_{01}}{2}\right)\left(\tfrac{-(c_1+c_2-c_3+c_4)s_3+(c_1+c_2+c_3-c_4)s_4}{2\pi i}, \tfrac{-(c_1+c_2)(s_3+s_4)}{\pi i}\right)\Bigg). \end{aligned}$$

Proof. The set of Descartes quadruples in $\mathcal{P}$ containing c_1, c_2 may be parametrized as follows:

$$\begin{aligned} &\{\mathbf{c} + n((n-1)(c_1+c_2) - c_3 + c_4)\alpha_3 + n((n+1)(c_1+c_2) - c_3 + c_4)\alpha_4 \mid n \text{ even}\} \\ &\cup \{\mathbf{c} + (n+1)(n(c_1+c_2) - c_3 + c_4)\alpha_3 + (n-1)(n(c_1+c_2) - c_3 + c_4)\alpha_4 \mid n \text{ odd}\}, \end{aligned}$$

where the first subset comes from applying words of even length in σ_3, σ_4 to (c_1, c_2, c_3, c_4), and the second comes from applying words of odd length. We can then write

$$\begin{aligned} Z_2^{\pm}(\mathbf{s}) = e^{-(\mathbf{c},\mathbf{s})} \Bigg(&\sum_{n\in\mathbb{Z}\text{ even}} e^{-n^2(c_1+c_2)(s_3+s_4)+n(c_1+c_2+c_3-c_4)s_3-n(c_1+c_2-c_3+c_4)s_4} \\ &\pm \sum_{n\in\mathbb{Z}\text{ odd}} e^{-n^2(c_1+c_2)(s_3+s_4)-n(c_1+c_2-c_3+c_4)s_3+n(c_1+c_2+c_3-c_4)s_4+(c_3-c_4)(s_3-s_4)}\Bigg) \end{aligned}$$

which is equivalent to the desired formula. □

It follows from this proposition that we may write:

(3.4)
$$\begin{aligned} Z(\mathbf{s}) = \sum_{c_1,c_2} e^{-(\mathbf{c},\mathbf{s})} \Bigg(&\left(\tfrac{\theta_{00}+\theta_{01}}{2}\right)\left(\tfrac{(c_1+c_2)(s_3-s_4)+(c_3-c_4)(s_3+s_4)}{2\pi i}, \tfrac{-(c_1+c_2)(s_3+s_4)}{\pi i}\right) \\ &+ e^{(c_3-c_4)(s_3-s_4)} \left(\tfrac{\theta_{00}-\theta_{01}}{2}\right)\left(\tfrac{(c_1+c_2)(s_4-s_3)+(c_3-c_4)(s_3+s_4)}{2\pi i}, \tfrac{-(c_1+c_2)(s_3+s_4)}{\pi i}\right)\Bigg), \end{aligned}$$

where the sum is over pairs c_1, c_2 such that some Descartes quadruple $\mathbf{c} = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$ appears in $\mathcal{P}$. It does not matter which quadruple $\mathbf{c}$ we choose to associate to c_1, c_2.

In the special case $c_3 = c_4$, which can occur if the packing $\mathcal{P}$ has a line of symmetry, $Z_2^+(\mathbf{s})$ and $Z_2^-(\mathbf{s})$ behave especially nicely. In this case, $Z_2^-(\mathbf{s})$ is essentially the Weyl denominator for $A_1^{(1)}$ rather than a general alternating sum over the Weyl group. We have

$$\begin{aligned} Z_2^+(\mathbf{s}) &= e^{-(\mathbf{c},\mathbf{s})}\theta_{00}\left(\frac{(c_1+c_2)(s_3-s_4)}{2\pi i}, \frac{-(c_1+c_2)(s_3+s_4)}{\pi i}\right), \\ Z_2^-(\mathbf{s}) &= e^{-(\mathbf{c},\mathbf{s})}\theta_{01}\left(\frac{(c_1+c_2)(s_3-s_4)}{2\pi i}, \frac{-(c_1+c_2)(s_3+s_4)}{\pi i}\right). \end{aligned} \tag{3.5}$$

The series $Z_2^+(\mathbf{s})$ and $Z_2^-(\mathbf{s})$ admit Jacobi triple product expressions and satisfy simpler transformation laws with respect to $\mathrm{GL}(2,\mathbb{Z})$. A related simplification occurs when $s_3 = s_4$, as we have in the integral transform of equation (2.8).

4. Relation to the Siegel modular form Δ_5

The Kac-Moody root system Φ has a principal rank 3 root subsystem $H_{71}^{(3)}$ with the Cartan matrix

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}. \tag{4.1}$$

This root system is hyperbolic and Lorentzian. Indeed, it is one of the original examples of a Lorentzian root system, studied by Gritsenko and Nikulin in [15]. They furnish an automorphic correction of this root system whose Weyl denominator is the Siegel automorphic form Δ_5 on $\mathrm{Sp}(4)$. In this section we will outline the relationship between $Z(\mathbf{s})$ and Δ_5.

An orbit of the Weyl group of $H_{71}^{(3)}$ in $\mathcal{P}$ is simply the collection of Descartes quadruples including a fixed circle. This collection plays an important role in the literature on Apollonian packings. After a change of variables, the Weyl group is isomorphic to the congruence subgroup $\Gamma_0(2)$ of $\mathrm{GL}(2,\mathbb{Z})$; its action on Descartes quadruples is isomorphic to the action of $\Gamma_0(2)$ on binary quadratic forms. As a consequence, one can show that the curvatures of circles tangent to a fixed circle in a packing $\mathcal{P}$ are precisely the values taken by a shifted binary quadratic form [28]. This has been a crucial tool for proving density results on the family of curvatures—see [4, 5].

