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FRACTIONAL ITO CALCULUS

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ABSTRACT. We derive Itô–type change of variable formulas for smooth functionals of irregular paths with nonzero pth variation along a sequence of partitions, where $p \geq 1$ is arbitrary, in terms of fractional derivative operators. Our results extend the results of the Föllmer–Itô calculus to the general case of paths with 'fractional' regularity. In the case where p is not an integer, we show that the change of variable formula may sometimes contain a nonzero 'fractional' Itô remainder term and provide a representation for this remainder term. These results are then extended to functionals of paths with nonzero ϕ-variation and multidimensional paths. Using these results, we derive an isometry property for the pathwise Föllmer integral in terms of ϕ-variation.

CONTENTS

Hans Föllmer derived [15] a pathwise formulation of the Itô formula and laid the grounds for the development of a pathwise approach to Itô calculus, which has been developed in different directions [1, 2, 5–8, 11, 12, 22, 28].

Föllmer's original approach focuses on functions of paths with finite *quadratic* variation along a sequence of partitions. In a recent work Cont and Perkowski [11] extended the Föllmer–Itô formula [15] to function(al)s of paths with variation of order $p \in 2\mathbb{N}$ along a sequence of partitions and obtained functional change of variable formulas, applicable to functionals of fractional Brownian motion and other fractional processes with arbitrarily low regularity (i.e., any Hurst exponent $H > 0$). These results involve pathwise integrals defined as limits of compensated left Riemann sums, which are in turn related to rough integrals associated with a "reduced" rough path [11].

As the notion of pth order variation may be defined for any $p > 0$, an interesting question is to investigate how the results in [11] extend to 'fractional' case $p \notin \mathbb{N}$. In particular one may ask whether the change of variable formula contains a fractional remainder term in this case and whether the definition of the compensated integral needs to be adjusted.

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We investigate these questions using the tools of fractional calculus [27]. Given that fractional derivative operators are (nonlocal) integral operators, one challenge is to obtain nonanticipative, 'local' formulas which have similar properties to those obtained in the integer case [11]. We are able to do so using a 'local' notion of fractional derivative and exhibit conditions under which these change of variable formulas contain (or not) a 'fractional Itô remainder term'. In most cases there is no remainder term; we also discuss some cases where a nonzero remainder term appears and give a representation for this term.

These results are first derived for smooth functions then extended to functionals, using the concept of vertical derivative [10]. We extend these results to the case of paths with finite ϕ-variation [19] for a class of functions ϕ and we obtain an isometry formula for the pathwise integral in terms of ϕ-variation, extending the results of [1, 11] to the fractional case. Finally, we extend these results to the multidimensional case.

Our change of variable formulas are purely analytical and pathwise in nature, but applicable to functionals of fractional Brownian motions and other fractional processes with arbitrary Hurst exponent, leading in this case to nonanticipative 'Itô' formulas for functionals of such processes. However, as probabilistic assumptions play no role in the derivation of our results, we have limited the discussion of such examples.

Zähle [30] defined a pathwise integral using a Young–type condition based on fractional regularity of the integrand and integrator. In the case of Hölder continuous functions this corresponds to Hölder exponents with sum strictly greater than 1, i.e., a Young–type condition. Our approach extends beyond the domain of validity of Young integration and we are able to treat borderline cases where the sum of Hölder exponents is one.

Outline. Section 1 recalls some results on pathwise calculus for functions of irregular paths (Section 1.1) and fractional derivative operators and associated fractional Taylor expansions (Section 1.2). Section 2 contains our main results on change of variable formulas for function(al)s of paths with fractional regularity. We first give a change of variable formula without remainder term for time-independent functions (Theorem 2.6), followed by a discussion of an example where a remainder term may appear (Example 2.8). We then provide a formula for computing this fractional remainder term using an auxiliary space (Theorem 2.11). Section 2.2 extends these results to the case of path-dependent functionals using the Dupire derivative, to the case where the pth variation is replaced by the more general concept of ϕ-variation [19]. In Section 3 we derive a pathwise isometry formula extending a result of [1] to the case of ϕ-variation. Finally, in Section 4 we discuss extensions to the multidimensional case. These extensions are not immediate, as the space $V_p(\pi)$ is not a vector space.

