CLOSED k-SCHUR KATALAN FUNCTIONS AS K-HOMOLOGY SCHUBERT REPRESENTATIVES OF THE AFFINE GRASSMANNIAN

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Dedicated to the memory of Bumsig Kim

ABSTRACT. Recently, Blasiak–Morse–Seelinger introduced symmetric functions called Katalan functions, and proved that the K-theoretic k-Schur functions due to Lam–Schilling–Shimozono form a subfamily of the Katalan functions. They conjectured that another subfamily of Katalan functions called closed k-Schur Katalan functions is identified with the Schubert structure sheaves in the K-homology of the affine Grassmannian. Our main result is a proof of this conjecture.

We also study a K-theoretic Peterson isomorphism that Ikeda, Iwao, and Maeno constructed, in a nongeometric manner, based on the unipotent solution of the relativistic Toda lattice of Ruijsenaars. We prove that the map sends a Schubert class of the quantum K-theory ring of the flag variety to a closed K-k-Schur Katalan function up to an explicit factor related to a translation element with respect to an antidominant coroot. In fact, we prove this map coincides with a map whose existence was conjectured by Lam, Li, Mihalcea, Shimozono, and proved by Kato, and more recently by Chow and Leung.

1. INTRODUCTION

The study of K-theoretic Schubert calculus attracts much attention in the last few decades. In this paper we focus on the K-theory version of the "quantum equals affine" phenomenon, which originally comes from an unpublished result by Peterson in 1997 for the case of (co)homology and has been developed by many authors. See the textbook [19] by Lam, Lapointe, Morse, Schilling, Shimozono, and Zabrocki on these topics.

Lam, Schilling, and Shimozono [21] identifies the K-homology $K_*(\mathrm{Gr})$ of the affine Grassmannian $\mathrm{Gr} = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ of $G = SL_{k+1}$, with a subring $\Lambda_{(k)} = \mathbb{C}[h_1, \ldots, h_k]$ of the ring of symmetric functions, where h_i is the ith complete symmetric function. In particular, the *K-theoretic k-Schur functions* $g_\lambda^{(k)}$, the *K-k-Schur functions* for short, were introduced in [21], which form a family of inhomogeneous symmetric functions in $\Lambda_{(k)}$. It was proved that the K-theoretic k-Schur functions, indexed by the partitions λ with $\lambda_1 \leq k$, are identified with a distinguished basis of $K_*(\mathrm{Gr})$. These functions form a basis of $\Lambda_{(k)}$, and are indexed by the partitions λ with $\lambda_1 \leq k$.

Although the K-k-Schur functions can be characterized by a Pieri type formula (see Definition 2.6), there was no explicit combinatorial formula until recently. Blasiak, Morse, and Seelinger [5] proved a raising operator formula for the K-k-Schur functions. In fact, they introduced a family of inhomogeneous symmetric functions called *K-theoretic Catalan functions*, *Katalan functions* for short, and proved that the K-theoretic k-Schur functions form a subfamily of the Katalan functions.

The Katalan functions $K(\Psi; M; \gamma)$ are indexed by triples (Ψ, M, γ), where Ψ is an upper order ideal in the set $\Delta_\ell^+ := \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq \ell\}$ of positive roots of type $A_{\ell-1}$, M is a multiset supported on $\{1, \ldots, \ell\}$, and $\gamma \in \mathbb{Z}^\ell$. For any root ideal $\mathcal{L} \subset \Delta_\ell^+$, set $L(\mathcal{L}) := \bigsqcup_{(i,j) \in \mathcal{L}} \{j\}$, where (i, j) is a shorthand notation for $\varepsilon_i - \varepsilon_j$. Let $\mathcal{P}^k$ denote the set of all partitions λ such that $\lambda_1 \leq k$. For $\lambda \in \mathcal{P}^k$, let $g_\lambda^{(k)}$ denote the corresponding K-k-Schur function (see §2.5 for the definition). In [5], it was proved that

(1.1) $$g_\lambda^{(k)} = K(\Delta^k(\lambda); L(\Delta^{k+1}(\lambda)); \lambda),$$

where

$$\Delta^k(\lambda) := \{(i, j) \in \Delta_\ell^+ \mid \lambda_i + j - i > k\},$$

and $\ell \geq \ell(\lambda)$, the length of λ. As one of the consequences of (1.1), a long-standing conjecture by Morse [31] was verified in [5]: for $\lambda \in \mathcal{P}^k$,

(1.2) $$\lambda_1 + \ell(\lambda) \leq k + 1 \Longrightarrow g_\lambda^{(k)} = g_\lambda,$$

where g_λ is the *dual stable Grothendieck polynomial*.

In [5], they introduced another subfamily of Katalan functions, called *closed k-Schur Katalan functions*, defined for $\lambda \in \mathcal{P}^k$ by

$$\tilde{\mathfrak{g}}_\lambda^{(k)} := K(\Delta^k(\lambda); L(\Delta^k(\lambda)); \lambda).$$

It is conjectured that the closed k-Schur Katalan function is related to the function

(1.3) $$\tilde{g}_\lambda^{(k)} := \sum_{\mu \underset{k}{\leq} \lambda} g_\mu^{(k)},$$

where $\lambda, \mu \in \mathcal{P}^k$, and $\underset{k}{\leq}$ denotes the order on $\mathcal{P}^k$ induced by the Bruhat order on the affine symmetric group $\tilde{S}_{k+1}$ (see §2.1 for details). We call $\tilde{g}_\lambda^{(k)}$ a *closed K-k-Schur function*. These functions are essential in the K-homology Schubert calculus because it is identified with the class of the Schubert structure sheaf $\mathcal{O}_\lambda^{\mathrm{Gr}}$ for the affine Grassmannian, whereas $g_\lambda^{(k)}$ is identified with the class of ideal sheaf of the boundary of the Schubert variety; see [20, Theorem 1] and [21, Theorems 5.4 and 7.17(1)]. It should be noted that Takigiku [39] proved a Pieri type formula for $\tilde{g}_\lambda^{(k)}$. Another important result in [39] is called the *k-rectangle factorization formula*. For $1 \leq i \leq k$, define

$$R_i = (\overbrace{i, \ldots, i}^{k+1-i}).$$

Takigiku showed

(1.4) $$\tilde{g}_{R_i} \cdot \tilde{g}_\lambda^{(k)} = \tilde{g}_{\lambda \cup R_i}^{(k)},$$

where $\lambda \cup R_i$ is the partition made by combining the parts of λ and those of R_i and then sorting them. This formula is natural from a geometric point of view (see [20]), and plays an important role in our construction.

Let σ be a ring automorphism of Λ given by $\sigma(h_i) = \sum_{j=0}^{i} h_j$ $(i \geq 1)$ with $h_0 = 1$. We can now state the main result of this paper confirming a conjecture [5, Conjecture 2.12(a)] by Blasiak, Morse, and Seelinger, which enable us to have a better explicit knowledge of the structure sheaf $\mathcal{O}_\lambda^{\mathrm{Gr}}$.

Theorem 1.1. *For $\lambda \in \mathcal{P}^k$, we have*
$$\tilde{g}_\lambda^{(k)} = \sigma(\tilde{\mathfrak{g}}_\lambda^{(k)}).$$

Knowing Takigiku's result (1.4), an immediate consequence is the following k-rectangle factorization formula for $\tilde{\mathfrak{g}}_\lambda^{(k)}$, which was first proved by Seelinger [37] by a more direct method involving generalized Katalan functions defined for arbitrary subsets of Δ_ℓ^+.

Corollary 1.2 (cf. [5, Conjecture 2.12(f)]). *For $1 \leq i \leq k$, we have*
$$\tilde{\mathfrak{g}}_{R_i}^{(k)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)} = \tilde{\mathfrak{g}}_{\lambda \cup R_i}^{(k)}.$$

Another motivation of our study is to clarify the connection between a map called the *K-theoretic Peterson isomorphism* constructed by Ikeda, Iwao, and Meano [13], and a map whose existence was conjectured by Lam, Li, Mihalcea, and Shimozono [20]; this conjecture was proved by Kato [15], and more recently by Chow and Leung [7, a] using different methods.

Let $QK(G/B)$ be the (small) quantum K-theory ring of the flag variety $G/B = \mathrm{SL}_{k+1}(\mathbb{C})/B$, a deformation of the Grothendieck ring of coherent sheaves on G/B studied by Givental and Lee [10] (see also the finiteness result [2] by Anderson, Chen, Tseng, and Iritani). This is a commutative associative algebra over the formal power series ring $\mathbb{C}[[Q]] := \mathbb{C}[[Q_1, \ldots, Q_k]]$ in the variables Q_i, called the Novikov variables. For $w \in S_{k+1}$, the symmetric group of degree $k+1$, the Schubert variety Ω^w in G/B is defined to be $\overline{B_- w B / B}$, where B_- is the opposite Borel subgroup. Let $\mathcal{O}_{G/B}^w$ denote (the class of) the structure sheaf of Ω^w. As a $\mathbb{C}[[Q]]$-module, $QK(G/B)$ has a basis consisting of $\mathcal{O}_{G/B}^w$ $(w \in S_{k+1})$. Let $K_*(\mathrm{Gr})$ be the K-homology group of the affine Grassmannian Gr, which has a ring structure by the Pontryagin product. The ring has a basis consisting of (the class of) the Schubert structure sheaves $\mathcal{O}_\lambda^{\mathrm{Gr}}$ indexed by the k-bounded partitions (see [21] for a more detailed description).

We represent $QK(G/B)$ as a quotient ring A_{k+1} of the polynomial ring $\mathbb{C}[[Q]][z_1, \ldots, z_{k+1}]$ (see §4 for details). According to Lam, Schilling, and Shimozono [21], we can identify $K_*(\mathrm{Gr})$ with $\Lambda_{(k)}$. More precisely, $\mathcal{O}_\lambda^{\mathrm{Gr}}$ is identified with $\sigma^{-1}(\tilde{g}_\lambda^{(k)})$ in our convention (see §4.3 for details). For any partition λ, let $g_\lambda \in \Lambda$ be the dual stable Grothendieck polynomial (see §2.4). We also set $\tilde{g}_\lambda := \sum_{\mu \subset \lambda} g_\mu$, following Takigiku [40]. We let A_{k+1}^{pol} be a $\mathbb{C}[Q]$-algebra which is a polynomial version of A_{k+1} (see §4). There is a ring isomorphism [13] given by:
$$\Phi_{k+1} : A_{k+1}^{\mathrm{pol}}[Q_i^{-1}(1 \leq i \leq k)] \to \Lambda_{(k)}[\tau_i^{-1}, (\tau_i^+)^{-1}(1 \leq i \leq k)],$$
$$\Phi_{k+1}(z_i) = \frac{\tau_i \tau_{i-1}^+}{\tau_i^+ \tau_{i-1}}, \quad \Phi_{k+1}(Q_i) = \frac{\tau_{i-1} \tau_{i+1}}{\tau_i^2},$$

where

$$\tau_i = g_{R_i}, \quad \tau_i^+ = \tilde{g}_{R_i}$$

for $1 \leq i \leq k$ and $\tau_0 = \tau_{k+1} = \tau_0^+ = \tau_{k+1}^+ = 1$. It should be emphasized that the nature of the construction of Φ_{k+1} is combinatorial rather than geometric, in the sense that it heavily depends on the explicit presentations of the rings involved. In fact, they used a nonlinear differential equation called the *relativistic Toda lattice* introduced by Ruijsenaars. The map Φ_{k+1} arises as a solution of the relativistic Toda lattice equation with its Lax matrix unipotent. For the affine side, the functions h_i $(1 \leq i \leq k)$ can be thought of as the coordinates of a certain abelian centralizer subgroup in $PSL_{k+1}(\mathbb{C})$.

There is a map $\theta_k : S_{k+1} \to \mathcal{P}^k$ explicitly described by Lam and Shimozono [24]. For $w \in S_{k+1}$, define $\mathrm{Des}(w) = \{1 \leq i \leq k \mid w(i) > w(i+1)\}$. The next result is a generalization of Theorem 1 in [24] by Lam and Shimozono for the (co)homology case to the K-theoretic setting.

Theorem 1.3. *For $w \in S_{k+1}$, we have*

$$\Phi_{k+1}(\mathcal{O}_{G/B}^w) = \frac{\sigma^{-1}(\tilde{g}_{\theta_k(w)})}{\prod_{i \in \mathrm{Des}(w)} \tau_i}.$$

As previously noted, $\sigma^{-1}(\tilde{g}_{\theta_k(w)}) = \tilde{\mathfrak{g}}_{\theta_k(w)}^{(k)}$ is identified with the structure sheaf $\mathcal{O}_{\theta_k(w)}^{\mathrm{Gr}}$, while τ_i is identified with the Schubert structure sheaf associated to an antidominant translation element in $\tilde{S}_{k+1}$ (see §4.3 for a more precise statement). Note also that for $w \in S_{k+1}$, we can take the *quantum Grothendieck polynomials* $\mathfrak{G}_w^Q \in \mathbb{C}[Q][z_1, \ldots, z_k]$ of Lenart and Maeno [27] (with the change of variables $x_i = 1 - z_i$) as a representative for $\mathcal{O}_{G/B}^w$ (see §4). The statement of Theorem 1.3 is a refinement of [13, Conjecture 1.8]. See also [5, Conjecture 2.12 (a) and (b)].

Future works and related results. Now we are able to say that the unipotent solution of the relativistic Toda lattice actually gives the canonical K-theoretic Peterson isomorphism, in type A. A natural question is how to generalize this fact to any semisimple algebraic group G. Furthermore, we would like to describe the K-theoretic Peterson isomorphisms in various types at the level of concrete polynomial representatives of Schubert classes. This leads us to a lot of interesting combinatorial problems related to the geometry of K-theoretic Gromov–Witten theory. We know from Kim [16] that the quantum cohomology ring of G/B is identified with the quotient ring of a polynomial ring by the ideal generated by the conserved quantities of the Toda lattice for the Langlands dual group $G^\vee$ with the nilpotent initial condition. For our purpose, one of the central tasks is to obtain an analogue of Kim's result in the context of quantum K-theory ring for G/B. Another possible clue in this direction is the work of Bezrukavnikov, Finkelberg, and Mirković [3] that studied a connection between the K-homology of the affine Grassmannian and certain generalized Toda lattice equations. However, we still do not understand how this work fits into the framework of the K-Peterson isomorphism.

For the affine Grassmannian side, Lam, Schilling, and Shimozono [22] defined the k-Schur functions for the symplectic groups. It is remarkable that Seelinger [37] made a conjecture that the symplectic k-Schur functions of [22] can be expressed by a raising operator formula. Pon [36] studied the case of the orthogonal group.

In particular, he gave a definition of "k-Schur functions" as the Schubert representatives for the homology of the affine Grassmannian in type B, and established the Pieri rule for types B and D. Further combinatorial researches on these functions are needed to explore the issue of giving explicitly the K-theoretic Peterson isomorphism.

Even in type A, there are a lot of things to do. We have a Chevalley type formula for $QK(G/B)$ ([28], [29], [27], [32]), which describes the multiplication by $\mathcal{O}^{s_i}_{G/B}$ ($1 \leq i \leq k$), where s_i is the transposition $(i, i+1)$. The image of $\mathcal{O}^{s_i}_{G/B}$ ($1 \leq i \leq k$) under Φ_{k+1} is $\sigma^{-1}(\tilde{g}^{(k)}_{R^*_i})/\tau_i$, where R^*_i is the partition obtained from R_i by removing the unique corner box. We expect a good combinatorial formula for $\tilde{g}^{(k)}_{R^*_i} \cdot \tilde{g}^{(k)}_{\lambda}$. Although we know a dictionary between quantum and affine Schubert classes, the problem of translating the Chevalley formula into its affine counterpart seems not to be so simple. For the homology case, the analogous issue was pursued by Dalal and Morse [9, Conjecture 39]. Another basic question is what formula on the quantum side is corresponding to $\tilde{g}^{(k)}_{(i)} \cdot \tilde{g}^{(k)}_{\lambda}$ and $\tilde{g}^{(k)}_{(1^i)} \cdot \tilde{g}^{(k)}_{\lambda}$ ($1 < i < k-1$). Furthermore, there is a conjecture[1] by Lenart and Maeno of a Pieri type formula for $\mathfrak{S}^Q_w$ [27, Conjecture 6.7]. We hope that the "quantum equals affine" phenomenon in K-theory would shed light on these questions. Also, it is natural to extend Theorem 1.3 torus equivariantly. In fact, Lam and Shimozono [23] proved the equivariant (co)homology version of the explicit Peterson isomorphism in [24]. It was shown that the double k-Schur functions of Lam–Shimozono [25], and the quantum double Schubert polynomials by Kirillov and Maeno [18] and by Ciocan–Fontanine and Fulton [8], can be obtained from each other by the map. There are arguments in [23, §4] on the centralizer family for SL_{k+1} in connection with Peterson's j-map, which would be useful in future studies.

Organization. In §2, we present some basic definitions and preliminary facts. In §3, we prove Theorem 1.1. In §4, we discuss the K-theoretic Peterson isomorphism. In Appendix A, we provide some results related to the parabolic quotient of a Coxeter group, which are used in the proof of Theorem 1.1. In Appendix B, we give a proof of a result (Lemma B.3) on Grassmannian permutations. In Appendix C, we record the vertical Pieri rule for the closed K-k-Schur functions.

2. Basic definitions

Let k be a positive integer. In this section, we fix the notation and explain the definitions and some properties of basic notions needed to understand Theorem 1.1.