From this change of variables, we can see directly that a sum over the Weyl group of $H_{71}^{(3)}$ has $\mathrm{GL}(2,\mathbb{Z})$ symmetries. The surprising fact is that such a sum may possess a larger group $\mathrm{Sp}(4,\mathbb{Z})$ of symmetries, in which $\mathrm{GL}(2,\mathbb{Z})$ is the subgroup of block diagonal matrices. This fact has not been used in the literature. Because of the technical details of automorphic correction, we will encounter some obstacles in applying this $\mathrm{Sp}(4)$ automorphicity to density and counting problems in packings. But new results may be possible, especially if we broaden to consider other Lorentzian root subsystems of Φ.

Let W_3 denote the subgroup $\langle \sigma_2, \sigma_3, \sigma_4 \rangle \subset W$. which is the Weyl group of an $H_{71}^{(3)}$ root subsystem. Fix a Descartes quadruple $\mathbf{c} = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 \in \mathcal{P}$. We assume from the start that $c_2 = c_3 = c_4$ and, by rescaling if necessary, that

$2c_1 + 2c_2 = 1$. This ensures that a sum over the Weyl orbit of $\mathbf{c}$ behaves like the Weyl denominator for $H_{71}^{(3)}$. These assumptions cannot be satisfied in an integral plane packing, but they can with the non-integral packing with D_3 symmetry whose base quadruple is $-\frac{\sqrt{3}}{4}\alpha_1 + \frac{1}{4}\left(2+\sqrt{3}\right)\alpha_2 + \frac{1}{4}\left(2+\sqrt{3}\right)\alpha_3 + \frac{1}{4}\left(2+\sqrt{3}\right)\alpha_4$. They can also be satisfied in the integral spherical and hyperbolic packings studied in [10].

As in Section 3, define

$$Z_3^+(\mathbf{s}) = \sum_{w\in W_3} e^{-(w\mathbf{c},\mathbf{s})}, \qquad Z_3^-(\mathbf{s}) = \sum_{w\in W_3} (-1)^{\ell(w)} e^{-(w\mathbf{c},\mathbf{s})}. \tag{4.2}$$

Our goal is to relate $Z_3^-(\mathbf{s})$ to the Siegel modular form Δ_5.

Let us fix some notation. The group $\mathrm{Sp}(4,\mathbb{Z})$ consists of integral 4×4 matrices $M = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$ such that ${}^tM\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}M = \begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}$. Here A, B, C, D, 0 and I denote 2×2 block matrices. The Siegel upper half plane $\mathbb{H}_2$ is the set of symmetric 2×2 complex matrices $Z = X + iY$ such that the imaginary part Y is a positive-definite matrix. $\mathrm{Sp}(4,\mathbb{Z})$ acts on $\mathbb{H}_2$ via

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix} Z = (AZ+B)(CZ+D)^{-1}. \tag{4.3}$$

A Siegel modular form f of weight $k\in\mathbb{Z}$ and character $\nu : \mathrm{Sp}(4,\mathbb{Z}) \to \mathbb{C}^\times$ is a holomorphic function on $\mathbb{H}_2$ satisfying

$$f(MZ) = \nu(M)\det(CZ+D)^k f(Z) \tag{4.4}$$

for all $M\in\mathrm{Sp}(4,\mathbb{Z})$.

The function $\Delta_5 : \mathbb{H}_2 \to \mathbb{C}$ is a Siegel cusp form of weight 5 with a nontrivial quadratic character ν. For full details on the construction of Δ_5, we refer the reader to [15]. Here we will work with the Fourier expansion of Δ_5:

$$\frac{1}{64}\Delta_5(Z) = \sum_{\substack{l,m,n \text{ odd} \\ m,n,4mn-l^2>0}} \sum_{d\mid \gcd(l,m,n)} d^4 g\left(\frac{mn}{d^2}.\frac{l}{d}\right) e^{\pi i(nz_1+lz_2+mz_3)}, \tag{4.5}$$

where $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3\end{pmatrix} \in \mathbb{H}_2$ [15, Equation 4.10]. The coefficients $g(k,l)$ are defined by the generating series

$$\sum_{k,l \text{ odd}} g(k,l)e^{\pi i(kz_1+lz_2)} = \eta(z_1)^9\theta_{11}(z_2,z_1), \tag{4.6}$$

where $\eta(z) = e^{\pi i z/12}\prod_{n=1}^{\infty}(1-e^{2\pi i n z})$ and $\theta_{11}(z,\tau) = \sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i(2n+1)z+\pi i(n+1/2)^2\tau}$. Then the Jacobi triple product formula yields:

$$\begin{aligned}&\sum_{k,l \text{ odd}} g(k,l)e^{\pi i(kz_1+lz_2)} \\ &= -e^{\pi i(z_1-z_2)}\prod_{n=1}^{\infty}(1-e^{2\pi i((n-1)z_1+z_2)})(1-e^{2\pi i(nz_1-z_2)})(1-e^{2\pi i n z_1})^{10}.\end{aligned} \tag{4.7}$$

In order to relate the action of $\mathrm{Sp}(4,\mathbb{Z})$ on $\mathbb{H}_2$ to the action of the Apollonian group, we introduce a new basis of $\mathbb{R}^4$ and change variables. Let $\beta_1 = (\alpha_3+\alpha_4)/2$, $\beta_2 = \alpha_4$, $\beta_3 = (\alpha_2+\alpha_4)/2$. Then $\mathbf{s} = s_1\omega_1 + s_2\omega_2 + s_3\omega_3 + s_4\omega_4$ can be rewritten as $z_0\omega_1 + z_1\beta_1 + z_2\beta_2 + z_3\beta_3$ for some $z_0, z_1, z_2, z_3 \in \mathbb{C}$. This is the same change of variables used to relate the action of the Apollonian group to the action of $\Gamma_0(2)$ on binary quadratic forms. We also let $\rho = -\frac{1}{2}(\alpha_2 + \alpha_3 + \alpha_4)$, the Weyl vector for the $H_{71}^{(3)}$ root subsystem.