1. Preliminaries

1.1. **Pathwise calculus for paths with finite pth variation.** We define, following [11, 15], the concept of pth variation along a sequence of partitions $\pi_n = \{t_0^n, \dots, t_{N(\pi_n)}^n\}$ with $t_0^n = 0 < \dots < t_k^n < \dots < t_{N(\pi_n)}^n = T$. Define the *oscillation* of $S \in C([0,T], \mathbb{R})$ along π_n as

$$\operatorname{osc}(S, \pi_n) := \max_{[t_j, t_{j+1}] \in \pi_n} \max_{r,s \in [t_j, t_{j+1}]} |S(s) - S(r)|.$$

We write $[t_j, t_{j+1}] \in \pi_n$ to indicate that t_j and t_{j+1} are immediate successors in π_n (i.e., $t_j < t_{j+1}$ and $\pi_n \cap (t_j, t_{j+1}) = \emptyset$).

Definition 1.1 (pth variation along a sequence of partitions)**.** Let $p > 0$. A continuous path $S \in C([0,T], \mathbb{R})$ is said to have a pth variation along a sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ if $\operatorname{osc}(S, \pi_n) \to 0$ and the sequence of measures

$$\mu^n := \sum_{[t_j,t_{j+1}]\in\pi_n} \delta(\cdot - t_j)|S(t_{j+1}) - S(t_j)|^p$$

converges weakly to a measure μ without atoms. In that case we write $S \in V_p(\pi)$ and $[S]^p(t) := \mu([0,t])$ for $t \in [0,T]$, and we call $[S]^p$ the *pth variation* of S.

Remark 1.2. Functions in $V_p(\pi)$ do not necessarily have finite p-variation in the usual sense. Recall that the p-variation of a function $f \in C([0,T], \mathbb{R})$ is defined as [13]

$$\|f\|_{\text{p-var}} := \Big(\sup_{\pi\in\Pi([0,T])} \sum_{[t_j,t_{j+1}]\in\pi} |f(t_{j+1}) - f(t_j)|^p \Big)^{1/p},$$

where the supremum is taken over the set $\Pi([0,T])$ of all partitions π of $[0,T]$. A typical example is the Brownian motion B, which has quadratic variation $[B]^2(t) = t$ along any refining sequence of partitions almost surely while at the same time having infinite 2-variation almost surely [13, 29]:

$$\mathbb{P}\left(\|B\|_{\text{2-var}} = \infty\right) = 1.$$

If $S \in V_p(\pi)$ and $q > p$, then $S \in V_q(\pi)$ with $[S]^q \equiv 0$.

$S \in C([0,T], \mathbb{R})$ belongs to $V_p(\pi)$ if and only if there exists a continuous function $[S]^p$ such that

$$\forall t \in [0,T], \qquad \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_j\leq t}} |S(t_{j+1}) - S(t_j)|^p \overset{n\to\infty}{\longrightarrow} [S]^p(t). \tag{1}$$

If this property holds, then the convergence in (1) is uniform.

Example 1.3. If B is a fractional Brownian motion with Hurst index $H \in (0,1)$ and $\pi_n = \{kT/n : k \in \mathbb{N}_0\} \cap [0,T]$, then $B \in V_{1/H}(\pi)$ and $[B]^{1/H}(t) = t\mathbb{E}[|B_1|^{1/H}]$, see [24, 26].

For $p \in 2\mathbb{N}$, the following change of variable formula for $f \in C^p(\mathbb{R}, \mathbb{R})$ was shown in [11]:

$$\forall S \in V_p(\pi), \qquad f(S(t)) - f(S(0)) = \int_0^t f'(S(s))\mathrm{d}S(s) + \frac{1}{p!}\int_0^t f^{(p)}(S(s))\mathrm{d}[S]^p(s),$$

where the integral

$$\int_0^t f'(S(s))\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!}(S(t_{j+1}\wedge t) - S(t_j\wedge t))^k \tag{2}$$

is defined as a (pointwise) limit of compensated Riemann sums.