2.1. Affine symmetric groups. The *affine symmetric group* $\tilde{S}_{k+1}$ is the group with generators $\{s_i \mid i \in I\}$ for $I = \{0, 1, \ldots, k\}$ subject to the relations:

$$s_i^2 = id, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \quad \text{for} \quad i - j \neq 0, \pm 1,$$

with indices considered modulo $k+1$. The *length* $\ell(w)$ of $w \in \tilde{S}_{k+1}$ is the minimum number m such that $w = s_{i_1} \cdots s_{i_m}$ for some $i_j \in I$; any such expression for w with $\ell(w)$ generators is said to be *reduced*. The set of *affine Grassmannian elements* $\tilde{S}^0_{k+1}$ is the minimal length coset representatives for $\tilde{S}_{k+1}/S_{k+1}$, where $S_{k+1} = \langle s_1, \ldots, s_k \rangle$.

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[1]After the first version of this paper was submitted, the conjecture has been proved in [33].

The *Bruhat order* (or *strong order*) on $\tilde{S}_{k+1}$ is denoted by $\leq$. It can be described by the subword property (see [4, §2]).

2.2. k-bounded partitions and affine Grassmannian elements. Let $\mathcal{P}^k_\ell := \{(\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Z}^\ell \mid k \geq \lambda_1 \geq \cdots \geq \lambda_\ell \geq 0\}$ denote the set of partitions contained in the $\ell \times k$ rectangle and let $\mathcal{P}^k$ be the set of partitions λ with $\lambda_1 \leq k$. For a partition λ, the *length* $\ell(\lambda)$ is the number of nonzero parts of λ. There is a bijection $\mathcal{P}^k \to \tilde{S}^0_{k+1}$ ($\lambda \mapsto x_\lambda$) due to Lapointe and Morse [26, Definition 45, Corollary 48] given by

$$(2.1) \qquad x_\lambda := (s_{\lambda_\ell - \ell} \cdots s_{-\ell+1}) \cdots (s_{\lambda_2 - 2} \cdots s_{-1})(s_{\lambda_1 - 1} \cdots s_0),$$

where $\ell = \ell(\lambda)$. To make the notation simple, we are omitting the dependency of x_λ on k.

Example 2.1. Let $k = 4, \lambda = (4, 3, 2)$. The corresponding affine Grassmannian element in $\tilde{S}^0_5$ is $x_\lambda = s_4 s_3 \cdot s_1 s_0 s_4 \cdot s_3 s_2 s_1 s_0$. This is obtained by reading the $(k+1)$-residues in each row of λ, from right to left, proceeding with bottom row to top.

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\\hline 4 & 0 & 1 \\\cline{1-3} 3 & 4 \\\cline{1-2}\end{array}.$$

For $\lambda, \mu \in \mathcal{P}^k$, we denote $\lambda \underset{k}{\leq} \mu$ if $x_\lambda \leq x_\mu$ holds, where $\leq$ is the Bruhat order on $\tilde{S}_{k+1}$. The following fundamental fact is included in the proof of [5, Proposition 2.16].

Lemma 2.2. *Suppose $\lambda \in \mathcal{P}^k$ satisfies $\lambda_1 + \ell(\lambda) \leq k+1$. For $\mu \in \mathcal{P}^k$, $\mu \underset{k}{\leq} \lambda \iff \mu \subset \lambda$.*

Remark 2.3. For $\lambda, \mu \in \mathcal{P}^k$, $\lambda \subset \mu$ implies $\lambda \underset{k}{\leq} \mu$. The reverse implication is not true in general. For example, in $\tilde{S}_3$ we have $(2, 2) \underset{2}{\leq} (2, 1, 1, 1)$ because the reduced expression $x_{(2,1,1,1)} = s_0 s_1 s_2 s_1 s_0$ has a subexpression $s_0 s_2 s_1 s_0 = x_{(2,2)}$.

2.3. Cyclically increasing elements in $\tilde{S}^0_{k+1}$. Let A be a proper subset of $I = \{0, 1, \ldots, k\}$. Set $|A| = r$. There is a sequence $(i_1, \ldots, i_r)$ consisting of the elements of A such that an index $i+1$ never occurs anywhere to the left of an index i (mod $k+1$). For any such sequence, the element $s_{i_1} \cdots s_{i_r} \in \tilde{S}_{k+1}$ depends only on the set of indices $A = \{i_1, \ldots, i_r\}$ (see also Remark B.2). Let us denote the element by u_A. Such an element in $\tilde{S}_{k+1}$ is called a *cyclically increasing element*. Similarly, for any $A \subsetneq I$, we can define the corresponding *cyclically decreasing element* denoted by d_A. We choose a sequence $(i_1, \ldots, i_r)$ such that an index i never occurs anywhere to the right of an index $i+1$ (mod $k+1$), and set $d_A := s_{i_1} \cdots s_{i_r}$.

Example 2.4. Let $k = 4$. Then for $A = \{0, 1, 3, 4\}$, $u_A = s_3 s_4 s_0 s_1$, $d_A = s_1 s_0 s_4 s_3$, and for $A = \{0, 2, 4\}$, $u_A = s_4 s_0 s_2 = s_4 s_2 s_0 = s_2 s_4 s_0$, $d_A = s_2 s_0 s_4 = s_0 s_2 s_4 = s_0 s_4 s_2$.

2.4. Dual stable Grothendieck polynomials. We work in the ring

$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots] = \mathbb{Z}[h_1, h_2, \ldots]$$

of symmetric functions in infinitely many variables $(x_1, x_2, \ldots)$, where $e_r = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$ and $h_r = \sum_{i_1 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_r}$. Set $h_0 = e_0 = 1$ and $h_r = 0$ for $r < 0$ by convention.

Let σ be the ring automorphism of Λ defined by $\sigma(h_i) = \tilde{h}_i$ ($i \geq 1$), where we set $\tilde{h}_i = \sum_{j=0}^{i} h_j$. Note that σ^{-1} sends h_i to $h_i - h_{i-1}$ for $i \geq 1$. For $i, m \in \mathbb{Z}$, define

$$h_i^{(m)} := \sigma^m(h_i) = \sum_{j=0}^{i} \binom{m+j-1}{j} h_{i-j},$$

where $\binom{n}{i} = n(n-1)\cdots(n-i+1)/i!$ for $n \in \mathbb{Z}$, $i \geq 1$ and $\binom{n}{0} = 1$ for $n \in \mathbb{Z}$. For $\gamma \in \mathbb{Z}^\ell$, define

(2.2) $$g_\gamma := \det(h_{\gamma_i + j - i}^{(i-j)})_{1 \leq i, j \leq \ell}.$$

When λ is a partition, g_λ is the *dual stable Grothendieck polynomials*. Although the defining formula for g_γ in [5] is $\det(h_{\gamma_i + j - i}^{(i-1)})_{1 \leq i, j \leq \ell}$, by some column operations, we see that their definition agrees with (2.2). Note that $h_i^{(m)}$ is denoted by $k_i^{(m)}$ in [5].

The following result is fundamental and used throughout the paper.

Proposition 2.5 (Takigiku, [40]). *For any partition λ, set $\tilde{g}_\lambda := \sum_{\mu \subset \lambda} g_\mu$. Then we have*

(2.3) $$\sigma(g_\lambda) = \tilde{g}_\lambda.$$

2.5. K-k-Schur functions. For $i \in I$, set for $w \in \tilde{S}_{k+1}$,

(2.4) $$s_i * w = \begin{cases} s_i w & (s_i w > w) \\ w & (s_i w < w) \end{cases}.$$

If we write $\phi_i : \tilde{S}_{k+1} \to \tilde{S}_{k+1}$ ($w \mapsto s_i * w$), then $\phi_i^2 = \phi_i$, $\phi_i \phi_{i+1} \phi_i = \phi_{i+1} \phi_i \phi_{i+1}$ (see the proof of [38, Proposition 2.1]). So for $v \in \tilde{S}_{k+1}$, we can define

$$v * w = s_{i_1} * (s_{i_2} * \cdots (s_{i_r} * w) \cdots) \quad (w \in \tilde{S}_{k+1}),$$

where $v = s_{i_1} \cdots s_{i_r}$ is an arbitrary reduced expression.

For $x \in \tilde{S}_{k+1}^0$, we write $g_x^{(k)}$ for $g_\lambda^{(k)}$ with $x = x_\lambda$, $\lambda \in \mathcal{P}^k$.

Definition 2.6. The K-k-Schur functions $\{g_\lambda^{(k)}\}_{\lambda \in \mathcal{P}^k}$ are the family of elements of $\Lambda_{(k)}$ such that $g_\varnothing^{(k)} = 1$ and

(2.5) $$h_r \cdot g_\lambda^{(k)} = \sum_{\substack{A \subset I, |A| = r \\ d_A * x_\lambda \in \tilde{S}_{k+1}^0}} (-1)^{|A| - \ell(d_A * x_\lambda) + \ell(x_\lambda)} g_{d_A * x_\lambda}^{(k)}$$

for $\lambda \in \mathcal{P}^k$ and $1 \leq r \leq k$.

Example 2.7. Let $k = 2$, $\lambda = (1, 1, 1)$, and $r = 2$. Then $x_\lambda = s_1 s_2 s_0$. There are three A's: $\{1, 0\}, \{0, 2\}, \{1, 2\}$. We compute

$$d_{\{1,0\}} * x_\lambda = (s_1 s_0) * (s_1 s_2 s_0) = s_1 s_0 s_1 s_2 s_0 = s_0 s_1 s_0 s_2 s_0 = s_0 s_1 s_2 s_0 s_2 \notin \tilde{S}_3^0,$$

$$d_{\{0,2\}} * x_\lambda = (s_0 s_2) * (s_1 s_2 s_0) = s_0 s_2 s_1 s_2 s_0 = x_{(2,1,1,1)} \in \tilde{S}_3^0,$$

$$d_{\{1,2\}} * x_\lambda = (s_2 s_1) * (s_1 s_2 s_0) = s_2 s_1 s_2 s_0 = x_{(2,1,1)} \in \tilde{S}_3^0,$$

and hence
$$h_2 \cdot g^{(2)}_{(1,1,1)} = g^{(2)}_{(2,1,1,1)} - g^{(2)}_{(2,1,1)}.$$

It is known that $\{g^{(k)}_\lambda\}_{\lambda \in \mathcal{P}^k}$ is a basis of $\Lambda_{(k)}$ [21].

Proposition 2.8 ([21])**.** *For* $1 \le r \le k$, $g^{(k)}_{(r)} = h_r$.

Recall that we set $\tilde{g}_\lambda := \sum_{\mu \subset \lambda} g_\mu$ for any partition λ.

Proposition 2.9. *For* $\lambda \in \mathcal{P}^k$ *such that* $\lambda_1 + \ell(\lambda) \le k + 1$,
$$\tilde{g}^{(k)}_\lambda = \tilde{g}_\lambda.$$

Proof. From Lemma 2.2, and (1.2), we see that
$$\tilde{g}^{(k)}_\lambda = \sum_{\mu \underset{k}{\le} \lambda} g^{(k)}_\mu = \sum_{\mu \subset \lambda} g^{(k)}_\mu = \sum_{\mu \subset \lambda} g_\mu = \tilde{g}_\lambda.$$

$\square$

2.6. Katalan functions.

Fix a positive integer ℓ. Let $\{\varepsilon_1, \ldots, \varepsilon_\ell\}$ be the standard basis of $\mathbb{Z}^\ell$. By a *positive root* β, we mean an element of the form $\varepsilon_i - \varepsilon_j \in \mathbb{Z}^\ell$ with $1 \le i < j \le \ell$, which is also denoted by (i, j). The set of all positive roots is denoted by Δ_ℓ^+. Although this is considered as the set of positive roots of type $A_{\ell-1}$, we use this notation Δ_ℓ^+ following [5] rather than $\Delta_{\ell-1}^+$.

A natural partial order $\le$ on Δ_ℓ^+ is defined by $\alpha \le \beta$ if $\beta - \alpha$ is a linear combination of positive roots with coefficients in $\mathbb{Z}_{\ge 0}$. An upper order ideal Ψ of Δ_ℓ^+ is called a *root ideal*.

Given a root ideal $\Psi \subset \Delta_\ell^+$, a multiset M supported on $\{1, \ldots, \ell\}$, and $\gamma \in \mathbb{Z}^\ell$, we call (Ψ, M, γ) a *Katalan triple*. Let $m_M : \{1, \ldots, \ell\} \to \mathbb{Z}_{\ge 0}$ denote the multiplicity function of M. Each Katalan triple (Ψ, M, γ) can be depicted by an $\ell \times \ell$ grid of square boxes (labeled by matrix-style coordinates) with the boxes of Ψ shaded, $m_M(a)$ •'s in column a (assuming $m_M(a) < a$), and the entries of γ written along the diagonal boxes.

Example 2.10. Let $\Psi = \{(1,2), (1,3), (1,4), (1,5), (2,4), (2,5), (3,5)\} \subset \Delta_5^+$, $M = \{3, 4, 5, 5\}$, and $\gamma = (3, 2, 0, 1, 0)$. The Katalan triple (Ψ, M, γ) is depicted by

We define the *Katalan function* associated to the triple (Ψ, M, γ) by
$$K(\Psi; M; \gamma) := \prod_{j \in M} (1 - L_j) \prod_{(i,j) \in \Delta_\ell^+ \setminus \Psi} (1 - R_{ij}) k_\gamma,$$
$$k_\gamma := h^{(0)}_{\gamma_1} h^{(1)}_{\gamma_2} \cdots h^{(\ell-1)}_{\gamma_\ell}.$$

Note that raising operators are not well-defined as linear transformations on Λ. They act on the subscript γ of k_γ rather than the function k_γ. A rigorous formulation can be found in [5, §3]. For any root ideal $\mathcal{L} \subset \Delta_\ell^+$, let
$$L(\mathcal{L}) = \bigsqcup_{(i,j) \in \mathcal{L}} \{j\}.$$

We also write $K(\Psi; L(\mathcal{L}); \gamma)$ simply as $K(\Psi; \mathcal{L}; \gamma)$.

In [5], Blasiak, Morse, and Seelinger introduced, for $\lambda \in \mathcal{P}_\ell^k$,

$$\mathfrak{g}_\lambda^{(k)} := K(\Delta^k(\lambda); \Delta^{k+1}(\lambda); \lambda),$$

$$\tilde{\mathfrak{g}}_\lambda^{(k)} := K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda).$$

$\mathfrak{g}_\lambda^{(k)}$ is called a *K-Schur Katalan function*, and $\tilde{\mathfrak{g}}_\lambda^{(k)}$ a *closed K-Schur Katalan function*. If we choose ℓ such that $\ell \geq \ell(\lambda)$, then $\mathfrak{g}_\lambda^{(k)}$ and $\tilde{\mathfrak{g}}_\lambda^{(k)}$ do not depend on ℓ ([5, Lemma 3.4 and Remark 3.5]).

The following simplified formula is available.

Proposition 2.11. *Let $\lambda \in \mathcal{P}_\ell^k$, then*

$$(2.6) \qquad \tilde{\mathfrak{g}}_\lambda^{(k)} = \prod_{(i,j) \in \Delta_\ell^+ \setminus \Delta^k(\lambda)} (1 - L_j)^{-1}(1 - R_{ij})h_\lambda,$$

where $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, and L_j, R_{ij} act on the subscript λ.

Proof. In the proof of [5, Proposition 2.3], it was shown that $\prod_{(i,j) \in \Delta_\ell^+}(1 - L_j)k_\gamma = h_\gamma$. (2.6) follows from this. □

Example 2.12. For $k = 3$, $\lambda = (2, 1, 1)$, $\tilde{\mathfrak{g}}_\lambda^{(3)}$ is depicted by . We have

$$\tilde{\mathfrak{g}}_{(2,1,1)}^{(3)} = (1 - L_2)^{-1}(1 - L_3)^{-1}(1 - R_{12})(1 - R_{23})h_{211}$$
$$= h_{211} + h_{201} - h_{220} - h_{301} + h_{310}$$
$$= h_1^2 h_2 + h_1 h_2 - h_2^2$$
$$= h_2(h_1^2 + h_1 - h_2).$$

One of the main results in [5] is

$$\mathfrak{g}_\lambda^{(k)} = g_\lambda^{(k)}.$$

The main result of the present paper (Theorem 1.1) is

$$(2.7) \qquad \tilde{\mathfrak{g}}_\lambda^{(k)} = \sigma^{-1}(\tilde{g}_\lambda^{(k)}).$$

A simple consequence of (2.7) is the following.

Corollary 2.13 ([5, Proposition 2.16(d)]). *If $\lambda_1 + \ell(\lambda) \leq k + 1$, then $\tilde{\mathfrak{g}}_\lambda^{(k)} = g_\lambda$.*

Proof. By Lemma 2.9 and Proposition 2.5,

$$\tilde{\mathfrak{g}}_\lambda^{(k)} = \sigma^{-1}(\tilde{g}_\lambda^{(k)}) = \sigma^{-1}(\tilde{g}_\lambda) = g_\lambda.$$

□

3. Proof of Theorem 1.1

Before we start the proof of Theorem 1.1, we gather some results on K-k-Schur functions and Katalan functions in the first three subsections. In §3.1, we explain some results on the k-conjugation. In §3.2, we collect some basic properties of Katalan functions used in §3.5. In §3.3, we introduce an action of the 0-Hecke algebras on $\Lambda_{(k)}$. The outline of the proof of Theorem 1.1 is given in §3.4. The last subsection is devoted to the proof of a key lemma (Lemma 3.17). With the help of

a general fact (Lemma 3.18) on the parabolic coset space of a Coxeter group, we complete the proof of Theorem 1.1.

3.1. k-conjugation. There is an automorphism ω_k of $\tilde{S}_{k+1}$ given by $\omega_k(s_i) = s_{-i} = s_{k+1-i}$ for $i \in I$. Note that ω_k fixes s_0. In fact, ω_k is an automorphism of a Coxeter group. So it is easy to see

$$(3.1) \qquad w \leq v \Longleftrightarrow \omega_k(w) \leq \omega_k(v).$$

The left weak order $\leq_L$ on $\tilde{S}_{k+1}$ is defined by the covering relation

$$w \lessdot_L v \Longleftrightarrow v = s_i w \text{ and } \ell(v) = \ell(w) + 1 \text{ for some } i \in I.$$

It is easy to see that $w \lessdot_L v \Longleftrightarrow \omega_k(w) \lessdot_L \omega_k(v)$, and hence we have

$$(3.2) \qquad w \leq_L v \Longleftrightarrow \omega_k(w) \leq_L \omega_k(v).$$

For $A = \{i_1, \dots, i_m\} \subsetneq I$, we have

$$\omega_k(u_A) = d_{\overline{A}}, \quad \omega_k(d_A) = u_{\overline{A}},$$

where $\overline{A} := \{-i_1, \dots, -i_m\}$.