Theorem 4.1. *For* $\mathbf{s} = s_1\omega_1 + s_2\omega_2 + s_3\omega_3 + s_4\omega_4 = z_0\omega_1 + z_1\beta_1 + z_2\beta_2 + z_3\beta_3$, *we have*

$$e^{(\mathbf{c},\mathbf{s})}\left(Z_3^-(\mathbf{s}) - \sum_{\alpha} m(\alpha) \sum_{w\in W_3} (-1)^{\ell(w)} e^{-(w(\mathbf{c}+\alpha),\mathbf{s})}\right) = \frac{e^{-(\rho,\mathbf{s})}}{64}\Delta_5\left(\frac{1}{\pi i}Z\right), \tag{4.8}$$

where the first sum is over $\alpha \in \mathbb{Z}_{\geq 0}\alpha_2 \oplus \mathbb{Z}_{\geq 0}\alpha_3 \oplus \mathbb{Z}_{\geq 0}\alpha_4$ *such that* $(\alpha, \alpha_i) \leq 0$ *for* $i = 2, 3, 4$, *and the* $m(\alpha)$ *are integer constants. Further,*

$$\frac{e^{-(\rho,\mathbf{s})}\Delta_5\left(\frac{1}{\pi i}Z\right)}{64e^{(\mathbf{c},\mathbf{s})}Z_3^-(\mathbf{s})} \tag{4.9}$$

is a series of exponentials of the form $e^{-(\beta,\mathbf{s})}$ *where each* β *is a nonnegative integer combination of* α_2, α_3, α_4 *satisfying* $(\beta, \beta) \leq 0$.

The first statement is the analog of Theorem 2.3 in [15]. The meaning of the second statement comes from comparing the product forms of the Weyl denominator for $H_{71}^{(3)}$, $e^{(\mathbf{c},\mathbf{s})}Z_3^-(\mathbf{s})$, to the Weyl denominator of its automorphic correction, $\frac{e^{-(\rho,\mathbf{s})}}{64}\Delta_5\left(\frac{1}{\pi i}Z\right)$. The original root system and the automorphic correction have the same real roots, and differ only by imaginary roots.

Proof. Note that $(\mathbf{c}, \alpha_i) = -(\rho, \alpha_i) = -1$ for $i = 2, 3, 4$ because $c_2 = c_3 = c_4$ and $2c_1 + 2c_2 = 1$. Therefore, for α as in the proposition, the bilinear pairing of $\mathbf{c}+\alpha$ with any positive root in the root subsystem will be a nonpositive integer. That is, $\mathbf{c}+\alpha$ behaves like an antidominant weight. Further, for $w \in W_3$, we have $\mathbf{c} - w(\mathbf{c}+\alpha) = -\rho - w(-\rho+\alpha)$, and this is a nonpositive integer combination of $\alpha_2, \alpha_3, \alpha_4$. Applying the bilinear form, we see that $(\mathbf{c}, \mathbf{s}) - (w(\mathbf{c}+\alpha), \mathbf{s}) = (-\rho, \mathbf{s}) - (w(-\rho+\alpha), \mathbf{s})$ is an even integer combination of z_1, z_2, z_3, with nonnegative coefficients of z_1 and z_3. Since $(\rho, \mathbf{s}) = z_1 + z_2 + z_3$, we can write:

$$-(w(-\rho+\alpha), \mathbf{s}) = nz_1 + lz_2 + mz_3.$$

With n, l, m odd, and $n, m > 0$. The condition that $4mn - l^2 > 0$ is equivalent to $(w(-\rho+\alpha), w(-\rho+\alpha)) < 0$, which holds because $-\rho+\alpha$ is positive and antidominant and w preserves the bilinear from. From this calculation and the Fourier expansion of Δ_5, we see that the two sides of (4.8) are exponential sums with the same support, and it suffices to compare the coefficients.

By the Fourier expansion, the coefficients on the right side of (4.8) are integers. Moreover, the constant coefficient, which corresponds to $l = m = n = 1$, is 1. We must show that the Fourier expansion on the right side is alternating with respect to the action of W_3. This action is generated as follows: if $-(w(-\rho+\alpha), \mathbf{s}) = nz_1 + lz_2 + mz_3$, then

$$\begin{aligned}
-(\sigma_2 w(-\rho+\alpha), \mathbf{s}) &= (n - 2l + 4m)z_1 + (4m - l)z_2 + mz_3\\
-(\sigma_3 w(-\rho+\alpha), \mathbf{s}) &= nz_1 + (4n - l)z_2 + (m - 2l + 4n)z_3\\
-(\sigma_4 w(-\rho+\alpha), \mathbf{s}) &= nz_1 - lz_2 + mz_3.
\end{aligned}$$

To show alternation for σ_2, we must check that

$$\sum_{d|\gcd(4m-l,m,n-2l+4m)} d^4 g\left(\frac{m(n-2l+4m)}{d^2}.\frac{4m-l}{d}\right) = -\sum_{d|\gcd(l,m,n)} d^4 g\left(\frac{mn}{d^2}.\frac{l}{d}\right).$$

Since the sum over d is the same on both sides, it is sufficient to check that $g(m(n-2l+4m), 4m-l) = -g(mn, l)$, or more generally $g(k-2lm+4m^2, 4m-l) = -g(k,l)$ for all $k, l, m \in \mathbb{Z}$. We can see from the definition that this property holds for coefficients of $\theta_{11}(z_2, z_1)$, and therefore it must hold for the coefficients $g(k,l)$ of $\eta(z_1)^9\theta_{11}(z_2, z_1)$. The proof of alternation for σ_3 is entirely parallel. For σ_4, it suffices to show that $g(nm, -l) = -g(nm, l)$, which again is apparent from the series definition of $\theta_{11}(z_2, z_1)$.