Remark 1.4 (Relation with Young integration and rough integration). As p-variation can be infinite for $S \in V_p(\pi)$, the pathwise integral appearing in the formula cannot be defined as a Young integral. Compensated Riemann sums such as (2) also appear in the construction of 'rough integrals' [16, 18]. For $X \in C^\alpha([0,T],\mathbb{R})$ with $\alpha \in (0,1), q = \lfloor \alpha^{-1} \rfloor$ one can define the pathwise integral (2) as a rough integral with respect to a 'reduced' rough path $(\mathbb{X}^1_{s,t}, \mathbb{X}^2_{s,t}, \dots, \mathbb{X}^q_{s,t})_{0\le s\le t\le T}$, where $\mathbb{X}^k_{s,t} := (X(t) - X(s))^k/k!$.

1.2. **Fractional derivatives and fractional Taylor expansions.** Several different notions of fractional derivatives exist in the literature [21, 25, 27] and it is not clear which ones are the right tools for a given context. Our goal here is to shed some light on the advantages of different notions of fractional derivative. Much of this material may be found in the literature [27]. We have provided proofs for some useful properties whose proof we have not been able to find in the literature.

Definition 1.5 (Riemann–Liouville fractional integral)**.** Let $\alpha > 0$. The left Riemann–Liouville fractional integral of order α is defined by

$$I^\alpha_{a^+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

for real functions f for which the integral is well-defined for $x > a \in \mathbb{R}$. Similarly, the right Riemann–Liouville fractional integral is given by

$$I^\alpha_{b^-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt$$

for $x < b \in \mathbb{R}$.

Remark 1.6. The fractional integral is always well-defined for $f \in L^1(a,b)$ with $I^\alpha_{a^+} f \in L^1(a,b)$. This allows for blow up at the boundaries a, b.

This may be used to define a (nonlocal) fractional derivative associated with some base point [21, 25].

Definition 1.7 (Riemann–Liouville fractional derivative)**.** Let $a < b$. Let $f \in L^1([a,b])$ and $n \le \alpha < n+1$ for some integer $n \in \mathbb{N}$. Then left Riemann–Liouville fractional derivative of order α with base point a on $[a,b]$ is defined by

$$D^\alpha_{a^+} f := \left(\frac{d}{dx}\right)^{n+1} I^{n+1-\alpha}_{a^+} f$$

if nth order derivative of $I^{n+1-\alpha}_{a^+} f$ exists and is absolutely continuous on $[a,b]$. Similarly, the right Riemann–Liouville fractional derivative of order α with base point b on $[a,b]$ is given by

$$D^\alpha_{b^-} f := \left(-\frac{d}{dx}\right)^{n+1} I^{n+1-\alpha}_{b^-} f.$$

Remark 1.8. When α is an integer, the fractional derivative coincides with the classical derivative, i.e., $D^n_{a^+} f = f^{(n)}$.

The Riemann–Liouville derivative has several shortcomings. One of them is that the fractional derivative of a constant is not zero. To overcome this, one can consider a modification of Riemann–Liouville fractional derivative which is called the *Caputo derivative*.

Definition 1.9 (Caputo derivative)**.** Suppose f is a real function and $n+1 \geq \alpha > n$. We define the left and right Caputo fractional derivatives of order α at $x \in (a,b)$ by

$$C_{a+}^{\alpha} f(x) = D_{a+}^{\alpha}\left[f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(t-a)^k\right](x),$$

$$C_{b-}^{\alpha} f(x) = D_{b-}^{\alpha}\left[f(t) - \sum_{k=0}^{n} \frac{(-1)^k f^{(k)}(b)}{k!}(b-t)^k\right](x)$$

if the derivatives exist. Here $f^{(k)}$ denotes the kth derivative of f.

We provide some basic properties of fractional derivatives in the appendix for the completeness. Associated with the Caputo derivative is a fractional Taylor expansion.