Definition 3.1. Let Ω be the ring morphism of Λ defined by

$$(3.3) \qquad \Omega(h_i) = g_{(1^i)} \quad (i \geq 1).$$

Proposition 3.2. Ω *is an involution on Λ and Ω commutes with σ.*

Proof. A proof of the fact that Ω is an involution can be found in [31, §8]. The commutativity follows from (2.3):

$$\Omega(\sigma(h_i)) = \Omega(\sum_{j=0}^{i} h_j) = \sum_{j=0}^{i} g_{(1^j)} = \sigma(g_{(1^i)}) = \sigma(\Omega(h_i)).$$

$\square$

It is easy to see that ω_k preserves $\tilde{S}_{k+1}^0$. Hence for $\lambda \in \mathcal{P}^k$, $\omega_k(x_\lambda) = x_\mu$ for some $\mu \in \mathcal{P}^k$. Then we define $\omega_k(\lambda) = \mu$. An explicit description of $\omega_k(\lambda)$, also denoted by λ^{ω_k} in [19], [31], is available (see [19, §1.3]).

Theorem 3.3 ([31]). *For $x \in \tilde{S}_{k+1}^0$, $\Omega(g_x^{(k)}) = g_{\omega_k(x)}^{(k)}$. Equivalently, for $\lambda \in \mathcal{P}^k$, $\Omega(g_\lambda^{(k)}) = g_{\omega_k(\lambda)}^{(k)}$.*

Corollary 3.4. *For $\lambda \in \mathcal{P}^k$, $\Omega(\tilde{g}_\lambda^{(k)}) = \tilde{g}_{\omega_k(\lambda)}^{(k)}$.*

Proof. For $x \in \tilde{S}_{k+1}^0$, $\tilde{g}_x^{(k)} = \sum_{y \leq x} g_y^{(k)}$, where y are the elements in $\tilde{S}_{k+1}^0$ such that $y \leq x$ in the Bruhat order. From Theorem 3.3, and the fact (3.1), we see that

$$\Omega(\tilde{g}_x^{(k)}) = \Omega(\sum_{y \leq x} g_y^{(k)}) = \sum_{y \leq x} \Omega(g_y^{(k)}) = \sum_{y \leq x} g_{\omega_k(y)}^{(k)} = \sum_{\omega_k(y) \leq \omega_k(x)} g_{\omega_k(y)}^{(k)} = \tilde{g}_{\omega_k(x)}^{(k)},$$

where the last equality holds since ω_k is an involution. Set $x = x_\lambda$. Then $\Omega(\tilde{g}_\lambda^{(k)}) = \tilde{g}_{\omega_k(\lambda)}^{(k)}$.

$\square$

3.2. **Basic properties of Katalan functions.** Let $\Psi \subset \Delta_\ell^+$ be a root ideal. A root $\alpha \in \Psi$ is a *removable root* of Ψ if $\Psi \setminus \alpha$ is a root ideal. A root $\beta \in \Delta_\ell^+$ is an *addable root* of Ψ if $\Psi \cup \alpha$ is a root ideal.

We define an oriented graph with $\{1, \ldots, \ell\}$ as the vertex set and the oriented edges $j \to i$ if (i, j) is a removable root in Ψ (in [5] the bounce graph is not considered as an oriented graph, but here it is). An edge of the bounce graph of Ψ is called simply a *bounce edge* of Ψ. Let $p \in \{1, \ldots, \ell\}$. If there is a bounce edge $p \to i$ of Ψ, then such i is unique by the construction, and we write $i = \mathrm{up}_\Psi(p)$.
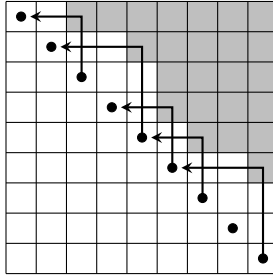

FIGURE 1. Bounce graph of Ψ

Each connected component of the bounce graph of Ψ is called a *bounce path* of Ψ. For $p \in \{1, \ldots, \ell\}$, $\mathrm{top}_\Psi(p)$ denotes the smallest element in the bounce path containing p (see [5]).

Example 3.5. For the root ideal Ψ in Figure 1, $\{1, 3\}$, $\{2, 5, 7\}$, $\{4, 6, 9\}, \{8\}$ are the bounce paths. We have, for example, $\mathrm{up}_\Psi(3) = 1$, $\mathrm{up}_\Psi(7) = 5$, $\mathrm{top}_\Psi(7) = \mathrm{top}_\Psi(5) = \mathrm{top}_\Psi(2) = 2$.

Definition 3.6 (Walls and ceilings). Let d be a positive integer. A root ideal Ψ is said to have a *wall* in rows $r, r+1, \ldots, r+d$ if the rows $r, r+1, \ldots, r+d$ of Ψ have the same length, and a *ceiling* in columns $c, c+1, \ldots, c+d$ if the columns $c, c+1, \ldots, c+d$ of Ψ have the same length.

Example 3.7. The root ideal Ψ in Figure 1 has a ceiling in columns $1, 2$, in columns $3, 4$, in columns $7, 8$, and has a wall in rows $3, 4$, in rows $7, 8, 9$.

Let us begin with an obvious remark.

Lemma 3.8. *Let Ψ be a root ideal of Δ_ℓ^+, and $p \in \{2, \ldots, \ell\}$. If $\mathrm{top}_\Psi(p) = p$, i.e., if there is no bounce edge starting from p, then Ψ has a ceiling in columns $p - 1, p$.*

Lemmas 3.9 and 3.11 are borrowed from [5, Lemma 4.4].

Lemma 3.9 (Adding or removing a root). *Let (Ψ, M, γ) be a Katalan triple.*

 (i) *For any addable root α of Ψ,*

$$K(\Psi; M; \gamma) = K(\Psi \cup \alpha; M; \gamma) - K(\Psi \cup \alpha; M; \gamma + \alpha).$$

 (ii) *For any removable root α of Ψ,*

$$K(\Psi; M; \gamma) = K(\Psi \setminus \alpha; M; \gamma) + K(\Psi; M; \gamma + \alpha).$$

Example 3.10. We apply Lemma 3.9(i) to the following Katalan triple, with $\alpha = \varepsilon_2 - \varepsilon_3$:

$$\begin{array}{c} \boxed{\text{grid}} \end{array} = \begin{array}{c} \boxed{\text{grid}} \end{array} - \begin{array}{c} \boxed{\text{grid}} \end{array}.$$

We apply Lemma 3.9(ii) to the following Katalan triple, with $\alpha = \varepsilon_3 - \varepsilon_4$:

$$\begin{array}{c} \boxed{\text{grid}} \end{array} = \begin{array}{c} \boxed{\text{grid}} \end{array} + \begin{array}{c} \boxed{\text{grid}} \end{array}.$$

Lemma 3.11 (Adding or removing a dot). *Let (Ψ, M, γ) be a Katalan triple.*

   (i) *For any $j \in M$, $K(\Psi; M; \gamma) = K(\Psi; M \setminus \{j\}; \gamma) - K(\Psi; M \setminus \{j\}; \gamma - \varepsilon_j)$,*
   (ii) *For any $1 \le j \le \ell$, $K(\Psi; M; \gamma) = K(\Psi; M \sqcup \{j\}; \gamma) + K(\Psi; M; \gamma - \varepsilon_j)$.*

Lemma 3.12 is [5, Lemma 3.3].

Lemma 3.12 (Alternating property). *Let $\Psi \subset \Delta_\ell^+$ be a root ideal, and M a multiset on $\{1, \ldots, \ell\}$. Suppose there is an index $1 \le i \le \ell - 1$ such that*

   (a) *Ψ has a ceiling in columns $i, i+1$,*
   (b) *Ψ has a wall in rows $i, i+1$,*
   (c) *$m_M(i+1) = m_M(i) + 1$.*

Then for any $\gamma \in \mathbb{Z}^\ell$,

$$(3.4) \qquad K(\Psi; M; \gamma) = -K(\Psi; M; s_i\gamma - \varepsilon_i + \varepsilon_{i+1}).$$

In particular, if $\gamma_{i+1} = \gamma_i + 1$ holds, then

$$(3.5) \qquad K(\Psi; M; \gamma) = 0.$$

3.3. The 0-Hecke algebra.

The 0-*Hecke algebra* H_{k+1} is the associative $\mathbb{C}$-algebra generated by $\{T_i \mid i \in I\}$ subject to the relations:

$$T_i^2 = -T_i, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad \text{for} \quad i - j \ne 0, \pm 1,$$

with indices considered modulo $k+1$. For $w \in \tilde{S}_{k+1}$, define $T_w = T_{i_1} \cdots T_{i_m}$ for any reduced expression $w = s_{i_1} \cdots s_{i_m}$. The elements T_w ($w \in \tilde{S}_{k+1}$) form a basis of H_{k+1}.

We introduce a family of symmetric functions $\overset{\circ}{\mathfrak{g}}{}_\lambda^{(k)}$ ($\lambda \in \mathcal{P}^k$) such that $\tilde{\mathfrak{g}}_\lambda^{(k)} = \sum_{\mu \underset{k}{\le} \lambda} \overset{\circ}{\mathfrak{g}}{}_\mu^{(k)}$ for all $\lambda \in \mathcal{P}^k$. Such functions uniquely exist since the transition matrix from $\{\overset{\circ}{\mathfrak{g}}{}_\lambda^{(k)}\}$ to $\{\mathfrak{g}_\lambda^{(k)}\}$ is upper unitriangular.

Proposition 3.13. *There is a left H_{k+1}-module structure on $\Lambda_{(k)}$ such that*

$$T_i \cdot \overset{\circ}{\mathfrak{g}}{}_\lambda^{(k)} = \begin{cases} \overset{\circ}{\mathfrak{g}}{}_{s_i x_\lambda}^{(k)} & (s_i x_\lambda > x_\lambda \text{ and } s_i x_\lambda \in \tilde{S}_{k+1}^0) \\ -\overset{\circ}{\mathfrak{g}}{}_\lambda^{(k)} & (s_i x_\lambda < x_\lambda) \\ 0 & (otherwise) \end{cases},$$

for $\lambda \in \mathcal{P}^k$ and $i \in I$. Moreover, we have

$$D_i \cdot \tilde{\mathfrak{g}}_\lambda^{(k)} = \begin{cases} \tilde{\mathfrak{g}}_{s_i x_\lambda}^{(k)} & (s_i x_\lambda > x_\lambda \text{ and } s_i x_\lambda \in \tilde{S}_{k+1}^0) \\ \tilde{\mathfrak{g}}_\lambda^{(k)} & (otherwise) \end{cases},$$

for $i \in I$ and $\lambda \in \mathcal{P}^k$, and $D_i := T_i + 1$.

A proof of this proposition is given in Appendix A.1.

Remark 3.14. Note that we will eventually show $\sigma(\overset{\circ}{\mathfrak{g}}{}_{\lambda}^{(k)}) = g_{\lambda}^{(k)}$. There is an action of H_{k+1} on $\Lambda_{(k)}$ used in [5]. If we denote the action of T_i in [5, §5.4] by T_i', then we have $T_i' = \sigma \circ T_i \circ \sigma^{-1}$.

3.4. **Outline of proof of Theorem 1.1.** The following characterization property for the closed K-k-Schur functions is available. Note that for $1 \le r \le k$, we have $\tilde{g}_{(1^r)}^{(k)} = \tilde{g}_{(1^r)}$ and $\tilde{g}_{(r)}^{(k)} = \tilde{g}_{(r)} = \tilde{h}_r$ (Proposition 2.9).

Lemma 3.15. *For $\lambda \in \mathcal{P}^k$ and $1 \le r \le k$, we have*

$$(3.6) \qquad \tilde{g}_{(1^r)} \cdot \tilde{g}_{\lambda}^{(k)} = \sum_{\substack{\mu \le \lambda \\ k}} \sum_{\substack{A \subset I, |A| \le r \\ u_A * x_\mu \in \tilde{S}_{k+1}^0}} (-1)^{|A| - \ell(u_A * \mu) + \ell(\mu)} g_{u_A * x_\mu}^{(k)}.$$

Moreover, the $\{\tilde{g}_{\lambda}^{(k)}\}_{\lambda \in \mathcal{P}^k}$ are the unique elements of $\Lambda_{(k)}$ satisfying (3.6) for $1 \le r \le k$.

Proof. By summing up (2.5) over $\mu \in \mathcal{P}^k$ such that $\mu \underset{k}{\le} \lambda$ and integers i with $0 \le i \le r$, we obtain

$$(3.7) \qquad \tilde{g}_{(r)} \cdot \tilde{g}_{\lambda}^{(k)} = \sum_{\substack{\mu \le \lambda \\ k}} \sum_{\substack{A \subset I, |A| \le r \\ d_A * x_\mu \in \tilde{S}_{k+1}^0}} (-1)^{|A| - \ell(d_A * \mu) + \ell(\mu)} g_{d_A * x_\mu}^{(k)}.$$

In fact, this identity appears in [39, p. 470]. We apply the k-conjugation to both sides of (3.7), and use Corollary 3.4, to have

$$\tilde{g}_{(1^r)} \cdot \tilde{g}_{\omega_k(\lambda)}^{(k)} = \sum_{\substack{\mu \le \lambda \\ k}} \sum_{\substack{A \subset I, |A| \le r \\ d_A * x_\mu \in \tilde{S}_{k+1}^0}} (-1)^{|A| - \ell(d_A * \mu) + \ell(\mu)} g_{u_{\overline{A}} * x_{\omega_k(\mu)}}^{(k)}.$$

This is equivalent to

$$(3.8) \quad \tilde{g}_{(1^r)} \cdot \tilde{g}_{\lambda}^{(k)} = \sum_{\substack{\omega_k(\mu) \le \omega_k(\lambda) \\ k}} \sum_{\substack{A \subset I, |A| \le r \\ d_A * x_{\omega_k(\mu)} \in \tilde{S}_{k+1}^0}} (-1)^{|A| - \ell(d_A * \omega_k(\mu)) + \ell(\omega_k(\mu))} g_{u_{\overline{A}} * x_\mu}^{(k)}.$$

Noting that $\omega_k(d_A * x_{\omega_k(\mu)}) = u_{\overline{A}} * x_\mu$, and $|\overline{A}| = |A|$, the right-hand side of (3.8) is identical to that of (3.6) because the involution ω_k preserves the Bruhat order and the length. The uniqueness follows from the unitriangularity of the transition matrix and the fact that $\tilde{g}_{(1^r)}^{(k)} = \tilde{g}_{(1^r)}$ for $1 \le r \le k$. $\qquad \square$

Theorem 1.1 will follow easily once we have Proposition 3.16.

Proposition 3.16. *For $\lambda \in \mathcal{P}^k$ and $1 \le r \le k$, we have*

$$(3.9) \qquad g_{(1^r)} \cdot \tilde{\mathfrak{g}}_{\lambda}^{(k)} = \sum_{\substack{\mu \le \lambda \\ k}} \sum_{\substack{A \subset I, |A| \le r \\ u_A * x_\mu \in \tilde{S}_{k+1}^0}} (-1)^{|A| - \ell(u_A * \mu) + \ell(\mu)} \overset{\circ}{\mathfrak{g}}{}_{u_A * x_\mu}^{(k)}.$$

We apply the ring automorphism σ of $\Lambda_{(k)}$ on both sides of (3.9) to obtain

$$(3.10) \qquad \tilde{g}_{(1^r)} \cdot \sigma(\tilde{\mathfrak{g}}_{\lambda}^{(k)}) = \sum_{\substack{\mu \le \lambda \\ k}} \sum_{\substack{A \subset I, |A| \le r \\ u_A * x_\mu \in \tilde{S}_{k+1}^0}} (-1)^{|A| - \ell(u_A * \mu) + \ell(\mu)} \sigma(\overset{\circ}{\mathfrak{g}}{}_{u_A * x_\mu}^{(k)}).$$

Note that when we express $\sigma(\overset{\circ}{\mathfrak{g}}{}_{u_A * x_\mu}^{(k)})$ as a linear combination of $\{\sigma(\tilde{\mathfrak{g}}_\lambda^{(k)})\}$, $g_{u_A * x_\mu}^{(k)}$ is also a linear combination of $\{\tilde{g}_\lambda^{(k)}\}$ with the same coefficients. Therefore we see that (3.6) and (3.10) are exactly the same equation via the correspondence $\sigma(\tilde{\mathfrak{g}}_\lambda^{(k)}) \mapsto \tilde{g}_\lambda^{(k)}$, and hence we have $\sigma(\tilde{\mathfrak{g}}_\lambda^{(k)}) = \tilde{g}_\lambda^{(k)}$ for all $\lambda \in \mathcal{P}^k$.

Here is the outline of the proof of Proposition 3.16. We compute $g_{(1^r)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)}$ by using the combinatorial theory of Katalan functions developed in [5] to show the following result, which is the technical heart of this paper (see §3.5 for the proof).