Now that we have the expected W_3 alternation on both sides of (4.8), note that each W_3 orbit contains a unique antidominant element, which can be written as $-\rho + \alpha$ where $-(-\rho + \alpha, \mathbf{s}) = nz_1 + lz_2 + mz_3$, and α is a nonnegative integer combination of α_2, α_3, α_4. In this case, set

$$m(\alpha) = -\sum_{d|\gcd(l,m,n)} d^4 g\left(\frac{mn}{d^2}.\frac{l}{d}\right) \tag{4.10}$$

and equation (4.8) follows.

To justify the final sentence of the theorem, use (4.8) to express (4.9) as a linear combination of terms of the form

$$\frac{\sum_{w\in W_3}(-1)^{\ell(w)}e^{(\mathbf{c}-w(\mathbf{c}+\alpha),\mathbf{s})}}{\sum_{w\in W_3}(-1)^{\ell(w)}e^{(\mathbf{c}-w(\mathbf{c}),\mathbf{s})}}.$$

Both the numerator and denominator are series of exponentials $e^{-(\beta,\mathbf{s})}$ where each β is a nonnegative integer combination of α_2, α_3, α_4. The denominator is a unit in the ring of such series. Finally, the quotient is W_3-invariant, so if it includes a term $e^{-(\beta,\mathbf{s})}$, then it also includes $e^{-(w(\beta),\mathbf{s})}$ for all $w \in W_3$. Thus every element in the W_3 orbit of β is a nonnegative integer combination of α_2, α_3, α_4. If $(\beta, \beta) > 0$, then $(\beta, \alpha_i) > 0$ for $i = 1$, 2, or 3, and we can apply σ_i to reduce the height of β. We can repeat this procedure until we have a negative coefficient of α_2, α_3, or α_4, a contradiction. We conclude that $(\beta, \beta) \leq 0$. □

In the group $\mathrm{Sp}(4,\mathbb{Z})$ of symmetries for Δ_5, W_3 is embedded as the subgroup of block matrices $\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1}\end{pmatrix}$ with $A \in \Gamma_0(2) \subset \mathrm{GL}(2,\mathbb{Z})$. The symmetry under $\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z})$ is of the kind needed to complete the Mellin inversion argument sketched in Section 2. The difference between $e^{(\mathbf{c},\mathbf{s})}Z_3^-(\mathbf{s})$ and its automorphic correction $\frac{e^{(\rho,\mathbf{s})}}{64}\Delta_5\left(\frac{1}{\pi i}Z\right)$ adds some technical complication, but this method can be used to estimate the density of curvatures tangent to a fixed circle in $\mathcal{P}$. We do not pursue this here because good estimates for the density of integers represented by a shifted binary quadratic form are already available [2]. Another promising approach is to consider alternate Lorentzian root subsystems of Φ.

In this section, we assumed that $\mathbf{c}$ has a particularly simple form, which causes $Z_3^-(\mathbf{s})$ to behave like the Weyl denominator for the $H_{71}^{(3)}$ root system. A related simplification occurs with specializations of $\mathbf{s}$, such as $s_2 = s_3 = s_4$. It is not clear whether an arbitrary sum over an orbit of W_3 can be automorphically corrected.

5. The Apollonian cone A

Since several different conelike objects appear below, we introduce some terminology here. We define the cone with apex $\mathbf{p} \in \mathbb{R}^n$ on a region $R \subset \mathbb{R}^n$ as the union of all rays originating at $\mathbf{p}$ and containing a point of R. The bounded cone with apex $\mathbf{p}$ on region R is the union of all line segments between $\mathbf{p}$ and a point of R. A simplicial cone is the cone on a simplex or finite union of simplices in $\mathbb{R}^n$.

We fix a bounded, integral packing $\mathcal{P}$, and ask where its associated generating function $Z(\mathbf{s})$ converges. It is known that the domain of absolute convergence of any series of exponentials with real coefficients is a convex tube domain (the analogous statement for power series is [25, Proposition 2.3.15]). The real parts of the variables s_i must lie in a convex region in $\mathbb{R}^4$, while the imaginary parts can be arbitrary. Proposition 5.1 establishes an initial domain of absolute convergence.

Proposition 5.1. *$Z(\mathbf{s})$ is absolutely convergent in the simplicial cone C_0 defined by the four inequalities $\Re(s_i) > 0$.*

Proof. Let c denote the negative curvature of the exterior circle in $\mathcal{P}$. Assume without loss of generality that the base quadruple is ordered from smallest to largest curvature. Then we have $c_1 \geq c$ and $c_2, c_3, c_4 \geq 0$ for all quadruples $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 \in \mathcal{P}$.

Any ordered quadruple appears for at most two Descartes configurations in $\mathcal{P}$. Indeed, if the same ordered quadruple of curvatures appears at distinct configurations, then the same sequence of moves in the Apollonian group can be applied to both quadruples to obtain two distinct copies of the base Descartes configuration in $\mathcal{P}$. This is possible if and only if the packing has symmetry type D_2. In this case there are exactly two copies of the ordered base quadruple in the packing, and thus two copies of any ordered quadruple. In any other case, there is only one copy of each ordered quadruple.