Proposition 1.10. *Let $f \in C^n([a,b])$ and $n+1 \geq \alpha > n$. If f admits a Caputo derivative of order α on $[a,b]$ and $C_{a+}^{\alpha} f \in L^1([a,b])$, then we have*

$$f(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_a^x C_{a+}^{\alpha} f(t)(x-t)^{\alpha-1}dt$$

and

$$f(x) = \sum_{k=0}^{n} f^{(k)}(b)\frac{(x-b)^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_x^b C_{b-}^{\alpha} f(t)(t-x)^{\alpha-1}dt.$$

As we were not able to find a proof of this expansion in the literature, we provide a detailed proof in the appendix.

1.3. **Local fractional derivative.** The above derivative operators are nonlocal operators. We now introduce the concept of *local* fractional derivative.

Definition 1.11 (Local fractional derivative)**.** Suppose f is left fractional differentiable of order α on $[a, a+\delta]$ for some positive δ, then the left local fractional derivative of order α of function f at point a is given by

$$f^{(\alpha+)}(a) = \lim_{y \to a, y \geq a} C_{a+}^{\alpha}(y),$$

when the limit exists. We can similarly define the right local fractional derivative of order α of function f at a point a by

$$f^{(\alpha-)}(a) = \lim_{y \to a, y \leq a} C_{a-}^{\alpha}(y),$$

when the limit exists.

Example 1.12. We give a simple example of Caputo fractional derivative and local derivative here. Consider $f(x) = |x|^\alpha$ and $0 < \beta \leq \alpha < 1$. Then we have

$$\begin{aligned}
C_{a+}^{\beta} f(x) &= \frac{d}{dx}\left(\frac{1}{\Gamma(1-\beta)}\int_a^x \frac{|t|^\alpha - |a|^\alpha}{(x-t)^\beta}dt\right) \\
&= \frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\left(\int_a^x |t|^\alpha (x-t)^{-\beta}dt - |a|^\alpha \frac{(x-a)^{1-\beta}}{1-\beta}\right) \\
&= \frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\left|\int_{\frac{a}{x}}^1 |t|^\alpha |1-t|^{-\beta}dt |x|^{\alpha-\beta+1}\right| - \frac{1}{\Gamma(1-\beta)}|a|^\alpha (x-a)^{-\beta} \\
&= \frac{1}{\Gamma(1-\beta)}\left(|a|^\alpha (x-a)^{-\beta}\frac{a-x}{x} + (\alpha-\beta+1)|x|^{\alpha-\beta}\int_{\frac{a}{x}}^1 |t|^\alpha |1-t|^{-\beta}dt\right).
\end{aligned}$$

So we can see directly that $f^{(\beta+)}(a) = 0$ for any $a \neq 0$ or $\beta < \alpha$ but $f^{(\alpha+)}(0) = \Gamma(\alpha+1)$. It can be seen further that for all $a \geq 0$, $C^{\alpha}_{a+}f$ is continuous on $[a,\infty]$ but for $a < 0$, $C^{\alpha}_{a+}f$ has singularity at point 0. In particular, for $\beta = \alpha$, we have

$$C^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(1-\beta)}\int_{\frac{a}{x}}^{1} g(t) - g\left(\frac{a}{x}\right) dt \quad \text{with} \quad g(t) = \left|\frac{t}{1-t}\right|^{\alpha}.$$

Remark 1.13. Using integration by parts formula, we actually have

$$D^{\alpha}_{a+}f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^{k-\alpha-n}}{\Gamma(k+2-\alpha)} + D^{(\alpha-n)}_{a+}f^{(n)}(x).$$

Hence $C^{\alpha}_{a+}f(x) = C^{(\alpha-n)}_{a+}f^{(n)}(x)$ so the existence of one side will imply the existence of the other side. Taking limits yields $f^{(\alpha+)}(x) = (f^{(n)})^{((n-\alpha)+)}(x)$. This is important in the proofs below.