Lemma 3.17 (Key lemma for Proposition 3.16).

$$(3.11) \qquad g_{(1^r)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)} = \sum_{A \subset I, \, |A| \leq r} T_{u_A} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)}$$

By the definition of $\overset{\circ}{\mathfrak{g}}{}_\lambda^{(k)}$ the right-hand side of (3.11) can be written as:

$$(3.12) \qquad \sum_{\mu \underset{k}{\leq} \lambda} \sum_{A \subset I, \, |A| \leq r} T_{u_A} \cdot \overset{\circ}{\mathfrak{g}}{}_\mu^{(k)} .$$

Thus, in order to complete the proof of Proposition 3.16, it suffices to show that

$$\sum_{A \subset I, \, |A| \leq r} T_{u_A} \cdot \overset{\circ}{\mathfrak{g}}{}_\mu^{(k)} = \sum_{\substack{A \subset I, \, |A| \leq r \\ u_A * x_\mu \in \tilde{S}_{k+1}^0}} (-1)^{|A| - \ell(u_A * \mu) + \ell(\mu)} \overset{\circ}{\mathfrak{g}}{}_{u_A * x_\mu}^{(k)} .$$

The final step of the proof of Proposition 3.16 is the following.

Lemma 3.18. *Let $A \subsetneq I$. Then*

$$T_{u_A} \cdot \overset{\circ}{\mathfrak{g}}{}_\mu^{(k)} = \begin{cases} (-1)^{|A| - \ell(u_A * x_\mu) + \ell(\mu)} \overset{\circ}{\mathfrak{g}}{}_{u_A * x_\mu}^{(k)} & (u_A * x_\mu \in \tilde{S}_{k+1}^0), \\ 0 & (\textit{otherwise}). \end{cases}$$

The proof of Lemma 3.18 is given by a general statement (Proposition A.6) that holds in the context of an arbitrary Coxeter group (W, S) and its parabolic quotient. A detailed discussion is given in Appendix A.2.

3.5. **Proof of Lemma 3.17.** This section is devoted to the proof of Lemma 3.17.

3.5.1. *Basic straightening rule.*

Definition 3.19 (Map $\mathfrak{r}$). Define a map $\mathfrak{r} : \{1, \ldots, \ell\} \to I = \mathbb{Z}/(k+1)\mathbb{Z}$ by $\mathfrak{r}(p) = -p + 1 \bmod k + 1$.

The following is the crucial combinatorial result, whose proof will be given in §3.5.4.

Lemma 3.20 (Basic straightening rule). *For $\lambda \in \mathcal{P}_m^k$ and an integer $\ell - k < p \leq \ell$ with $p \geq m + 2$,*

$$K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda + \varepsilon_p) = D_{\mathfrak{r}(p)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)}.$$

Furthermore, we have

$$D_{\mathfrak{r}(p)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)} = \tilde{\mathfrak{g}}_{s_{\mathfrak{r}(p)} x_\lambda}^{(k)} \iff s_{\mathfrak{r}(p)} x_\lambda > x_\lambda \textit{ and } s_{\mathfrak{r}(p)} x_\lambda \in \tilde{S}_{k+1}^0$$

$$(3.13) \qquad\qquad\qquad\qquad \iff \mathrm{top}_\Psi(p) < \mathrm{top}_\Psi(p-1),$$

with $\Psi = \Delta^k(\lambda)$.

Corollary 3.21. *For* $\lambda \in \mathcal{P}_m^k$ *and an integer* $\ell - k < p \le \ell$ *with* $p \ge m + 2$,

$$K(\Delta^k(\lambda); L(\Delta^k(\lambda)) \sqcup \{p\}; \lambda + \varepsilon_p) \quad = \quad T_{\mathfrak{r}(p)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)}.$$
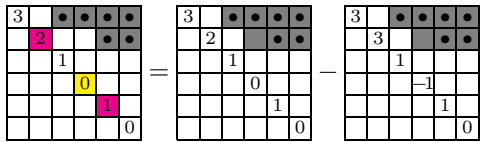
Proof. We apply Lemma 3.11(i) to deduce that

$$K(\Delta^k(\lambda); L(\Delta^k(\lambda)) \sqcup \{p\}; \lambda + \varepsilon_p)$$
$$= K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda + \varepsilon_p) - K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda)$$
$$= D_{\mathfrak{r}(i)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)} - \tilde{\mathfrak{g}}_\lambda^{(k)} \quad \text{(by Lemma 3.20)}$$
$$= T_{\mathfrak{r}(i)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)}.$$

$\square$

For the reader's convenience, we give some examples showing how the proof of Lemma 3.20 goes.

Example 3.22. Let $k = 4$, $\lambda = (3, 2, 1, 0, 0, 0)$, $\Psi := \Delta^k(\lambda)$, $M = L(\Psi)$. Note that $\beta = (2, 4)$ is an addable root of Ψ. Consider $\gamma = \lambda + \varepsilon_5 = (3, 2, 1, 0, 1, 0)$. By using Lemma 3.9(i), we have

$$(3.14) \qquad K(\Psi; M; \gamma) = K(\Psi \cup \beta; M; \gamma) - K(\Psi \cup \beta; M; \gamma + \varepsilon_2 - \varepsilon_4).$$

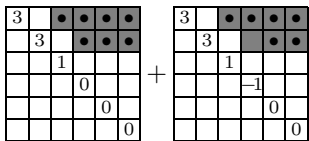The first term of (3.14) vanishes by (3.5) with $i = 4$. The second term equals

$$(-1)^2 K(\Psi \cup \beta; M; s_4(\gamma + \varepsilon_2 - \varepsilon_4) - \varepsilon_4 + \varepsilon_5) = K(\Psi \cup \beta; M; \mu)$$

with $\mu := (3, 3, 1, 0, 0, 0)$, by (3.4) with $i = 4$. Hence we have

$$K(\Psi; M; \lambda + \varepsilon_5) = K(\Psi \cup \beta; M; \mu) = $$
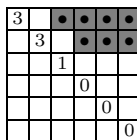

Next, we apply Lemma 3.11(ii) to it with $j = 4$ to get

$$(3.15) \qquad K(\Psi \cup \beta; M \sqcup \{4\}; \mu) + K(\Psi \cup \beta; M; (3, 3, 1, -1, 0, 0)).$$

The second term of (3.15) vanishes by (3.5) with $i = 4$. Noting $\Psi \cup \beta = \Delta^4(\mu)$ and $M \sqcup \{4\} = L(\Delta^4(\mu))$, we finally obtain

$$K(\Psi \cup \beta; M \sqcup \{4\}; \mu) = K(\Delta^4(\mu); L(\Delta^4(\mu)); \mu) = \tilde{\mathfrak{g}}_{(3,3,1)}^{(4)}.$$

Example 3.23. Let $k = 4$, and (Ψ, M, λ) the same as Example 3.22. Consider $\gamma = \lambda + \varepsilon_6 = (3, 2, 1, 0, 0, 1)$. We can apply Lemma 3.11(i) with $j = 5$ to get

$$(3.16) \qquad K(\Psi; M; \gamma) = K(\Psi; M \setminus \{5\}; \gamma) - K(\Psi; M \setminus \{5\}; \gamma - \varepsilon_5),$$

depicted by

The first term vanishes by (3.5) with $i = 5$. Hence we have

$$K(\Psi; M; \gamma) = -K(\Psi; M \setminus \{5\}; \gamma - \varepsilon_5) = -$$

where we used (3.4) for the last equality with $i = 5$. Thus, we obtain

where we again used (3.5) with $i = 5$ for the last equality.

3.5.2. *Some lemmas.*

Lemma 3.24 (Bounce-up lemma)**.** *Let (Ψ, M, γ) be a Katalan triple, and $p \to q$ a bounce edge of Ψ, such that*

   (a) $\beta := (q, p - 1)$ *is an addable root of Ψ,*
   (b) $\gamma_p = \gamma_{p-1} + 1$,
   (c) $m_M(p) = m_M(p - 1) + 1$,
   (d) Ψ *has a wall in rows $p - 1, p$.*

Then, we have

$$(3.17) \qquad K(\Psi; M; \gamma) = K(\Psi \cup \beta; M; \gamma + \varepsilon_q - \varepsilon_p)$$
$$(3.18) \qquad\qquad\quad = K(\Psi \cup \beta; M \sqcup \{p - 1\}; \gamma + \varepsilon_q - \varepsilon_p).$$

Proof. Since β is a removable root, we have, by Lemma 3.9(i),

$$(3.19) \qquad K(\Psi; M; \gamma) = K(\Psi \cup \beta; M; \gamma) - K(\Psi \cup \beta; M; \gamma + \beta).$$

Note that $\Psi \cup \beta$ has a ceiling in columns $p - 1, p$. Hence, by (c) and (d), we can apply (3.5) to the first term of (3.19), which then vanishes. Applying (3.4) to the second term, we obtain (3.17).

By Lemma 3.11(ii) with $j = p - 1$, the right-hand side of (3.17) equals

$$K(\Psi \cup \beta; M \sqcup \{p - 1\}; \gamma + \varepsilon_q - \varepsilon_p) + K(\Psi \cup \beta; M; \gamma + \varepsilon_q - \varepsilon_{p-1} - \varepsilon_p),$$

in which the second term vanishes by (3.5) with $i = p - 1$. Therefore, we obtain (3.18). $\qquad\square$

Example 3.25. The following equation is given by applying Lemma 3.24 with $p = 6$. Here, $q = 3$ and $\beta = (3, 5)$.
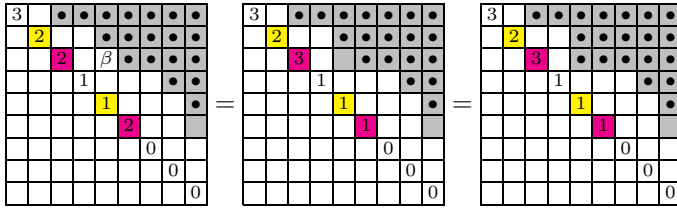
Lemma 3.26 (Absorption lemma, [5, Lemma 4.4])**.** *Let* (Ψ, M, γ) *be a Katalan triple, and* $p \geq 2$ *such that*

    (a) $\text{top}_\Psi(p) = p$,
    (b) $\gamma_p = \gamma_{p-1} + 1$,
    (c) $m_M(p) = m_M(p - 1)$,
    (d) Ψ *has a wall in rows* $p - 1, p$.

Then,

$$K(\Psi; M; \gamma) = K(\Psi; M; \gamma - \varepsilon_p).$$

Proof. By (a), Ψ has a ceiling in columns $p - 1, p$. From Lemma 3.11(i) with $j = p - 1$, it follows that

$$(3.20) \qquad K(\Psi; M; \gamma) = K(\Psi; M \setminus \{p - 1\}; \gamma) - K(\Psi; M \setminus \{p - 1\}; \gamma - \varepsilon_{p-1}).$$

Let $M' = M \setminus \{p - 1\}$. Then, we have $m_{M'}(p) = m_{M'}(p - 1) + 1$, which implies that we can apply (3.5) with $i = p - 1$ to the first term of (3.20). Hence we obtain
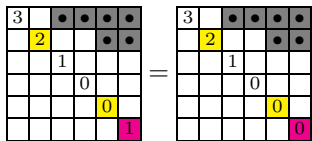
$$K(\Psi; M; \gamma) = -K(\Psi; M'; \gamma - \varepsilon_{p-1}) = K(\Psi; M'; \gamma - \varepsilon_p),$$

where we used (3.4) with $i = p - 1$ for the last equality. By Lemma 3.11(ii) with $j = p - 1$, we deduce that

$$K(\Psi; M'; \gamma - \varepsilon_p) = K(\Psi; M; \gamma - \varepsilon_p) + K(\Psi; M'; \gamma - \varepsilon_p - \varepsilon_{p-1}),$$

in which the second term vanishes by (3.5) with $i = p - 1$. $\qquad\square$

Example 3.27. From Lemma 3.26 with $p = 6$, we have

3.5.3. *Mirror edges and mirror paths.*

Definition 3.28 (Mirror edge, mirror path, mirror top)**.** Let Ψ be a root ideal, and $e := (p \to q)$ a bounce edge of Ψ. We say that e is a *mirror edge* if $p - 1 \to q - 1$ is also a bounce edge of Ψ. A bounce path $p = p_0 \to p_1 \to \cdots \to p_L$ is a *mirror path* of *length* L of Ψ if $p_i \to p_{i+1}$ is a mirror edge for each $0 \leq i \leq L - 1$. If such L is maximal, then we define $\text{mtop}_\Psi(p) = p_L$ and call it the *mirror top* of p. In particular, if p is not contained in any mirror path, we have $\text{mtop}_\Psi(p) = p$.

Example 3.29. For the root ideal Ψ illustrated in the following picture, $10 \to 6 \to 3$ is a mirror path having the maximal length 2. Hence, we have $\mathrm{mtop}_\Psi(10) = 3$.
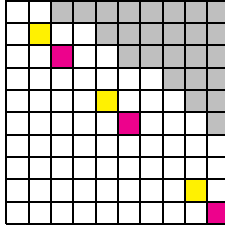
It is easy to see the following.

Lemma 3.30. *Let Ψ be a root ideal, and $p \to q$ a mirror edge of Ψ. Then $\beta := (q, p-1)$ is an addable root of Ψ. Furthermore, $\Psi \cup \{\beta\}$ has a wall in rows $q-1, q$, and a ceiling in columns $p-1, p$.*

Lemma 3.31. *Let Ψ be a root ideal, and z the lowest nonempty row of Ψ. Assume that Ψ is wall-free in nonempty rows, i.e., Ψ has no wall between the 1st and zth rows. Then, for $p \geq 2$, we have*

$$\mathrm{mtop}_\Psi(p) = \max\{\mathrm{top}_\Psi(p), \mathrm{top}_\Psi(p-1) + 1\}.$$

Proof. Let $p \to q$ be a bounce edge of Ψ. Note that, whenever $\mathrm{up}_\Psi(p-1)$ exists, it satisfies $\mathrm{up}_\Psi(p-1) = q-1$; otherwise, Ψ has a wall in rows $q-1, q$.

Let $p = p_0 \to p_1 \to \cdots \to p_L$ and $p-1 = q_0 \to q_1 \to \cdots \to q_{L'}$ be bounce paths of maximal lengths. By what we noted in the previous paragraph, we have $q_i = p_i - 1$ for $0 \leq i \leq \min(L, L')$. If $L > L'$, then we have $\mathrm{mtop}_\Psi(p) = p_{L'} = q_{L'} + 1 = \mathrm{top}_\Psi(p-1) + 1$, while $\mathrm{mtop}_\Psi(p) > \mathrm{top}_\Psi(p)$. If $L = L'$, then $\mathrm{mtop}_\Psi(p) = \mathrm{top}_\Psi(p) = \mathrm{top}_\Psi(p-1) + 1$. If $L < L'$, then $\mathrm{mtop}_\Psi(p) = \mathrm{top}_\Psi(p)$, while $\mathrm{mtop}_\Psi(p) = \mathrm{top}_\Psi(p) = q_L + 1 > \mathrm{top}_\Psi(p-1) + 1$. $\square$

Note that $\Psi = \Delta^k(\lambda)$ is wall-free in nonempty rows, then we can apply the previous result to it.

Lemma 3.32. *Let $\lambda \in \mathcal{P}^k$. Set $\Psi := \Delta^k(\lambda)$, $M = L(\Psi)$. Let $e = (p \to q)$ be a bounce edge of Ψ such that $q \geq 2$.*

    (1) *e is a mirror edge of Ψ if and only if $\lambda_{q-1} = \lambda_q$.*
    (2) *If e is a mirror edge of Ψ, then $m_M(p) = m_M(p-1) + 1$.*
    (3) *If e is not a mirror edge, then $\lambda_{q-1} > \lambda_q$.*

Proof. (1) and (2) are clear from the definitions of Ψ, M. For (3), if e is not a mirror edge, then it follows from (1) that $\lambda_{q-1} \neq \lambda_q$. Since λ is a partition, we have $\lambda_{q-1} > \lambda_q$. $\square$

Lemma 3.33. *Let $\lambda \in \mathcal{P}^k$, and $\Psi := \Delta^k(\lambda) \subset \Delta_\ell^+$. Then for $p \in \{1, \ldots, \ell\}$,*

$$\mathfrak{r}\left(\mathrm{mtop}_\Psi(p)\right) = \mathfrak{r}(p).$$

Proof. This is [5, Lemma 5.7]. $\square$

In the final step of the proof of Lemma 3.20, we use the following.

Lemma 3.34 (Cleaning Lemma, [5, Lemma 4.7])**.** *Let (Ψ, M, γ) be a Katalan triple such that*

    (a) *$\beta = (q, p-1)$ is a removable root of Ψ,*

(b) Ψ *has a wall in rows* $p - 1, p$,
(c) Ψ *has a ceiling in columns* $p - 1, p$,
(d) $\gamma_q = \gamma_{q-1}$,
(e) $m_M(p) = m_M(p - 1) + 1$.