Therefore $Z(\mathbf{s})$ can be compared to the product of four geometric series

$$\frac{2e^{-cs_1}}{(1-e^{-s_1})(1-e^{-s_2})(1-e^{-s_3})(1-e^{-s_4})}$$

which converges absolutely in this region. $\square$

A similar argument, counting triples of circles in the packing instead of quadruples, can be used to prove Proposition 5.2, which gives a larger domain of convergence.

Proposition 5.2. *$Z(\mathbf{s})$ is absolutely convergent in the simplicial cone C_1 defined by the 12 inequalities $\Re(s_i) > -2\Re(s_j)$.*

In fact, the domain of convergence is even larger. The defining property of $Z(\mathbf{s})$ is its invariance under the Apollonian group $W = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$. Applying $w \in W$ to $\mathbf{s}$ simply permutes the summands of $Z(\mathbf{s})$, preserving absolute convergence.

Applying σ_i to the cone C_0 yields a simplicial cone defined by the four inequalities $\Re(s_i) < 0$ and $\Re(s_j) > -2\Re(s_i)$ for $j \neq i$. From this we see that C_1 is the convex hull of the set $C_0 \cup \sigma_1(C_0) \cup \sigma_2(C_0) \cup \sigma_3(C_0) \cup \sigma_4(C_0)$. In particular, this set includes the faces of $\bar{C}_0$ where exactly one of the $\Re(s_i)$ is 0, but does not include the 2-skeleton of $\bar{C}_0$. Indeed, if the real parts of two of the s_i are 0, then $Z(\mathbf{s})$ certainly diverges because there exist infinitely many quadruples in the packing with two circles fixed.

Let $C = \{\mathbf{s} \in \mathbb{C}^4 \mid \text{all } \Re(s_i) \geq 0, \text{ at most one } \Re(s_i) = 0\}$. Then we have the following:

Theorem 5.3. *The domain of absolute convergence of* $Z(\mathbf{s})$ *is* $A = \bigcup_{w\in W} w(C)$.

This domain is the interior of the Tits cone, $\bigcup_{w\in W} w(\bar{C})$, and is independent of the Apollonian packing $\mathcal{P}$ that we have fixed.

Proof. We may assume for simplicity that our point of convergence $\mathbf{s}$ lies in $\mathbb{R}^4$, with the understanding that the real domain of absolute convergence determines the complex domain.

It is clear from the invariance of $Z(\mathbf{s})$ under W and from the previous propositions that the domain of absolute convergence contains A. We must show that if $Z(\mathbf{s})$ converges absolutely at $\mathbf{s}$, then this point belongs to A. If any W-translate of $\mathbf{s}$ has two or more nonpositive coordinates, then $Z(\mathbf{s})$ diverges, because the series will contain infinitely many terms with absolute value bounded below by some positive constant. If some W-translate has all nonnegative coordinates and at most one zero coordinate, then the point lies in A by definition.

The only remaining possibility is that every W-translate of $\mathbf{s}$ has three positive coordinates and one negative. Let B be the set of points with this property, and suppose $\mathbf{s} \in B$. Define the height of $\mathbf{s}$ as $s_1 + s_2 + s_3 + s_4$. Note that applying σ_i adds $4s_i$ to the height of a point. Every point in B has positive height, because there is only one negative coordinate s_i, and two of the three other coordinates must exceed $-2s_i$. Moreover, for a point $\mathbf{s} \in B$, there is a unique sequence of W-translates:

$$\mathbf{s}, \quad \sigma_{i_1}\mathbf{s}, \quad \sigma_{i_2}\sigma_{i_1}\mathbf{s}, \quad \sigma_{i_3}\sigma_{i_2}\sigma_{i_1}\mathbf{s}, \quad \ldots$$

with height decreasing monotonically. The heights of this sequence must converge to a lower bound h, so the negative coordinates of the sequence must converge to 0. It follows that the sequence of points eventually lies in the compact region $h \leq s_1 + s_2 + s_3 + s_4 \leq h + \epsilon$, $s_1, s_2, s_3, s_4 \geq -\epsilon$. We can choose a convergent subsequence, denoted $(\mathbf{s}_n)$. Because the negative coordinate s_i of $\mathbf{s}_n$ converges to 0, and one other coordinate s_j is bounded above by $-2s_i$, the point $\lim_{n\to\infty} \mathbf{s}_n$ must lie on the 2-skeleton of the simplicial cone $\bar{C}_0$.

Note that the function $Z(\mathbf{s})$ on $\mathbb{R}^4$ is defined by a sum of positive terms. If it converges at $\mathbf{s}$, then it converges to the same value at each $\mathbf{s}_n$. This implies that the partial sums at each $\mathbf{s}_n$ are uniformly bounded above, which in turn implies that the partial sums at $\lim_{n\to\infty} \mathbf{s}_n$ are bounded above. We would then have that $Z(\mathbf{s})$ converges at $\lim_{n\to\infty} \mathbf{s}_n$, a contradiction. □

6. The geometry of A

What does the domain of absolute convergence A look like? The question is of interest because there are few explicit examples in the literature of Tits cones for Kac-Moody groups beyond the affine and hyperbolic cases.

The geometry of the Tits cone is well-understood for hyperbolic root systems. The bilinear form $(\, , \,)$ determines a cone J of timelike vectors satisfying $(\mathbf{s}, \mathbf{s}) < 0$. One half of this cone, J^+, intersects the dominant cone $\bar{C}$. In the hyperbolic case, $\bar{C}$ is a polytope of finite hyperbolic volume in J^+ with vertices at the fundamental weights, and its W-translates cover J^+, so $A = J^+$.