Then

(3.21) $K(\Psi; M; \gamma) = K(\Psi \setminus \beta; M; \gamma).$

3.5.4. *Proof of Lemma 3.20.*

Proof of Lemma 3.20. Let $\lambda \in \mathcal{P}^k$, and $\Psi := \Delta^k(\lambda) \subset \Delta_\ell^+$. Set $\mathcal{T} := (\Psi, M, \gamma) = (\Delta^k(\lambda), L(\Delta^k(\lambda)), \lambda + \varepsilon_p)$. Let $p = p_0 \to p_1 \to \cdots \to p_L$ be a mirror path starting from p of maximal length $L \geq 0$. We will prove that we can successively apply Lemma 3.24 to $\mathcal{T}$ and obtain the sequence of Katalan triples:

(3.22) $\mathcal{T} =: \mathcal{T}_0 \xrightarrow{p_0} \mathcal{T}_1 \xrightarrow{p_1} \cdots \xrightarrow{p_{L-1}} \mathcal{T}_L =: \mathcal{T}'.$

Here $\mathcal{T}_i \xrightarrow{p_i} \mathcal{T}_{i+1}$ means that $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ by applying equation (3.17) in Lemma 3.24 with respect to the mirror edge $p_i \to p_{i+1}$ of $\mathcal{T}_i$. If $L = 0$, it suffices to put $\mathcal{T}' = \mathcal{T}$. Assume $L \geq 1$. We can check immediately that $\mathcal{T} = \mathcal{T}_0$ satisfies assumptions (a)–(d) of Lemma 3.24: (a) holds by Lemma 3.30; (b) holds from $\lambda_{p-1} = \lambda_p = 0$ (since $p \geq m + 2$) and $\gamma = \lambda + \varepsilon_p$; (c) holds by Lemma 3.32(2); (d) holds because there are no roots of $\Psi = \Delta^k(\lambda)$ in rows $p - 1, p$ (since $p > \ell - k$). Then, we obtain a new Katalan triple $\mathcal{T}_1 = (\Psi^{(1)}, M^{(1)}, \gamma^{(1)})$ by applying (3.17) to $\mathcal{T}_0$. When $L > 1$, we have $\lambda_{p_1 - 1} = \lambda_{p_1}$ from Lemma 3.32(1), which implies $\gamma_{p_1}^{(1)} = \gamma_{p_1 - 1}^{(1)} + 1$. By Lemma 3.30, $\Psi^{(1)}$ has a wall in rows $p_1 - 1, p_1$. Hence, $\mathcal{T}_1$ satisfies (b), (d) of Lemma 3.24. The remaining conditions (a) and (c) hold for $\mathcal{T}_1$ for the same reasons as in the case of $\mathcal{T}_0$: the multiset $M^{(1)}$ is the same as M; the root ideals $\Psi^{(1)}$ and Ψ coincide with each other in columns $\leq p_1$. Therefore, we can apply equation (3.17) in Lemma 3.24 to $\mathcal{T}_1$ with respect to the mirror edge $p_1 \to p_2$, and obtain $\mathcal{T}_2$. The previous procedure can be repeated L times to obtain the sequence in (3.22). Let $\mathcal{T}' = (\Psi', M', \gamma')$. Then, we have

(3.23) $\Psi' = \Psi \cup \{\beta_1, \ldots, \beta_L\}, \quad M' = M, \quad \gamma' = \gamma + \varepsilon_{\mathrm{mtop}_\Psi(p)} - \varepsilon_p (= \lambda + \varepsilon_{\mathrm{mtop}_\Psi(p)}),$

where $\beta_i := (p_i, p_{i-1} - 1)$ $(1 \leq i \leq L)$. Note that Ψ' has a wall in rows $p_i - 1, p_i$ for each i. The following procedure depends on whether (i) $\mathrm{top}_\Psi(p) < \mathrm{top}_\Psi(p - 1)$ or (ii) $\mathrm{top}_\Psi(p) > \mathrm{top}_\Psi(p - 1)$.

If $\mathrm{top}_\Psi(p) < \mathrm{top}_\Psi(p - 1)$, then we have $\mathrm{mtop}_\Psi(p) = \mathrm{top}_\Psi(p - 1) + 1 > \mathrm{top}_\Psi(p)$ by Lemma 3.31, which implies that there is a bounce edge $\mathrm{mtop}_\Psi(p) \to q$ of Ψ for some q. Note that $\mathrm{mtop}_\Psi(p) \to q$ is also a bounce edge of Ψ'. Therefore, we can apply Lemma 3.24 to $\mathcal{T}'$ with respect to this bounce edge. From (3.18), it follows that

$K(\Psi'; M'; \gamma') = K(\Psi' \cup \beta; M' \sqcup \{\mathrm{mtop}_\Psi(p) - 1\}; \gamma' + \varepsilon_q - \varepsilon_{\mathrm{mtop}_\Psi(p)}),$

where

$$\beta := (q, \mathrm{mtop}_\Psi(p) - 1).$$

Note that

$$\lambda' := \gamma' + \varepsilon_q - \varepsilon_{\mathrm{mtop}_\Psi(p)} = \lambda + \varepsilon_q$$

and $M' \sqcup \{\mathrm{mtop}_\Psi(p) - 1\} = L(\Delta^k(\lambda'))$. Moreover, λ' is a partition by Lemma 3.32(3). To prove that λ' is k-bounded, it suffices to consider the case that $\lambda_1 = k$

and $q = 1$. However, one sees that this does not occur in view of Lemma 3.31. We also note that

$$\Psi \cup \beta = (\Psi' \cup \beta) \setminus \{\beta_1, \ldots, \beta_L\} = \Delta^k(\lambda').$$

By using Lemma 3.34 recursively, we can remove $\beta_L, \ldots, \beta_2, \beta_1$ from Ψ', and obtain

$$K(\Delta^k(\lambda'); L(\Delta^k(\lambda')); \lambda') = \tilde{\mathfrak{g}}_{\lambda'}^{(k)}.$$

From Lemma 3.33, we deduce that

$$q = \mathrm{up}_\Psi(\mathrm{mtop}_{\mathcal{T}}(p)) \equiv \mathrm{mtop}_{\mathcal{T}}(p) \equiv \mathfrak{r}(p) \bmod k + 1,$$

and $s_{\mathfrak{r}(p)} x_\lambda = x_{\lambda'}$.

Next assume that $\mathrm{top}_\Psi(p) > \mathrm{top}_\Psi(p - 1)$. We check that $\mathcal{T}'$ satisfies conditions (a)–(d) of Lemma 3.26 with $p = \mathrm{mtop}_\Psi(p)$. From Lemma 3.31, we have $\mathrm{mtop}_\Psi(p) = \mathrm{top}(p)$, which implies (a). Condition (b) follows from Lemma 3.32(1). Condition (c) follows from the fact that there is a ceiling in the columns $p - 1, p$. Condition (d) follows from the definition of Ψ'.

From Lemma 3.26, we have

$$K(\Psi'; M'; \gamma') = K(\Psi'; M'; \gamma' - \varepsilon_{\mathrm{mtop}_\Psi(p)}).$$

In view of (3.23), we can rewrite

$$K(\Psi'; M'; \gamma' - \varepsilon_{\mathrm{mtop}_\Psi(p)}) = K(\Delta^k(\lambda) \cup \{\beta_1, \ldots, \beta_L\}; \Delta^k(\lambda); \lambda).$$

By Lemma 3.34, we can remove $\beta_L, \ldots, \beta_2, \beta_1$ successively to obtain

$$K(\Delta^k(\lambda); L(\Delta^k(\lambda)); \lambda) = \tilde{\mathfrak{g}}_\lambda^{(k)}.$$
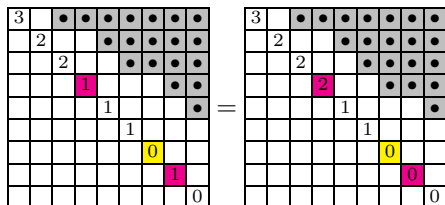
Again by Lemma 3.33, we deduce that

$$q = \mathrm{up}_\Psi(\mathrm{mtop}_{\mathcal{T}}(p)) \equiv \mathrm{mtop}_{\mathcal{T}}(p) \equiv \mathfrak{r}(p) \bmod k + 1,$$

and $s_{\mathfrak{r}(p)} x_\lambda = x_\lambda$. $\qquad\square$
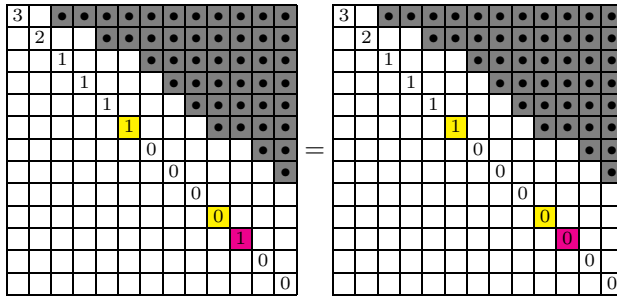
Here are some examples.

Example 3.35. Case 1 ($p = 8$): $\mathrm{top}(p) = \mathrm{mtop}_\Psi(p) = 4 < \mathrm{top}(p - 1) = 7$ with $p = 8$. In this case, we have $L = 0$.
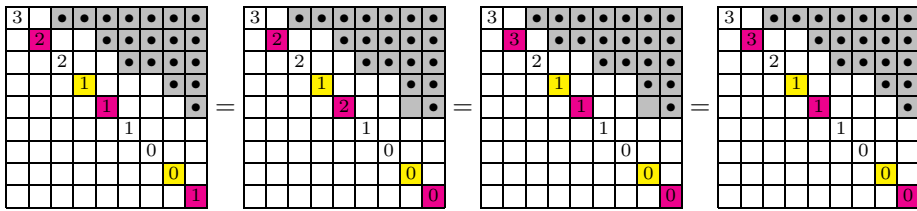
(1) Lemma 3.24 equation (3.18) with $p = 8$.

Example 3.36. Case 2 ($p = 11$): $\mathrm{top}(p) = 11 > \mathrm{top}(p - 1) = 6$ and $L = 0$.
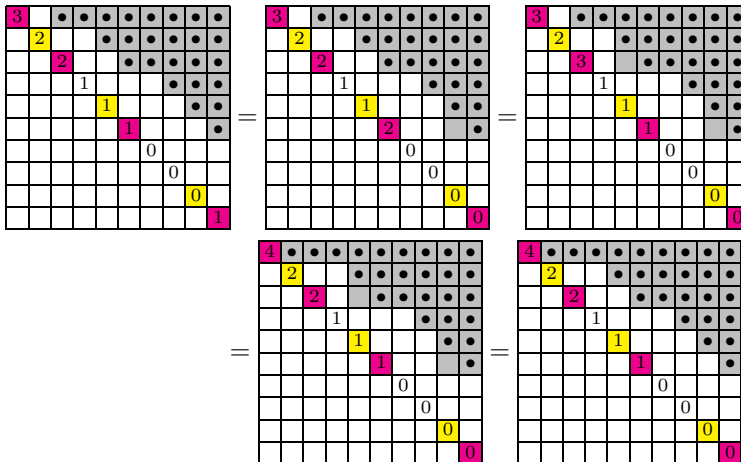
(1) Lemma 3.26 with $p = 11$.

Example 3.37. Case 1 ($p = 9$): $\mathrm{top}(p) = 2 < \mathrm{top}(p-1) = 4$.

(1) Lemma 3.24 equation (3.17) with $p = 9$.
(2) Lemma 3.24 equation (3.18) with $p = 5$.
(3) Cleaning Lemma with $q = 5$.

Example 3.38. Case 1 ($p = 10$): $\mathrm{top}(p) = 1 < \mathrm{top}(p-1) = 2$ and $\mathfrak{r}(p) = 1$.

(1) Lemma 3.24 equation (3.17) with $p = 10$.
(2) Lemma 3.24 equation (3.17) with $p = 6$.
(3) Lemma 3.24 equation (3.18) with $p = 3$.
(4) Cleaning Lemma twice with $q = 3, 5$.

Example 3.39. Case 2 ($p = 10$): $\mathrm{top}(p) = 6 > \mathrm{top}(p-1) = 2$.

(1) Lemma 3.24 equation (3.17) with $p = 10$.
(2) Lemma 3.26 with $p = 6$.
(3) Cleaning Lemma with $q = 6$.

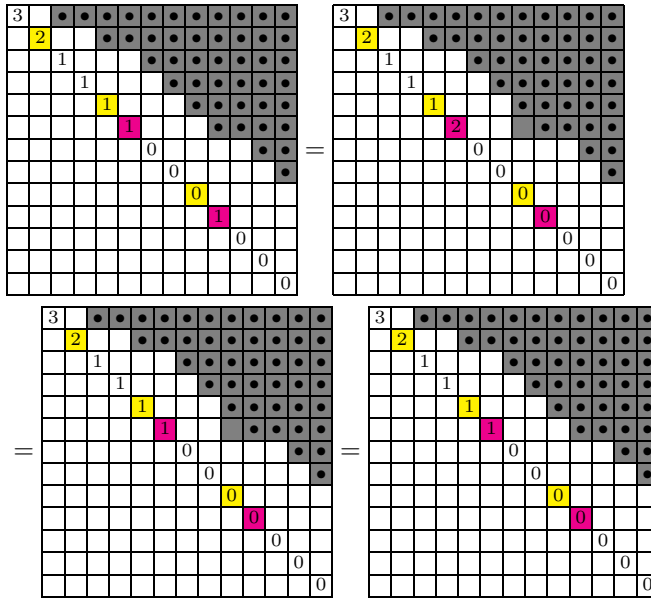*Remark* 3.40. From the proof of Lemma 3.20, we can slightly improve the statement of the lemma as follows: For $\lambda \in \mathcal{P}_m^k$, an integer $\ell - k < p \leq \ell$ with $p \geq m+2$, and $\kappa \in \mathbb{Z}^\ell$ with $p < \operatorname{supp}(\kappa) \leq \ell$, we have

$$K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda + \varepsilon_p + \kappa) = K(\Delta^k(\mu); \Delta^k(\mu); \mu + \kappa),$$

where $\tilde{\mathfrak{g}}_\mu^{(k)} = D_{\mathfrak{r}(p)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)}$. Indeed, throughout the straightening process introduced in §3.5.4, κ is left unchanged because the process affects only topmost p rows of the diagram.

From Remark 3.40, we obtain the following key formula: For $\lambda \in \mathcal{P}_m^k$, and a set of integers $\ell - k < p_1 < \cdots < p_r \leq \ell$ with $p_1 \geq m+2$, we have

$$(3.24) \qquad K(\Delta^k(\lambda); \Delta^k(\lambda); \lambda + \varepsilon_{p_1} + \cdots + \varepsilon_{p_r}) = D_{\mathfrak{r}(p_r)} \cdots D_{\mathfrak{r}(p_1)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)}.$$

3.5.5. *Proof of Lemma* 3.17. Let Ψ_1, Ψ_2 be root ideals of $\Delta_{\ell_1}^+$ and $\Delta_{\ell_2}^+$, respectively. Let $\Psi_1 \uplus \Psi_2$ be the subset of $\Delta_{\ell_1+\ell_2}^+$ defined by:

$$\{(i,j) \in \Delta_{\ell_1+\ell_2}^+ \mid (i,j) \in \Psi_1 \text{ or } (i-\ell_1, j-\ell_1) \in \Psi_2 \text{ or } (i \leq \ell_1 \text{ and } j > \ell_1)\}.$$

Let $\lambda \in \mathcal{P}_\ell^k$. Set $\Psi = \Delta^k(\lambda)$. By the product rule [5, Lemma 3.8], we have

$$(3.25) \qquad g_{(1^r)} \cdot \tilde{\mathfrak{g}}_\lambda^{(k)} = K(\Psi \uplus \varnothing_r; \Psi \uplus \varnothing_r; \lambda + \varepsilon_{\{\ell+1,\ldots,\ell+r\}}).$$

By applying Diagonal Removable Lemma [5, Lemma 4.13], one shows that the right-hand side of (3.25) is identical to

$$(3.26) \qquad\qquad K(\Psi'; \Psi'; \lambda + \varepsilon_{\{\ell+1,\ldots,\ell+r\}}),$$

where $\Psi' = (\Delta^k(\lambda) \uplus \varnothing_r) \setminus \{(i,j) \in \Delta_{\ell+r}^+ \mid i \leq \ell, \ j \geq \ell, j - i \leq r - 1\}$. We can apply [5, Lemma 5.3] with $x = \ell - k + 1$, $h = k - r + 1$ (in the notation there) to show that (3.26) is identical to

$$(3.27) \qquad \sum_{a=0}^r \sum_{\substack{S \subset \{\ell-k+1+r-a,\ldots,\ell\} \\ |S|=a}} K(\Psi''; L(\Psi'') \sqcup S; \lambda + \varepsilon_S + \varepsilon_{\{\ell+1,\ldots,\ell+r-a\}}),$$

where $\Psi'' = \Psi' \setminus \{(i,j) \mid i \leq \ell, j \geq \ell, j - i \leq k\} = \Delta^k(\lambda) \subset \Delta^+_{\ell+r}$ (see also the last part of the proof of [5, Proposition 5.4]). By Lemma 3.11(i) and (3.24), for $S = \{p_1 < p_2 < \cdots < p_a\}$, we have

$$
\begin{aligned}
&K(\Psi''; L(\Psi'') \sqcup S; \lambda + \varepsilon_S + \varepsilon_{\{\ell+1,\ldots,\ell+r-a\}}) \\
&\quad = \sum_{S' \subset S} (-1)^{|S \setminus S'|} K(\Psi''; L(\Psi''); \lambda + \varepsilon_{S'} + \varepsilon_{\{\ell+1,\ldots,\ell+r-a\}}) \\
&\quad = \sum_{b=0}^{a} \sum_{\ell-k+1+r-a \leq p_{i_1} < p_{i_2} < \cdots < p_{i_b} \leq \ell} (-1)^{a-b} D_{\mathfrak{r}(\ell+r-a)} \cdots D_{\mathfrak{r}(\ell+1)} D_{\mathfrak{r}(p_{i_b})} \cdots D_{\mathfrak{r}(p_{i_1})} \cdot \tilde{\mathfrak{g}}_{\lambda}^{(k)} \\
&\quad = D_{\mathfrak{r}(\ell+r-a)} \cdots D_{\mathfrak{r}(\ell+1)} T_{\mathfrak{r}(p_a)} \cdots T_{\mathfrak{r}(p_1)} \cdot \tilde{\mathfrak{g}}_{\lambda}^{(k)}.
\end{aligned}
$$

Therefore, (3.27) is equal to

$$
(3.28) \qquad \sum_{a=0}^{r} \sum_{\ell-k+1+r-a \leq p_1 < \cdots < p_a \leq \ell} D_{\mathfrak{r}(\ell+r-a)} \cdots D_{\mathfrak{r}(\ell+1)} T_{\mathfrak{r}(p_a)} \cdots T_{\mathfrak{r}(p_1)} \cdot \tilde{\mathfrak{g}}_{\lambda}^{(k)}.
$$

Proposition 3.41. *For $1 \leq r \leq k$, and $\ell \in \mathbb{Z}$, we have*

$$
\sum_{a=0}^{r} \sum_{\ell-k+1+r-a \leq p_1 < \cdots < p_a \leq \ell} D_{\mathfrak{r}(\ell+r-a)} \cdots D_{\mathfrak{r}(\ell+1)} T_{\mathfrak{r}(p_a)} \cdots T_{\mathfrak{r}(p_1)} = \sum_{A \subset I, \, |A| \leq r} T_{u_A}.
$$

Proof. We first note that for any subsequence A_1 of $(\mathfrak{r}(\ell+r-a), \ldots, \mathfrak{r}(\ell+1))$ such that $0 \leq |A_1| \leq r - a$, and A_2 of $(\mathfrak{r}(\ell), \mathfrak{r}(\ell-1), \ldots, \mathfrak{r}(\ell-k+1+r-a))$ such that $|A_2| = a$, the concatenation $A_1 \cdot A_2$ is a cyclically increasing sequence of length less than or equal to r.

For $0 \leq a \leq n \leq r$, let

$$
X_a^n = \left\{ (p_1, \ldots, p_n) \in \mathbb{Z}^n \; \middle| \; \begin{array}{c} \ell - k + r - a < p_1 < \cdots < p_a \leq \ell \\ < p_{a+1} < \cdots < p_n \leq \ell + r - a \end{array} \right\}.
$$

Substituting $D_p = T_p + 1$ into the left-hand side of the equation in the proposition, we see that

$$
\begin{aligned}
&\sum_{a=0}^{r} \sum_{\ell-k+1+r-a \leq p_1 < \cdots < p_a \leq \ell} (T_{\mathfrak{r}(\ell+r-a)} + 1) \ldots (T_{\mathfrak{r}(\ell+1)} + 1) T_{\mathfrak{r}(p_a)} \ldots T_{\mathfrak{r}(p_1)} \\
&\qquad = \sum_{a=0}^{r} \sum_{n=a}^{r} \sum_{(p_1,\ldots,p_n) \in X_a^n} T_{u_{\{\mathfrak{r}(p_1),\ldots,\mathfrak{r}(p_n)\}}} \\
&\qquad = \sum_{n=0}^{r} \sum_{a=0}^{n} \sum_{(p_1,\ldots,p_n) \in X_a^n} T_{u_{\{\mathfrak{r}(p_1),\ldots,\mathfrak{r}(p_n)\}}}.
\end{aligned}
$$

Then it suffices to show that the map

$$
\mathrm{rm} : \bigsqcup_{a=0}^{n} X_a^n \to \{A \subset I \,;\, |A| = n\}, \quad (p_1, \ldots, p_n) \mapsto \{\overline{p_1}, \ldots, \overline{p_n}\},
$$

is bijective. For this, we identify a subset $A \subset I$ with a 01-sequence $\eta_1 \eta_2 \cdots \eta_{k+1} \in \{0,1\}^{k+1}$ by letting $\eta_p = 0$ if $\overline{p + \ell} \in A$ and $\eta_p = 1$ otherwise. Then the image of

X_a^n by rm is contained in the set

$$S_a^n := \left\{ \eta_1 \eta_2 \ldots \eta_{k+1} \in \{0,1\}^{k+1} \left| \begin{array}{l} \sum_{p=1}^{k+1} \eta_p = k+1-n, \\ \sum_{p=1}^{r-a} \eta_p = r-n \\ \eta_{r-a+1} = 1 \end{array} \right. \right\}.$$

For any $\eta_1 \eta_2 \ldots \eta_{k+1} \in \bigcup_{a=0}^n S_a^n$, we have

$$a = r + 1 - \min\{p \,|\, \eta_1 + \cdots + \eta_p = r - n + 1\},$$

which implies $S_a^n \cap S_{a'}^n = \emptyset \iff a \neq a'$. This implies that rm is injective as its restriction to X_a^n is injective. Because $\sum_{a=0}^n |X_a^n| = \sum_{a=0}^n \binom{k-r+a}{a}\binom{r-a}{n-a} = \binom{k+1}{n} = \sharp\{A \subset I \,;\, |A| = n\}$, the map rm is also surjective. $\square$

This completes the proof of Lemma 3.17.

4. RELATION TO QUANTUM K-THEORY

4.1. **Quantum K-theory ring of the flag variety.**

Theorem 4.1 ([17],[27],[28],[34],[35])**.** *$QK(G/B)$ can be identified with the quotient ring*

$$(4.1) \qquad A_{k+1} := \mathbb{C}[[Q]][z_1, \ldots, z_{k+1}]/I_{k+1},$$

where I_{k+1} is the ideal of $\mathbb{C}[[Q]][z_1, \ldots, z_{k+1}]$ generated by

$$(4.2) \qquad \sum_{\substack{I \subset \{1,\ldots,k+1\} \\ |I|=i}} \prod_{j \in I} z_j \prod_{\substack{j \in I \\ j+1 \notin I}} (1 - Q_j) - \binom{k+1}{i} \quad (1 \leq i \leq k+1),$$

in such a way that for $w \in S_{k+1}$ the quantum Grothendieck polynomial $\mathfrak{G}_w^Q$ of Lenart and Maeno, with the change of variables $x_i = 1 - z_i$, represents $\mathcal{O}_{G/B}^w$ in $QK(G/B_{k+1})$.

Let $QK^{\mathrm{pol}}(G/B)$ denote the $\mathbb{C}[Q]$-module spanned by $\mathcal{O}_{G/B}^w$ ($w \in S_{k+1}$). Let S be the multiplicative subset $1 + (Q)$ of $\mathbb{C}[Q]$. For a $\mathbb{C}[Q]$-module M, we denote by M_S the localization by S. By a result of Anderson, Chen, Tseng, and Iritani [2], $QK^{\mathrm{pol}}(G/B)_S$ forms a subring of $QK(G/B)$.

Let A_{k+1}^{pol} be the quotient ring $\mathbb{C}[Q][z_1, \ldots, z_{k+1}]/I_{k+1}^{\mathrm{pol}}$, where I_{k+1}^{pol} is the ideal of $\mathbb{C}[Q][z_1, \ldots, z_{k+1}]$ generated by polynomials in (4.2). The following result is a modified version of Kirillov–Maeno's (conjectural) presentation, given as [28, Remark 52]. For the reader's convenience, we give a sketch of its proof; see also [34, Remark 6.2].

Proposition 4.2. *There exists an isomorphism of $\mathbb{C}[Q]_S$ algebras*

$$(4.3) \qquad (A_{k+1}^{\mathrm{pol}})_S \longrightarrow QK^{\mathrm{pol}}(G/B)_S.$$

Moreover, the image of $\mathfrak{G}_w^Q$ in $(A_{k+1}^{\mathrm{pol}})_S$ is sent to $\mathcal{O}_{G/B}^w$ for $w \in S_{k+1}$.

Proof. By [1, Chapter 3, Exercise 2], we see that the localized ideal $(Q)_S$ is contained in the Jacobson radical of $\mathbb{C}[Q]_S$. By [34, Corollary B.3(2)], we see that the right-hand side of (4.3) is finitely generated as a $\mathbb{C}[Q]_S$-module. Then we can apply Nakayama–type arguments ([11, Proposition A.3 and Remark A.6]) to obtain the isomorphism by the same arguments as in [34, Theorem 6.1]. The second statement follows from [28, Theorem 51] or [35, Theorem 4.4]. $\square$

4.2. Map Φ_{k+1}. Recall that $\tau_i := g_{R_i}$, $\tau_i^+ := \tilde{g}_{R_i}$ with $R_i = (\overbrace{i, \ldots, i}^{k+1-i})$ for $1 \le i \le k$. Note that the notation R_i is different from the one used in [13], and the indices of τ_i are also switched from the ones in [13] by $i \mapsto k+1-i$ (see Remark 4.4).

Theorem 4.3 ([13]). *There is a ring isomorphism*

$$(4.4) \qquad \Phi_{k+1} : A_{k+1}^{\mathrm{pol}}[Q_i^{-1}] \longrightarrow \Lambda_{(k)}[\tau_i^{-1}, (\tau_i^+)^{-1}],$$

where $1 \le i \le k$, such that

$$(4.5) \qquad z_i \mapsto \frac{\tau_i \tau_{i-1}^+}{\tau_i^+ \tau_{i-1}} \; (1 \le i \le k+1), \quad Q_i \mapsto \frac{\tau_{i-1} \tau_{i+1}}{\tau_i^2} \; (1 \le i \le k).$$

The map Φ_{k+1} was constructed by solving the relativistic Toda lattice equation with the initial condition that the Lax matrix is unipotent. Although the construction of Φ_{k+1} has no apparent geometric meaning, it is expected that the map sends a Schubert structure sheaf $\mathcal{O}_{G/B}^w$ to an element in $K_*(\mathrm{Gr})_{\mathrm{loc}}$ related to a Schubert class.

Remark 4.4. Our convention for Φ_{k+1} is slightly different from the one in [13]. Let $\overline{\Omega}$ be the automorphism of $\Lambda_{(k)}[\tau_i^{-1}, (\tau_i^+)^{-1}]$ given as the natural extension of Ω (see §3.1). One can check that $\Phi'_{k+1} = \overline{\Omega} \circ \Phi_{k+1}$ coincides with the map introduced in [13] by replacing σ_i in their notation with τ_i^+.

Corollary 4.5. *There is an injective $\mathbb{C}[Q]$-algebra homomorpshism*

$$\phi : A_{k+1}^{\mathrm{pol}}[Q_i^{-1}] \hookrightarrow QK(G/B)[Q_i^{-1}]$$

such that the image of $\mathfrak{S}_w^Q$ in $A_{k+1}^{\mathrm{pol}}[Q_i^{-1}]$ is sent to $\mathcal{O}_{G/B}^w$.

Proof. We know from Theorem 4.3 that $A_{k+1}^{\mathrm{pol}}[Q_i^{-1}]$ is an integral domain, and hence $A_{k+1}^{\mathrm{pol}}[Q_i^{-1}]$ is a $\mathbb{C}[Q]$-subalgebra of $(A_{k+1}^{\mathrm{pol}}[Q_i^{-1}])_S$, where $S = 1 + (Q)$. Consider the composition of ring homomorphisms

$$
\begin{aligned}
A_{k+1}^{\mathrm{pol}}[Q_i^{-1}] &\hookrightarrow (A_{k+1}^{\mathrm{pol}}[Q_i^{-1}])_S \\
&= ((A_{k+1}^{\mathrm{pol}})_S)[Q_i^{-1}] \\
&\cong (QK^{\mathrm{pol}}(G/B)_S)[Q_i^{-1}] \quad (\text{Proposition 4.2}) \\
&\hookrightarrow QK(G/B)[Q_i^{-1}].
\end{aligned}
$$

The statement on $\mathfrak{S}_w^Q$ follows from that of Proposition 4.2. $\qquad\square$

4.3. K-homology of the affine Grassmannian. For $x \in \tilde{S}_{k+1}^0$, let $\xi_x^0 \in K_*(\mathrm{Gr})$ be the element defined in [21, §6.3]. The nonequivariant K-theoretic k-Schur function $g_x^{(k)}$ in [21, Theorem 7.17(2)] corresponds to ξ_x^0. Let us denote this isomorphism by $\alpha_\circ : K_*(\mathrm{Gr}) \to \Lambda_{(k)}$. It holds that $\alpha_\circ(\mathcal{O}_x^{\mathrm{Gr}}) = \tilde{g}_x^{(k)}$ ([39], [20, Lemma 2(ii)]). We define a twisted isomorphism $\alpha : K_*(\mathrm{Gr}) \to \Lambda_{(k)}$ by $\alpha := \sigma^{-1} \circ \alpha_\circ$. Thus we have

$$(4.6) \qquad \alpha(\mathcal{O}_x^{\mathrm{Gr}}) = \sigma^{-1}(\tilde{g}_x^{(k)}).$$

Proposition 4.6. *If $\lambda \subset R_i$ for some $1 \le i \le k$, equivalently $\lambda_1 + \ell(\lambda) \le k+1$, then we have*

$$(4.7) \qquad \alpha(\mathcal{O}_\lambda^{\mathrm{Gr}}) = g_\lambda.$$

In particular, we have

$$(4.8) \qquad \alpha(\mathcal{O}_{R_i}^{\mathrm{Gr}}) = \tau_i.$$

Proof. We know from Proposition 2.9 that $\tilde{g}_\lambda^{(k)} = \tilde{g}_\lambda$. Therefore, using Proposition 2.5, we see that

$$\alpha(\mathcal{O}_\lambda^{\mathrm{Gr}}) = \sigma^{-1}(\tilde{g}_\lambda^{(k)}) = \sigma^{-1}(\tilde{g}_\lambda) = g_\lambda.$$

$\square$

4.4. Correspondence of Schubert bases.

We consider the extended affine symmetric group $\hat{S}_{k+1}$, which is generated by $\{s_i \mid i \in I\} \cup \{\pi\}$ satisfying the same relations among s_i's and

$$\pi^{k+1} = id, \quad \pi s_i = s_{i+1}\pi.$$

We have $\hat{S}_{k+1} \cong S_{k+1} \ltimes P^\vee$, where $P^\vee$ is the coweight lattice of $\mathrm{SL}_{k+1}(\mathbb{C})$. The translation element associated to $-\varpi_i^\vee \in P^\vee$ is explicitly given by

$$t_{-\varpi_i^\vee} = \pi^{-i} x_{R_i}.$$

Example 4.7. For $k = 3$,

$$(4.9) \qquad t_{-\varpi_1^\vee} = \pi^{-1} s_2 s_3 s_0, \quad t_{-\varpi_2^\vee} = \pi^{-2} s_0 s_3 s_1 s_0, \quad t_{-\varpi_3^\vee} = \pi^{-3} s_2 s_1 s_0.$$

Let $w \in S_{k+1}$ be an *i-Grassmannian permutation*, i.e., $\mathrm{Des}(w) = \{i\}$. The set of all *i*-Grassmannian permutations in S_{k+1} is in bijection with $\mathcal{P}_i^{k+1-i}$. Explicitly, for an *i*-Grassmannian permutation w in S_{k+1}, the corresponding partition $\lambda \in \mathcal{P}_i^{k+1-i}$ is given by

$$(4.10) \qquad \lambda_{i+1-j} = w(j) - j \quad (1 \le j \le i).$$

For each partition λ in $\mathcal{P}_i^{k+1-i}$, we denote the corresponding *i*-Grassmannian permutation by $w_{\lambda,i}$. For $\lambda \in \mathcal{P}_i^{k+1-i}$, the *dual* partition of λ is the element $\lambda^\vee$ in $\mathcal{P}_i^{k+1-i}$ defined by $\lambda_j^\vee = k + 1 - i - \lambda_{i+1-j}$ $(1 \le j \le i)$.

Proposition 4.8 ([13]). *Let $w_{\lambda,i} \in S_{k+1}$ be an i-Grassmannian permutation. Then*

$$(4.11) \qquad \Phi_{k+1}(\mathfrak{G}_{\lambda,i}) = \frac{g_{(\lambda^\vee)'}}{\tau_i},$$

where $(\lambda^\vee)'$ is the conjugate of $\lambda^\vee$.

Proof. Let Φ'_{k+1} be the map in [13]. Recall that for $w \in S_{k+1}$, $\mathcal{O}_{G/B}^w$ is identified with $\mathfrak{G}_w^Q \bmod I_{k+1}$. [13, Theorem 7.1] reads as $\Phi'_{k+1}(\mathfrak{G}_{w_{\lambda,i}}^Q \bmod I_{k+1}) = g_{\lambda^\vee}/g_{(k+1-i)^i}$. From Remark 4.4, we deduce (4.11). $\square$

In order to describe the image $\Phi_{k+1}(\mathcal{O}_{G/B}^w)$, we need a map $S_{k+1} \to \mathcal{P}^k$, $w \mapsto \theta_k(w)$, due to Lam and Shimozono [24, Lemma 11].

Proposition 4.9 ([24]). *Let $w \in S_{k+1}$. There is a k-bounded partition $\theta_k(w)$ such that*

$$(4.12) \qquad wt_{-\sum_{i \in \mathrm{Des}(w)} \varpi_i^\vee} = \pi^{-\sum_{i \in \mathrm{Des}(w)} i} \cdot x_{\theta_k(w)}.$$

Proof. One can show that the left-hand side of (4.12) is an affine Grassmannian element in $\hat{S}_{k+1} = \langle \pi \rangle \ltimes \tilde{S}_{k+1}$ (see the first part of the proof of [24, Lemma 6.1]). Then such an element can be uniquely written in the form on the right-hand side of (4.12) for a *k*-bounded partition, which we denote by $\theta_k(w)$. $\square$

For an i-Grassmannian permutation $w = w_{\lambda,i}$, we have $\theta_k(w) = (\lambda^\vee)'$ ([13, Lemma 7.1]). See Appendix B, for another direct proof of this fact.

The next result is a refined version of [13, Conjecture 1.8].

Theorem 4.10. *For $w \in S_{k+1}$, we have*

$$(4.13) \qquad \Phi_{k+1}(\mathfrak{G}_w) = \frac{\sigma^{-1}(\tilde{g}^{(k)}_{\theta_k(w)})}{\prod_{i \in \mathrm{Des}(w)} \tau_i}.$$

Proof. Let $Q^\vee$ denote the coroot lattice of $\mathrm{SL}_{k+1}(\mathbb{C})$. Then we have $\tilde{S}_{k+1} \cong S_{k+1} \ltimes Q^\vee$. We denote by $t_\beta \in \tilde{S}_{k+1}$ the translation element corresponding to $\beta \in Q^\vee$. Let $K_*(\mathrm{Gr})_{\mathrm{loc}}$ be the localization of $K_*(\mathrm{Gr})$ by the multiplicative set generated by $\mathcal{O}^{\mathrm{Gr}}_{t_\beta}$ $(\beta \in Q^\vee)$.