We will see, however, that none of this holds in Φ. Each fundamental weight ω_i satisfies $(\omega_i, \omega_i) = \frac{1}{8}$ so these vectors are spacelike, not timelike. Thus $\bar{C}$ extends beyond the cone J^+, and has infinite hyperbolic volume. The Tits cone $\bar{A}$ includes J^+ as a proper subset, and is much more geometrically complicated than J^+. We will see that the intricate structure of Apollonian packings carries over to A.

In order to visualize A, we must cut down its dimension. First, since A is a tube domain, it suffices to describe the real part. Because the real part of each region $w(C)$ for $w \in W$ is a simplicial cone defined by a system of homogeneous linear inequalities, the real part of A is a cone through the origin in $\mathbb{R}^4$. It is easily shown that the real part of A lies in the half-space $s_1 + s_2 + s_3 + s_4 \geq 0$. A cone on the origin in this half-space can be considered as a subset of $\mathbb{RP}^3$. For the rest of this section, by abuse of notation, we will consider C, all $w(C)$, and A as subsets of $\mathbb{RP}^3$. The graphics in this section depict the affine section of A along the hyperspace $s_1 + s_2 + s_3 + s_4 = 1$.

Figure 3, generated in Mathematica [20], shows the domains $\bigcup_{\substack{w \in W \\ \ell(w) \leq \ell}} w(C)$ for $\ell = 0, 1, 2, 3$ in $\mathbb{RP}^3$. Each $w(C)$ is a 3-simplex.

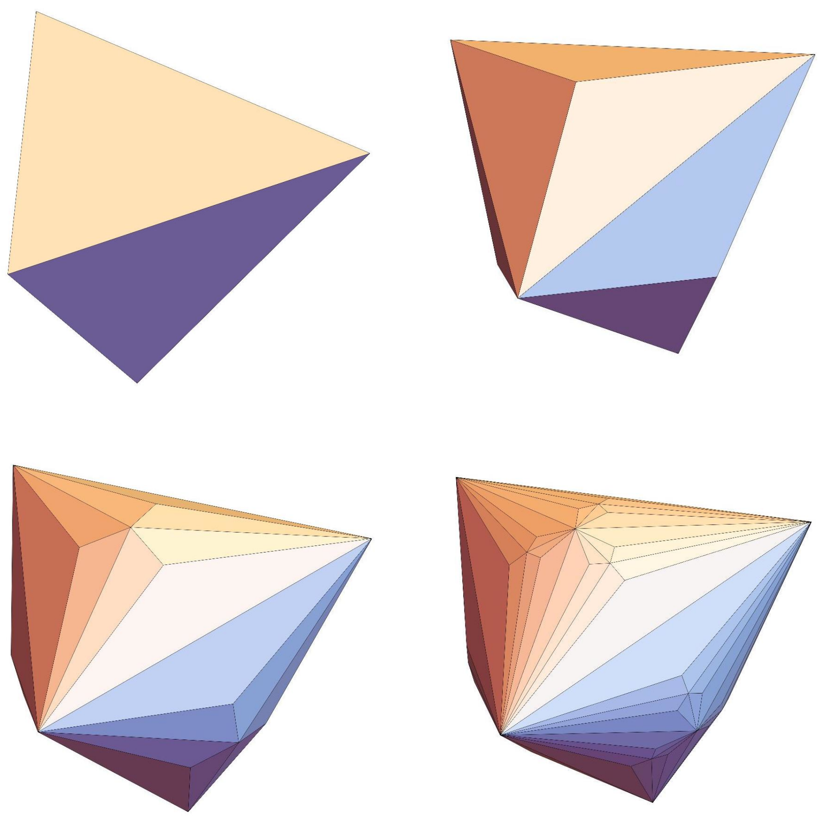

FIGURE 3. Preliminary domains of convergence for $Z(\mathbf{s})$

The boundary of the timelike cone $J = \{\mathbf{s} \in \mathbb{RP}^3 \mid (\mathbf{s}, \mathbf{s}) < 0\}$ is the lightlike cone $N = \{\mathbf{s} \in \mathbb{RP}^3 \mid (\mathbf{s}, \mathbf{s}) = 0\}$. N is a sphere in projective space, and J is the ball enclosed by the sphere. Both are W-invariant.

The 3-simplex C is has vertices ω_1, ω_2, ω_3, ω_4. The six edges are segments along $s_i = s_j = 0$ for some i, j. Each edge is tangent to N. If we fix a vertex ω_i, the three points of tangency on edges through ω_i lie in the plane $s_1 + s_2 + s_3 + s_4 - 2s_i = 0$. The four planes $s_1 + s_2 + s_3 + s_4 - 2s_i = 0$ intersect N in four mutually tangent circles.

We use these circles as a base quadruple to generate an Apollonian packing $\mathcal{T}$ on N. Then, for each circle $S \in \mathcal{T}$, let O_S be the open spherical cap on N, with boundary S (chosen so that all the sets O_S are disjoint). There is a unique point $\mathbf{p}_S$ such that the cone on S with apex $\mathbf{p}_S$ is tangent to N. Let C_S be the bounded cone on O_S with apex $\mathbf{p}_S$, and let C'_S be the unbounded cone. An image of the region $J \cup \bigcup_{S \in \mathcal{T}} C_S \subset \mathbb{RP}^3$ is shown in Figure 4.