Kato [15, Corollary 4.15] constructed an injective ring homomorphism

$$\kappa : K_*(\mathrm{Gr})_{\mathrm{loc}} \hookrightarrow QK(G/B)_{\mathrm{loc}}$$

such that

$$(4.14) \qquad \kappa\left(\mathcal{O}^{\mathrm{Gr}}_{wt_\beta} \cdot (\mathcal{O}^{\mathrm{Gr}}_{t_\gamma})^{-1}\right) = Q^{\beta - \gamma} \mathcal{O}^w_{G/B} \quad (w \in S_{k+1}),$$

where $\beta, \gamma \in -Q^\vee$ such that $wt_\beta \in \tilde{S}^0_{k+1}$ and γ is strictly antidominant, and $Q^{\beta - \gamma}$ is defined by identifying $Q^{\alpha_i^\vee}$ with Q_i.

Since $\mathcal{O}^{\mathrm{Gr}}_{R_i}$ corresponds to $\tau_i = g_{R_i}$ by α (Proposition 4.6), the isomorphism α yields $K_*(\mathrm{Gr})_{\mathrm{loc}} \cong \Lambda_{(k)}[\tau_i^{-1} \ (1 \le i \le k)]$. Let κ' be the composition

$$\begin{aligned}
\kappa' : K_*(\mathrm{Gr})_{\mathrm{loc}} &\cong \Lambda_{(k)}[\tau_i^{-1}] \quad \text{(induced by } \alpha) \\
&\hookrightarrow \Lambda_{(k)}[\tau_i^{-1}, (\tau_i^+)^{-1}] \\
&\underset{\Phi_{k+1}^{-1}}{\cong} A^{\mathrm{pol}}_{k+1}[Q_i^{-1}] \quad \text{(Theorem 4.3)} \\
&\underset{\phi}{\hookrightarrow} QK(G/B)[Q_i^{-1}] \quad \text{(Corollary 4.5)}.
\end{aligned}$$

We claim that $\kappa' = \kappa$. Note that $K_*(\mathrm{Gr})$ is generated by $\mathcal{O}^{\mathrm{Gr}}_{s_{i-1} \cdots s_1 s_0} = \mathcal{O}^{\mathrm{Gr}}_{(i)}$ $(1 \le i \le k)$, and that $\alpha(\mathcal{O}^{\mathrm{Gr}}_{(i)}) = g_{(i)}$. For $1 \le i \le k$, let $u_i \in S_{k+1}$ be the k-Grassmannian permutation of shape (1^{k-i}). So by Proposition 4.8, we have

$$\Phi_{k+1}(\mathfrak{G}^Q_{u_i}) = \frac{g_{(i)}}{\tau_k}.$$

Hence it follows that

$$\kappa'(\mathcal{O}^{\mathrm{Gr}}_{(i)} \cdot (\mathcal{O}^{\mathrm{Gr}}_{R_k})^{-1}) = \phi(\mathfrak{G}^Q_{u_i}) = \mathcal{O}^{u_i}_{G/B}.$$

By Lemma B.3, we have $\theta_k(u_i) = (i)$. Therefore, we have $\mathcal{O}^{\mathrm{Gr}}_{u_i t_{-\varpi_k^\vee}} = \mathcal{O}^{\mathrm{Gr}}_{(i)}$, and hence by the definition of κ

$$\kappa(\mathcal{O}^{\mathrm{Gr}}_{(i)} \cdot (\mathcal{O}^{\mathrm{Gr}}_{R_k})^{-1}) = \mathcal{O}^{u_i}_{G/B}.$$

Since $(\mathcal{O}^{\mathrm{Gr}}_{R_k})^{-1}$ is an invertible element, we have $\kappa'(\mathcal{O}^{\mathrm{Gr}}_{(i)}) = \kappa(\mathcal{O}^{\mathrm{Gr}}_{(i)})$, and hence $\kappa' = \kappa$.

For $w \in S_{k+1}$, we deduce that $\mathcal{O}^{\mathrm{Gr}}_{wt_{-\sum_{i \in \mathrm{Des}(w)} \varpi_i^{\vee}}} = \mathcal{O}^{\mathrm{Gr}}_{\theta_k(w)}$ by Proposition 4.9, and hence see that

$$
\phi(\mathfrak{S}^Q_w) = \mathcal{O}^w_{G/B} = \kappa \left(\mathcal{O}^{\mathrm{Gr}}_{\theta_k(w)} \cdot \prod_{i \in \mathrm{Des}(w)} (\mathcal{O}^{\mathrm{Gr}}_{R_i})^{-1} \right)
$$

$$
= \kappa' \left(\mathcal{O}^{\mathrm{Gr}}_{\theta_k(w)} \cdot \prod_{i \in \mathrm{Des}(w)} (\mathcal{O}^{\mathrm{Gr}}_{R_i})^{-1} \right)
$$

$$
= (\phi \circ \Phi^{-1}_{k+1}) \left(\sigma^{-1}(\tilde{g}_{\theta_k(w)}) \cdot \prod_{i \in \mathrm{Des}(w)} \tau_i^{-1} \right) \quad \text{(by (4.6))}.
$$

Since ϕ is injective we obtain (4.13). $\square$

4.5. **Localizations of $K_*(\mathrm{Gr})$.** The isomorphism conjectured by Lam, Li, Mihalcea, and Shimozono in [20] is different from Φ_{k+1} in the way of localization of $K_*(\mathrm{Gr})$. The localization of $K_*(\mathrm{Gr})$ in [20] can be identified with $\Lambda_{(k)}[\tau_i^{-1} \ (1 \leq i \leq k)]$, while our version is $\Lambda_{(k)}[\tau_i^{-1}, \ (\tau_i^+)^{-1} \ (1 \leq i \leq k)]$. The aim of this section is to clarify the geometric meaning of $\tau_i^+ = \tilde{g}_{R_i} \in \Lambda_{(k)}$.

Lemma 4.11. $\tau_i^2 - \tau_{i-1}\tau_{i+1} = \tau_i^+ \cdot \tau_i^-$.

Proof. This is the discrete Toda equation given by Hirota [12]. We can show this by comparing the construction of Φ_{k+1} [13] and the Lax formalism for the discrete Toda equation given in [14, §1]. $\square$

Lemma 4.12. *For $1 \leq i \leq k$, $\Phi_{k+1}(1 - Q_i) = \dfrac{\tau_i^+ \cdot \tau_i^-}{\tau_i^2}$.*

Proof. This follows from Lemma 4.11. $\square$

Lemma 4.13. *For $1 \leq i \leq k+1$,*

$$
\tau_i = g_{R_i^*} + \tau_i^-.
$$

Proof. It suffices to show that $\sigma(\tau_i - g_{R_i^*}) = \tau_i$. This follows from (2.3), because R_i^* is the unique maximal proper element among the partitions $\mu \subset R_i$. $\square$

Proposition 4.14. *For $1 \leq i \leq k$,*

$$
(1 - Q_i)z_1 \cdots z_i = 1 - \mathcal{O}^{s_i}_{G/B}
$$

in $QK(G/B)$.

Proof. By the equality $\Phi_{k+1}(\mathcal{O}^{s_i}) = g_{R_i^*}/\tau_i$ [13, Theorem 1.7], and Lemma 4.13, the image of the right-hand side by Φ_{k+1} is

$$
1 - \frac{g_{R_i^*}}{\tau_i} = \frac{\tau_i - g_{R_i^*}}{\tau_i} = \frac{\tau_i^-}{\tau_i}
$$

under the isomorphism of Proposition 4.2. On the other hand,

$$
\Phi_{k+1}((1 - Q_i)z_1 \cdots z_i) = \frac{\tau_i^- \tau_i^+}{\tau_i^2} \prod_{j=1}^{i} \frac{\tau_j \tau_{j-1}^+}{\tau_j^+ \tau_{j-1}} = \frac{\tau_i^-}{\tau_i}.
$$

$\square$

Remark 4.15. In view of Proposition 4.2, the previous result corresponds to [27, Corollary 3.33]; note also that

$$1 - \mathcal{O}_{G/B}^{s_i} = \mathcal{O}_{G/B}(-\varpi_i).$$

Corollary 4.16. *The following element*

$$Q_{t_{\varpi_i^\vee}}^{-1} (1 - Q_i)^{-1} (1 - \mathcal{O}_{G/B}^{s_i})$$

in $QK(G/B)_{\mathrm{loc}}$ is sent to $1/\tau_i^+$ by Φ_{k+1}, where $Q_{t_{\varpi_i^\vee}}$ is the element such that $\Phi_{k+1}(Q_{t_{\varpi_i^\vee}}) = \tau_i$.

Note that the factor $Q_{t_{\varpi_i^\vee}}$ is invertible in $QK(G/B)_{\mathrm{loc}}$. Hence, up to this invertible factor, $1/\tau_i^+$ corresponds to the element $(1 - Q_i)^{-1}(1 - \mathcal{O}_{G/B}^{s_i})$ of $QK(G/B) \subset QK(G/B)_{\mathrm{loc}}$.

Appendix A. Parabolic quotient of Coxeter groups

We discuss some properties of a coset space of a Coxeter group. Our basic reference is Bjorner–Brenti [4]. Let (W, S) be a *Coxeter system* (see [4] for the definition), where W is the group generated by $S = \{s_i \mid i \in I\}$ with index set I. The *Bruhat order* on W (see [4, Chapter 2]) is denoted by $\leq$. Let J be any subset of I. Let W_J be the subgroup of W generated by s_i $(i \in J)$. The *minimal coset representatives* W^J of the quotient W/W_J is defined to be $W^J := \{w \in W \mid ws_i > w \text{ for all } i \in J\}$. Any element w of W is expressed uniquely as $w = w^J w_J$, with $w_J \in W_J$ and $w^J \in W^J$ [4, Proposition 2.4.4].

The following result is well-known and used throughout this section.

Lemma A.1. *Let $x \in W^J$ and $i \in I$. Then, $s_i x < x \Longrightarrow s_i x \in W^J$.*

A.1. **Proof of Proposition 3.13.** The 0-*Hecke algebra* H_W is the associative $\mathbb{C}$-algebra generated by $\{T_i \mid i \in I\}$ subject to the same relations as those for W except $T_i^2 = -T_i$ in place of $s_i^2 = id$. For $w \in W$, define $T_w = T_{i_1} \cdots T_{i_m}$ for any reduced expression $w = s_{i_1} \cdots s_{i_m}$. The elements T_w $(w \in W)$ form a basis of H_W.

Proposition A.2. *Let V^J be a left H_W module given by*

$$V^J = H_W e_J, \quad e_J := \sum_{w \in W_J} T_w.$$

Then, $V^J = \bigoplus_{x \in W^J} \mathbb{C} a_x$ with $a_x := T_x e_J$, and for $i \in I$,

$$(\mathrm{A.1}) \qquad T_i \cdot a_x = \begin{cases} a_{s_i x} & (s_i x > x \text{ and } s_i x \in W^J) \\ -a_x & (s_i x < x) \\ 0 & (s_i x > x \text{ and } s_i x \notin W^J) \end{cases}.$$

Proof. For $i \in J$, we will prove $T_i e_J = 0$. Let $X^+ = \{v \in W \mid s_i v > v\}$ and $X^- = \{v \in W \mid s_i v < v\}$. We have $s_i(X^\pm) = X^\mp$. If $v \in X^+$, then $T_i T_v = T_{s_i v}$. If $v \in X^-$, then $T_v = T_i T_{s_i v}$, and hence $T_i T_v = T_i^2 T_{s_i v} = -T_i T_{s_i v} = -T_v$. Therefore,

$$T_i e_J = \sum_{v \in X^+} T_i T_v + \sum_{v \in X^-} T_i T_v = \sum_{v \in X^+} T_{s_i v} - \sum_{v \in X^-} T_v = 0.$$

If $v \notin W^J$, then it is easy to see that there is a reduced expression $v = s_{i_1} \cdots s_{i_m}$ with $i_m \in J$. Hence $T_v e_J = T_{s_{i_1} \cdots s_{i_{m-1}}} T_{i_m} e_J = 0$. It follows that $\{a_x\}_{x \in W^J}$

spans $V^J = H_W e_J$. Note that for $x \in W^J$ and $v \in W_J$, we have $T_x T_v = T_{xv}$ since $\ell(xv) = \ell(x) + \ell(v)$ ([4, Proposition 2.4.4(2)]); the linear independence of $\{a_x\}_{x \in W^J}$ follows from this fact.

Let $x \in W^J$. If $s_i x > x$, then

$$T_i \cdot a_x = T_i \cdot T_x e_J = T_{s_i x} e_J = \begin{cases} a_{s_i x} & (s_i x \in W^J) \\ 0 & (s_i x \notin W^J) \end{cases}.$$

If $s_i x < x$, then $T_i \cdot a_x = T_i T_x e_J = -T_x e_J = -a_x$.   □

Proposition A.3. *For $x \in W^J$, let $b_x := \sum_{y \leq x} a_y$. Set $D_i = T_i + 1$ ($i \in I$). Then*

$$(A.2) \qquad D_i \cdot b_x = \begin{cases} b_{s_i x} & (s_i x \in W^J, \ s_i x > x) \\ b_x & (\text{otherwise}) \end{cases}.$$

Lemma A.4 (Z-lemma). *Let $w, v \in W$ and $i \in I$. Suppose $s_i w > w$ and $s_i v > v$. Then the following conditions are equivalent:*

$$(1) \ w \leq v, \quad (2) \ s_i w \leq s_i v, \quad (3) \ w \leq s_i v.$$

Proof. [30, Proposition 5.4.3].   □

Lemma A.5. *Let $x \in W^J$ and $i \in I$. We set*

$$X^+_{\leq x} := \{y \in W^J \mid y \leq x, \ s_i y \in W^J, \ s_i y > y\},$$
$$X^-_{\leq x} := \{y \in W^J \mid y \leq x, \ s_i y \in W^J, \ s_i y < y\},$$
$$X^0_{\leq x} := \{y \in W^J \mid y \leq x, \ s_i y \notin W^J\}.$$

(1) *If $s_i x \in W^J$, $s_i x > x$, then $s_i(X^+_{\leq x}) = X^-_{\leq s_i x}$, $X^+_{\leq x} = X^+_{\leq s_i x}$, $X^0_{\leq x} = X^0_{\leq s_i x}$.*

(2) *If $s_i x \in W^J$, $s_i x < x$, then $s_i(X^+_{\leq x}) = X^-_{\leq x}$.*

(3) *If $s_i x \notin W^J$, then $s_i(X^+_{\leq x}) = X^-_{\leq x}$.*

Proof. (1) and (2) follow immediately from Lemma A.4. For (3), the inclusion $s_i(X^+_{\leq x}) \supset X^-_{\leq x}$ follows from Lemma A.4. We will show that $s_i(X^+_{\leq x}) \subset X^-_{\leq x}$. Take arbitrary $z \in s_i(X^+_{\leq x})$. Write $z = s_i y$ with $y \in X^+_{\leq x}$. Since $s_i z = s_i^2 y = y < s_i y = z$, it suffices to prove $s_i y \leq x$. Since $x \in W^J$ and $s_i x \notin W^J$, we have $s_i x > x$ by Lemma A.1. Hence it follows from Lemma A.4 that $s_i x \geq s_i y$. Also, it follows from [4, Corollary 2.5.2] that there is $v \in W_J$ such that $s_i x = xv$; in fact, we can take $v = s_j$ for some $j \in J$. Hence we have $xv \geq s_i y$. Here note that $s_i y \in W^J$ as $y \in X^+_{\leq x}$. It follows that

$$s_i y = (s_i y)^J \leq (xv)^J = x^J = x;$$

here we used the fact that $w \leq v$ for $w, v \in W$ implies $w^J \leq v^J$ [4, Proposition 2.5.1].   □

Proof of Proposition A.3. We have $b_x = \sum_{y \in X^+_{\leq x}} a_y + \sum_{y \in X^-_{\leq x}} a_y + \sum_{y \in X^0_{\leq x}} a_y$. Using (A.1), it is straightforward to verify

$$D_i \cdot b_x = \sum_{y \in X^+_{\leq x}} a_{s_i y} + \sum_{y \in X^+_{\leq x}} a_y + \sum_{y \in X^0_{\leq x}} a_y.$$

Consider first the case that $s_i x \in W^J$, $s_i x > x$. From Lemma A.5(1), we see that

$$D_i \cdot b_x = \sum_{y \in X^-_{\leq s_i x}} a_y + \sum_{y \in X^+_{\leq s_i x}} a_y + \sum_{y \in X^0_{\leq s_i x}} a_y = b_{s_i x}.$$

Next consider the case that $s_i x \in W^J$, $s_i x < x$. We see that $s_i(X^+_{\leq x}) = X^-_{\leq x}$ by Lemma A.5(2), and hence we see that

$$D_i \cdot b_x = \sum_{y \in X^-_{\leq x}} a_y + \sum_{y \in X^+_{\leq x}} a_y + \sum_{y \in X^0_{\leq x}} a_y = b_x.$$

Finally we consider the case that $s_i x \notin W^J$. We have $s_i(X^+_{\leq x}) = X^-_{\leq x}$ by Lemma A.5(3), and hence $D_i \cdot b_x = b_x$ by exactly the same reasoning as in the previous case. $\qquad\square$

Proof of Proposition 3.13. We apply Proposition A.2 to $W = \tilde{S}_{k+1} = \langle s_i \mid i = 0, 1, \ldots, k \rangle$, $W_J = S_{k+1}$, $W^J = \tilde{S}^0_{k+1}$ with $J = \{1, 2, \ldots, k\}$, and define an isomorphism $V^J \to \Lambda_{(k)}$ of vector spaces by $a_x \mapsto \overset{\circ}{\mathfrak{g}}{}^{(k)}_\lambda$, $b_x \mapsto \tilde{\mathfrak{g}}^{(k)}_\lambda$ under the bijection $W^J = \tilde{S}^0_{k+1} \ni x \mapsto \lambda \in \mathcal{P}^k$. Then we obtain Proposition 3.13. $\qquad\square$

A.2. **Proof of Lemma 3.18.** For $w \in W$, and $i \in I$, we define $s_i * w$ in the same way as in (2.4). Let $\mathcal{S} = (i_1, \ldots, i_r)$ be a sequence of elements of I, and $w \in S$. We define

$$(A.3) \qquad \mathcal{S} * w = s_{i_1} * (s_{i_2} * \cdots * (s_{i_r} * w) \cdots).$$

Proposition A.6. *Let $\mathcal{S} = (i_1, \ldots, i_r)$ be a sequence of elements of I. Set $|\mathcal{S}| = r$, and $T_\mathcal{S} := T_{i_1} \cdots T_{i_r} \in H_W$. Then for $x \in W^J$, we have*

$$T_\mathcal{S} \cdot a_x = \begin{cases} (-1)^{|\mathcal{S}| - \ell(\mathcal{S}*x) + \ell(x)} a_{\mathcal{S}*x} & (\mathcal{S} * x \in W^J) \\ 0 & (\mathcal{S} * x \notin W^J) \end{cases}.$$

Proof. For $1 \leq p \leq r$, set $\mathcal{S}_p = (i_p, \ldots, i_r)$, and $\mathcal{S}_{r+1} = \varnothing$. We use decreasing induction on p. If $p = r + 1$, then the assertion is obvious. Suppose $1 \leq p < r + 1$. We first consider the case that $\mathcal{S}_p * x \notin W^J$. Let $q\ (\geq p)$ be the maximal integer such that $\mathcal{S}_q * x \notin W^J$. Then, we have $\mathcal{S}_{q+1} * x \in W^J$, $\mathcal{S}_q * x = s_{i_q} * (\mathcal{S}_{q+1} * x) = s_{i_q}(\mathcal{S}_{q+1} * x) \notin W^J$, and so $s_{i_q}(\mathcal{S}_{q+1} * x) > \mathcal{S}_{q+1} * x$ by Lemma A.1. Therefore, $T_{i_q} \cdot a_{\mathcal{S}_{q+1}*x} = 0$. By the inductive hypothesis, we deduce that $T_{i_{q+1}} \cdots T_{i_r} \cdot a_x = \pm a_{\mathcal{S}_{q+1}*x}$, and hence

$$T_{\mathcal{S}_p} \cdot a_x = T_{i_p} \cdots T_{i_q}(T_{i_{q+1}} \cdots T_{i_r} \cdot a_x) = \pm T_{i_p} \cdots T_{i_q} \cdot a_{\mathcal{S}_{q+1}*x} = 0.$$

Next we consider the case when $\mathcal{S}_p * x \in W^J$. Note that, in view of Lemma A.1, we have $\mathcal{S}_q * x \in W^J$ for $p \leq q \leq r + 1$. By the inductive hypothesis, we have $T_{i_{p+1}} \cdots T_{i_r} a_x = (-1)^{r-p-\ell(\mathcal{S}_{p+1}*x) + \ell(x)} \cdot a_{\mathcal{S}_{p+1}*x}$, so

$$(A.4) \qquad T_{\mathcal{S}_p} \cdot a_x = T_{i_p} \cdot (T_{i_{p+1}} \cdots T_{i_r} a_x) = (-1)^{r-p-\ell(\mathcal{S}_{p+1}*x)+\ell(x)} T_{i_p} \cdot a_{\mathcal{S}_{p+1}*x}.$$

Now we consider two cases: (a) $s_{i_p}(\mathcal{S}_{p+1} * x) > \mathcal{S}_{p+1} * x$, (b) $s_{i_p}(\mathcal{S}_{p+1} * x) < \mathcal{S}_{p+1} * x$. If (a) holds, then $s_{i_p}(\mathcal{S}_{p+1} * x) = s_{i_p} * (\mathcal{S}_{p+1} * x) = \mathcal{S}_p * x \in W^J$. Therefore,

$$T_{i_p} \cdot a_{\mathcal{S}_{p+1}*x} = a_{s_{i_p}(\mathcal{S}_{p+1}*x)} = a_{\mathcal{S}_p*x},$$

and hence by (A.4),

$$T_{\mathcal{S}_p} \cdot a_x = (-1)^{r-p-\ell(\mathcal{S}_{p+1}*x)+\ell(x)} T_{i_p} a_{\mathcal{S}_{p+1}*x} = (-1)^{r-p-\ell(\mathcal{S}_{p+1}*x)+\ell(x)} a_{\mathcal{S}_p*x};$$

the sign is correct since we have $\ell(\mathcal{S}_{p+1} * x) = \ell(\mathcal{S}_p * x) - 1$. Suppose (b) holds. Note that $\mathcal{S}_p * x = \mathcal{S}_{p+1} * x \in W^J$ by (A.3). By the definition of T_{i_p},

$$T_{i_p} \cdot a_{\mathcal{S}_{p+1}*x} = -a_{\mathcal{S}_{p+1}*x} = -a_{\mathcal{S}_p*x}.$$

Hence we deduce that

$$T_{\mathcal{S}_p} \cdot a_x = (-1)^{r-p-\ell(\mathcal{S}_{p+1}*x)+\ell(x)}(-a_{\mathcal{S}_p*x}) = (-1)^{r-p+1-\ell(\mathcal{S}_p*x)+\ell(x)}a_{\mathcal{S}_p*x},$$

where we again used $\mathcal{S}_p * x = \mathcal{S}_{p+1} * x$. This completes the proof. □

Proof of Lemma 3.18. Let $A \subsetneq I$, with $|A| = r$, and take a reduced expression $s_{i_1} \cdots s_{i_r}$ for u_A. We apply Proposition A.6 to the sequence $\mathcal{S} = (i_1, \ldots, i_r)$. Then $T_{\mathcal{S}} = T_{u_A}$ and we obtain Lemma 3.18. □

APPENDIX B. GRASSMANNIAN PERMUTATIONS

An explicit description of $\theta_k(w)$ is available (see [24, §6]). For the reader's convenience, we include a simple direct proof when w is a Grassmannian element $w_{\lambda,i}$ (Lemma B.3).

Proposition B.1. *Let $\lambda \in \mathcal{P}^k$ be a k-bounded partition of size r such that $\lambda_1 + \ell(\lambda) \leq k+1$. Take any standard tableau T of shape λ. We denote the box of λ with entry i in T by $b_T(i)$. Then $s_{\mathrm{res}(b_T(r))} \cdots s_{\mathrm{res}(b_T(2))} s_{\mathrm{res}(b_T(1))}$ is a reduced expression for x_λ.*

Proof. We note first that the corresponding fact is well-known for the i-Grassmannian permutation $w_{\lambda,i}$. The reader can consult [6, §3.1] for an exposition of this fact in a more general setting.

Since $\lambda_1 + \ell(\lambda) \leq k+1$, there is $1 \leq i \leq k$ such that $\lambda \subset R_{k+1-i}$. Then in the reduced expression of x_λ given by (2.1), s_{k+1-i} does not appear. Hence $x_\lambda \in \langle s_{-i+1}, \ldots, s_{-i+k} \rangle \cong S_{k+1}$; the isomorphism of groups $\phi \colon S_{k+1} \to \langle s_{-i+1}, \ldots, s_{-i+k} \rangle$ is given by $\phi(s_j) = s_{-i+j}$. Then, the result follows from the case for the i-Grassmannian permutation. □

Remark B.2. An element in a Coxeter group W is *fully commutative* if any two of its reduced expressions are related by a series of transpositions of adjacent commuting generators. It is well-known that any i-Grassmannian element in S_{k+1} is fully commutative. From the proof of Proposition B.1, we see that if $\lambda_1 + \ell(\lambda) \leq k+1$, then x_λ is fully commutative.

Lemma B.3. *Let λ be a partition contained in $R_{k+1-i} = (k+1-i)^i$, and $w_{\lambda,i} \in S_{k+1}$ the corresponding i-Grassmannian permutation. Then $\theta_k(w_{\lambda,i}) = (\lambda^\vee)'$.*

Proof. In the extended affine symmetric group $\hat{S}_{k+1}$, we compute $w_{\lambda,i} t_{-\varpi_i^\vee} = w_{\lambda,i} \pi^{-i} x_{R_i} = \pi^{-i}(\pi^i w_{\lambda,i} \pi^{-i}) x_{R_i}$. A reduced expression for $y_\lambda := \pi^i w_{\lambda,i} \pi^{-i}$ is obtained by replacing s_j with s_{j+i} in $w_{\lambda,i}$. It is straightforward to see when we reflect the tableau of y_λ along the line with a slope of 1, it fits inside the tableau of shape R_i filled with $(k+1)$-residues adjusted to the south-east corner (see Example B.4). Let T be the standard tableau with shape R_i that is obtained by filling positive integers into each row of $(\lambda^\vee)'$ from left to right, with rows taken from top to bottom, and then into each column of the remaining boxes from top to bottom, with columns taken from left to right. Then apply Proposition B.1 to T. The obtained reduced expression for x_{R_i} shows $x_{R_i} = y_\lambda^{-1} \cdot x_{(\lambda^\vee)'}$, and hence $w_{\lambda,i} t_{-\varpi_i^\vee} = \pi^{-i} x_{(\lambda^\vee)'}$, showing $\theta_k(w_{\lambda,i}) = (\lambda^\vee)'$. □

Example B.4. For $k = 6, i = 3$, and $\lambda = (3,2)$, the corresponding 3-Grassmannian element is $w_{\lambda,3} = s_3 s_2 \cdot s_5 s_4 s_3$. We have

$$y_\lambda := \pi^3(w_{\lambda,3})\pi^{-3} = s_6 s_5 \cdot s_1 s_0 s_6 : \quad \begin{array}{|c|c|c|} \hline 6 & 0 & 1 \\ \hline 5 & 6 & \\ \hline \end{array}.$$

We read the entries of the tableau of shape R_3 filled with 7-residues $\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 6 & 0 & 1 \\ \hline 5 & 6 & 0 \\ \hline 4 & 5 & 6 \\ \hline \end{array}$ according

to the order given by $T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 10 \\ \hline 6 & 8 & 11 \\ \hline 7 & 9 & 12 \\ \hline \end{array}$ to obtain

$$x_{R_3} = (s_6 s_0 s_1 \cdot s_5 s_6) \cdot s_4 s_5 s_0 s_6 s_2 s_1 s_0.$$

This is $y_\lambda^{-1} x_{(\lambda^\vee)'}$. The shaded boxes correspond to y_λ.

APPENDIX C. VERTICAL PIERI RULE FOR THE CLOSED K-k-SCHUR FUNCTIONS

A Pieri rule for $\tilde{g}_\lambda^{(k)}$ was proved by Takigiku [40]. We record here the vertical version of Takigiku's formula, which should be known to experts but is missing in the literature. We do not use this result in the main part of this paper.

For $\lambda \in \mathcal{P}^k$, $1 \leq r \leq r$, define

$$\mathcal{H}_{\lambda,r}^{(k)} := \{A \subsetneq I \ \big| \ |A| = r, \ d_A x_\lambda \in \tilde{S}_{k+1}^0, \ d_A x_\lambda \geq_L x_\lambda\} ,$$

$$\mathcal{V}_{\lambda,r}^{(k)} := \{A \subsetneq I \ \big| \ |A| = r, \ u_A x_\lambda \in \tilde{S}_{k+1}^0, \ u_A x_\lambda \geq_L x_\lambda\} .$$

For an element A in $\mathcal{H}_{\lambda,r}^{(k)}$, we write $d_A x_\lambda = x_\kappa$ for an element $\kappa \in \mathcal{P}^k$, and denote this κ by $d_A \lambda$. Similarly, for an element A in $\mathcal{V}_{\lambda,r}^{(k)}$, we write $u_A x_\lambda = x_\kappa$ for an element $\kappa \in \mathcal{P}^k$, and denote this κ by $u_A \lambda$.

Example C.1. For $k = 3$, $\lambda = (2,1) \in \mathcal{P}^3$, and $r = 2$, we have (see Figure 2)

$$\mathcal{H}_{\lambda,r}^{(k)} = \{\{2,3\}, \{0,2\}\}, \quad d_{\{2,3\}} = s_3 s_2, \quad d_{\{0,2\}} = s_0 s_2 = s_2 s_0,$$

$$\mathcal{V}_{\lambda,r}^{(k)} = \{\{1,2\}, \{0,2\}\}, \quad u_{\{1,2\}} = s_1 s_2, \quad u_{\{0,2\}} = s_0 s_2 = s_2 s_0.$$
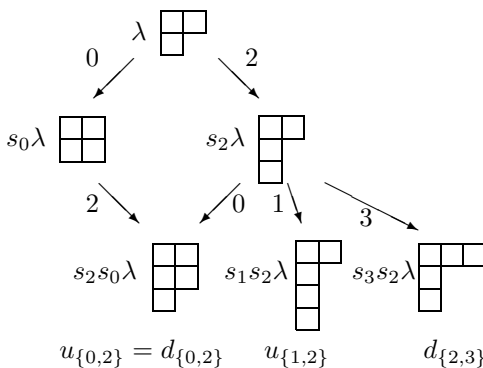

FIGURE 2. $k = 3$, $\lambda = (2,1)$, $r = 2$.

The corresponding weak (horizontal and vertical) strips are given by

$$d_{\{2,3\}}\lambda = (3,1,1), \quad d_{\{0,2\}}\lambda = u_{\{0,2\}}\lambda = (2,1,1), \quad u_{\{1,2\}}\lambda = (2,1,1,1).$$

For $A_1, \ldots, A_m \in \mathcal{H}_{\lambda,r}^{(k)}$, it is known by Takigiku [40, Corollary 4.8] that $A_1 \cap \cdots \cap A_m \in \mathcal{H}_{\lambda,r'}^{(k)}$, with $r' = |A_1 \cap \cdots \cap A_m|$ and hence $d_{A_1 \cap \cdots \cap A_m}\lambda \in \mathcal{P}^k$ is defined.

Proposition C.2. *Let $A \mapsto \overline{A}$ be the map given by sending $i \in A$ to $-i \in \overline{A}$. Then for $\lambda \in \mathcal{P}^k$, and $1 \le r \le k$,*

$$\overline{\mathcal{H}_{\lambda,r}^{(k)}} = \mathcal{V}_{\lambda^{\omega_k},r}^{(k)}.$$

Proof. Let $A \in \mathcal{H}_{\lambda,r}^{(k)}$. Recall that ω_k is an automorphism of the group $\tilde{S}_{k+1}$ preserving the left weak order. Therefore, it follows that

$$u_{\overline{A}}x_{\lambda^{\omega_k}} = \omega_k(d_A)\omega_k(x_\lambda) = \omega_k(d_A x_\lambda) \ge_L \omega_k(x_\lambda) = x_{\lambda^{\omega_k}},$$

and hence $\overline{A} \in \mathcal{V}_{\lambda,r}^{(k)}$. Thus $\overline{\mathcal{H}_{\lambda,r}^{(k)}} \subset \mathcal{V}_{\lambda^{\omega_k},r}^{(k)}$. Similarly we have $\overline{\mathcal{H}_{\lambda,r}^{(k)}} \supset \mathcal{V}_{\lambda^{\omega_k},r}^{(k)}$. $\square$

Theorem C.3 ([39]). *Let $\lambda \in \mathcal{P}^k$ and $1 \le r \le k$. Let*

$$\mathcal{H}_{\lambda,r}^{(k)} = \{A_1, \ldots, A_m\}, \quad \mathcal{V}_{\lambda,r}^{(k)} = \{B_1, \ldots, B_n\}.$$

Then

(C.1) $$\tilde{g}_{(r)} \cdot \tilde{g}_\lambda^{(k)} = \sum_{i=1}^{m}(-1)^{i-1} \sum_{1 \le a_1 < \cdots < a_i \le m} \tilde{g}_{d_{A_{a_1} \cap \cdots \cap A_{a_i}}\lambda}^{(k)},$$

(C.2) $$\tilde{g}_{(1^r)} \cdot \tilde{g}_\lambda^{(k)} = \sum_{i=1}^{n}(-1)^{i-1} \sum_{1 \le a_1 < \cdots < a_i \le n} \tilde{g}_{u_{B_{a_1} \cap \cdots \cap B_{a_i}}\lambda}^{(k)}.$$

Proof. (C.1) is due to Takigiku [39]. From Proposition C.2, it follows that $\{\overline{A_1}, \ldots, \overline{A_m}\} = \mathcal{V}_{\lambda^{\omega_k},r}^{(k)}$. By applying Ω to both sides of the equation, we obtain

$$\tilde{g}_{(1^r)} \cdot \tilde{g}_{\lambda^{\omega_k}}^{(k)} = \sum_{i=1}^{m}(-1)^{i-1} \sum_{1 \le a_1 < \cdots < a_i \le m} \tilde{g}_{u_{\overline{A_{a_1}} \cap \cdots \cap \overline{A_{a_i}}}\lambda^{\omega_k}}^{(k)},$$

where we used $\Omega(\tilde{g}_{(r)}) = \tilde{g}_{(1^r)}$ and

$$\Omega(\tilde{g}_{d_{A_{a_1} \cap \cdots \cap A_{a_i}}\lambda}^{(k)}) = \tilde{g}_{u_{\overline{A_{a_1} \cap \cdots \cap A_{a_i}}}\lambda^{\omega_k}}^{(k)} = \tilde{g}_{u_{\overline{A_{a_1}} \cap \cdots \cap \overline{A_{a_i}}}\lambda^{\omega_k}}^{(k)}.$$

Thus we have (C.2). $\square$

Example C.4. According to Example C.1, we have

$$\tilde{g}_{(2)}^{(3)}\tilde{g}_{(2,1)}^{(3)} = \tilde{g}_{(3,1,1)}^{(3)} + \tilde{g}_{(2,2,1)}^{(3)} - \tilde{g}_{(2,1,1)}^{(3)},$$
$$\tilde{g}_{(1,1)}^{(3)}\tilde{g}_{(2,1)}^{(3)} = \tilde{g}_{(2,1,1,1)}^{(3)} + \tilde{g}_{(2,2,1)}^{(3)} - \tilde{g}_{(2,1,1)}^{(3)}.$$

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