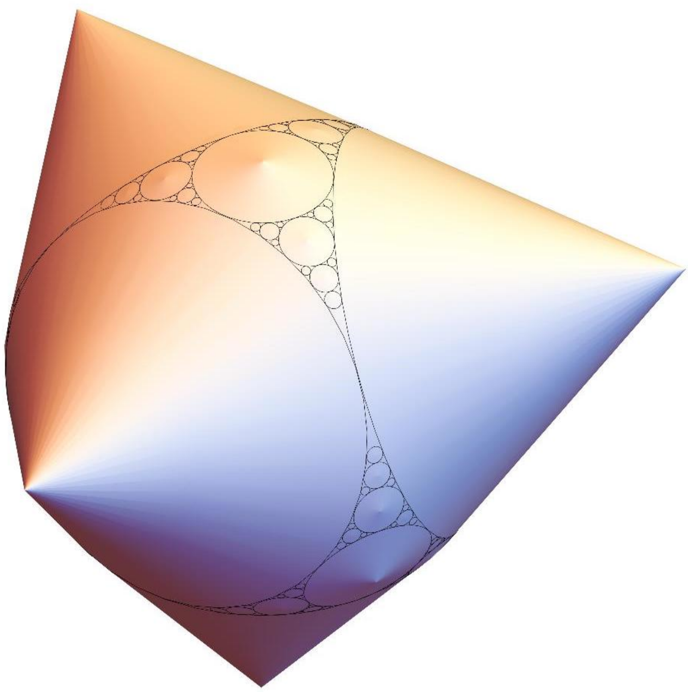

FIGURE 4. Full domain of convergence for $Z(\mathbf{s})$

Because the boundary of each cone C_S is tangent to the sphere, the line segments connecting apexes $\mathbf{p}_S$ for tangent circles $S \in \mathcal{T}$ are contained in the boundary of both cones, and tangent to the sphere. Such line segments are dense in the boundary of the region $J \cup \bigcup_{S \in \mathcal{T}} C_S$. The line segments connecting apexes of non-adjacent cones must intersect J; otherwise the two cones would intersect each other.

Let $F \subset N$ denote the fractal set of points in N not on any circle $S \in \mathcal{T}$ or in any spherical cap O_S.

Proposition 6.1.

$$\bigcap_{S\in\mathcal{T}} C'_S = J \cup F \cup \bigcup_{S\in\mathcal{T}} C_S. \tag{6.1}$$

Proof. Given two points $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{RP}^3 \setminus J$, we may form two cones tangent to N with these points as apexes. We will denote the open regions enclosed by these cones as C'_1 and C'_2, and the circles of tangency as S_1 and S_2 (For points on N, the cone becomes a half-space, and the circle becomes a single point.) If the circles S_1 and S_2 are externally disjoint, then each point is contained in the interior of the other cone. If the circles are externally tangent, then each point is on the boundary of the other cone. If the circles intersect non-tangentially, then each point is outside the other cone. If the circles are internally tangent, then the inner point is on the boundary of the outer cone. If one circle is internal to the other, then the inner point is enclosed by the outer cone.

First we will show that the union on the right side is contained in the intersection on the left. The ball J is contained in every region C'_S. The set F is also contained in every C'_S because it does not intersect the boundary of any of these sets. Finally, a pair of circles in the packing $\mathcal{T}$ must be externally disjoint or externally tangent. It follows that each point $\mathbf{p}_S$ is contained in all the regions $\bar{C}'_S$, and therefore that every bounded region C_S is contained in all the unbounded regions C'_S.

Next we show the opposite inclusion. For a point $\mathbf{s}$ contained in every region C'_S, either $\mathbf{s} \in J$ or we may form the cone tangent to N with $\mathbf{s}$ at its apex. The circle of tangency cannot intersect any circle $S \in \mathcal{T}$. This is possible only if the circle is internal to some S, in which case $\mathbf{s} \in C_S$, or if the circle degenerates to a single point in F. $\square$

We now come to the main result of this section.

Theorem 6.2. *We have*

$$A = J \cup \bigcup_{S\in\mathcal{T}} C_S \tag{6.2}$$

in $\mathbb{RP}^3$. Moreover, each vertex of each simplex $w(\bar{C})$ is $\mathbf{p}_S$ for some $S \in \mathcal{T}$, and each edge of each $w(\bar{C})$ is the line segment connecting the apexes of two adjacent cones in the packing.

Proof. The four points ω_i are the apexes of four cones tangent to N, which intersect N in the base quadruple of $\mathcal{T}$. The four reflections σ_i, considered as transformations of $\mathbb{RP}^3$, map N to itself. Because each circle on N is contained in a unique plane, and σ_i maps planes to planes, σ_i must map circles to circles on N. Each σ_i fixes the plane $s_i = 0$. For $j \neq i$, σ_i maps the plane $s_1 + s_2 + s_3 + s_4 - 2s_j = 0$ to itself. Thus σ_i maps three of the four circles in the base quadruple to themselves, and moves the fourth circle. Because σ_i must preserve tangency, it maps the packing $\mathcal{T}$ to itself. From the action on the base quadruple, we can see that W acts on $\mathcal{T}$ in the standard way.

Because the transformations σ_i map the packing $\mathcal{T}$ to itself, and because they preserve tangency, they must map the collection of cones C_S for $S \in \mathcal{T}$ to itself. In particular, the set of apexes $\mathbf{p}_S$ for $S \in \mathcal{T}$ is the orbit of W on the four initial points ω_1, ω_2, ω_3, ω_4. This set is the zero-skeleton of A. Each $w(\bar{C})$ is a simplex with vertices at the four points $\mathbf{p}_S$ corresponding to a Descartes quadruple of circles

$S \in \mathcal{T}$. The edges of this simplex are line segments, tangent to N, connecting apexes of adjacent cones—these form the one-skeleton of A. The faces and interior of the simplex lie in the union of J with the four cones of the Descartes quadruple.

It remains to show that $A \supseteq J \cup \bigcup_{S \in \mathcal{T}} C_S$. We have remarked that the boundary of A, specifically the 2-skeleton, is dense in the boundary of each cone C_S. Since A is convex, it follows that C_S is contained in A. Since the cones C_S intersect the boundary of J in another dense set, it follows that J is contained in A. □

Note that even conditional convergence is impossible along the 2-skeleton of A. Conditional convergence may be possible at other points of ∂A. The question of conditional convergence will not be discussed further here.

Recall that the set B from the proof of Theorem 5.3 consists of points $\mathbf{s}$ such that every W-translate $w(\mathbf{s})$ has three positive coordinates and one negative. We can classify a point $\mathbf{s} \in \mathbb{RP}^3$, assuming $s_1 + s_2 + s_3 + s_4 \geq 0$, based on the number of negative coordinates of its W-translates.

Proposition 6.3. *Any point of $\mathbb{RP}^3 \setminus \bar{A}$ has a W-translate with three negative coordinates, or two negative coordinates and one zero coordinate.*

Proof. Given a point $\mathbf{s} \in \mathbb{RP}^3 \setminus \bar{A}$, we can form the cone tangent to N with apex $\mathbf{s}$. As in the proof of 6.1, the circle of tangency cannot be internal to any circle in $\mathcal{T}$. It must contain some point of the boundary of some circle of $\mathcal{T}$ internally, and then, because it contains a neighborhood of that point, it must in fact contain a circle $S \in \mathcal{T}$ internally. It follows that the point $\mathbf{s}$ must be contained in the region bounded by the cone opposite to C'_S, i.e. the cone with apex $\mathbf{p}_S$ and half-lines in the opposite direction of those in C'_S.

As demonstrated in Theorem 6.2, the region C_S is contained in a union of simplices $w(C)$ for those $w \in W$ which map the base quadruple to a quadruple involving S. Each of these simplices is defined by four inequalities. If we drop the inequality which is strictly satisfied by $\mathbf{p}_S$, we obtain an unbounded triangular cone with apex $\mathbf{p}_S$. The region C'_S is the union of the regions enclosed by these unbounded triangular cones. A point in the cone opposite C'_S must lie in the opposite of one of the these triangular cones. Thus, it satisfies three inequalities opposite to those which define the triangular cone. If it lies opposite the interior of a triangular cone, it satisfies the three inequalities strictly.

By applying a W-translation which maps S to a circle of the base quadruple, we find that some $w(\mathbf{s})$ has three nonpositive coordinates, with at most one coordinate equal to zero. □

Proposition 6.4 describes the region B.

Proposition 6.4. *Let $\mathbf{s}$ be a point of ∂A which is not in the 2-skeleton of A. Then every W-translate of $\mathbf{s}$ has exactly three positive coordinates and one negative. Thus B is the boundary ∂A with the two-skeleton of A removed.*

Proof. It suffices to show that $\mathbf{s}$ satisfies exactly three of the four inequalities which define each simplex $w(C)$ for $w \in W$. Then we see that the translate of $\mathbf{s}$ by every w^{-1} has three positive coordinates and one negative. The simplex $w(C)$ has vertices $\mathbf{p}_{S_1}$, $\mathbf{p}_{S_2}$, $\mathbf{p}_{S_3}$, $\mathbf{p}_{S_4}$ for a Descartes quadruple of circles $S_1, S_2, S_3, S_4 \in \mathcal{T}$. There are two cases: if $\mathbf{s}$ does not lie on ∂C_{S_i} for $i = 1, 2, 3, 4$, then since $\mathbf{s} \in \partial A$, it must lie on the surface in the interstitial region between three of the circles S_i. Assume that $\mathbf{s}$ lies between S_1, S_2, and S_3. Then $\mathbf{s}$ is on the right side of the plane through $\mathbf{p}_{S_1}$, $\mathbf{p}_{S_2}$, $\mathbf{p}_{S_4}$, the plane through $\mathbf{p}_{S_1}$, $\mathbf{p}_{S_3}$, $\mathbf{p}_{S_4}$, and the plane through $\mathbf{p}_{S_2}$, $\mathbf{p}_{S_3}$, $\mathbf{p}_{S_4}$, but it is on the wrong side of the plane through $\mathbf{p}_{S_1}$, $\mathbf{p}_{S_2}$, $\mathbf{p}_{S_3}$.

For the other case, assume that $\mathbf{s}$ lies on ∂C_{S_1}. Then there are three line segments connecting $\mathbf{p}_{S_1}$ to $\mathbf{p}_{S_2}$, $\mathbf{p}_{S_3}$, and $\mathbf{p}_{S_4}$. The point $\mathbf{s}$ must lie between two of these segments. Assume that it lies between the segment to $\mathbf{p}_{S_2}$ and the segment to $\mathbf{p}_{S_3}$. Then $\mathbf{s}$ is on the right side of the plane through $\mathbf{p}_{S_1}$, $\mathbf{p}_{S_2}$, $\mathbf{p}_{S_4}$, the plane through $\mathbf{p}_{S_1}$, $\mathbf{p}_{S_3}$, $\mathbf{p}_{S_4}$, and the plane through $\mathbf{p}_{S_2}$, $\mathbf{p}_{S_3}$, $\mathbf{p}_{S_4}$, but it is on the wrong side of the plane through $\mathbf{p}_{S_1}$, $\mathbf{p}_{S_2}$, $\mathbf{p}_{S_3}$. $\square$

The results of this section allow us to rediscover the geometry of Apollonian packings with just the Descartes quadratic form as a starting point. This raises the natural question: what generalizations of Apollonian packings can we obtain if we start with a different quadratic form?

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