Corollary 1.14 (Fractional Taylor formula)**.** *Let $n+1 \geq \alpha > n$ and $f \in C^n([a,b])$ admitting a left (resp. right) local fractional derivative of order α at a. There for $x \in [a,b]$*

$$f(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} + \frac{1}{\Gamma(\alpha+1)}f^{(\alpha+)}(a)(x-a)^{\alpha} + o(|x-a|^{\alpha}), \tag{3}$$

$$f(x) = \sum_{k=0}^{n} f^{(k)}(b)\frac{(x-b)^k}{k!} + \frac{1}{\Gamma(\alpha+1)}f^{(\alpha-)}(b)(b-x)^{\alpha} + o(|x-b|^{\alpha}). \tag{4}$$

The proof is given in Appendix A.2. A similar mean value theorem holds for the nonlocal fractional derivative. The following result, which we state for completeness, is a consequence of Proposition 1.10.

Corollary 1.15. *Let $f \in C^n([a,b])$ and $n+1 \geq \alpha > n$. Suppose Caputo fractional derivative of order α of f is continuous on $[a,b]$. Then for $x \in [a,b]$, there exists $\xi \in [a,x]$ such that*

$$f(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} + \frac{1}{\Gamma(\alpha+1)}C^{\alpha}_{a+}f(\xi)(x-a)^{\alpha}.$$

A similar formula holds for the Caputo right derivative.

Proposition 1.16. *Let $\alpha \notin \mathbb{N}$ and $f \in C^{\alpha}([a,b])$. If $f^{(\alpha+)}$ exists everywhere on $[a,b]$ then $f^{(\alpha+)} = 0$ almost everywhere on $[a,b]$.*

A rather complex proof of this proposition is given in [4, Corollary 3]. We here give a simple proof of this property using only properties of monotone functions.

Proof. We first consider the case $0 < \alpha < 1$. By Corollary 1.14, we actually have

$$f(y) = f(x) + g(x)(y-x)^{\alpha} + o(|y-x|^p) \qquad \text{with} \quad g = \frac{1}{\Gamma(\alpha+1)}f^{(\alpha+)}.$$

Hence for any sequence $y_n \to x, y_n > x$, we have

$$\lim_{n\to\infty} \frac{f(y_n) - f(x)}{(y_n - x)^{\alpha}} = g(x).$$

Now let $E^+ = \{x \in \mathbb{R} : g(x) > 0\}$. If $\mathcal{L}(E^+) > 0$, then there exists a compact set $K \in E^+$ with $\mathcal{L}(K) > 0$ where $\mathcal{L}$ denotes the Lebesgue measure on the real line.

Consider the open set $\cup_{x\in K}(x, x+\delta_x)$ in $\mathbb{R}$, where δ_x is the largest number so that $f(y) > f(x), \forall y \in (x, x+\delta_x)$. We can then write $\cup_{x\in K}(x, x+\delta_x) = \cup_{k=1}^{\infty} I_k$ for some open interval I_k. There exists I_r such that $\mathcal{L}(K \cap \bar{I}_r) > 0$. In fact, for each point in K, it is either in the set $\cup_{k=1}^{\infty} I_k$ or it is a boundary point of some interval I_k. We may augment I_k to an interval $\tilde{I}_k$ including zero or one of both of its boundary points so that $\tilde{I}_k$ are disjoint and $K \subset \cup_{k=1}^{\infty} \tilde{I}_k$, and such that the right boundary point is not in $\tilde{I}_k$ if it belongs to K. Then we find I_r such that $\mathcal{L}(K \cap \tilde{I}_r) > 0$. Hence $\mathcal{L}(K \cap \bar{I}_r) > 0$. We may also assume that there are no points in K isolated from the right, since the set of all such points is of measure zero.

For any $\bar{x} < \bar{y} \in K \cap \bar{I}_r$, there exists $x_0 \in K \cap [\bar{x}, \bar{y}]$ such that

$$x_0 = \underset{x\in K\cap[\bar{x},\bar{y}]}{\operatorname{argmax}} \{f(x)\}.$$

If $x_0 \neq \bar{y}$, then there exists $y_0 \in (x_0, x_0 + \delta_{x_0}) \cap K \cap [\bar{x}, \bar{y}]$ since we assume there are no right-isolated points in K. Then we have $f(y_0) > f(x_0)$, which is a contradiction. Hence $x_0 = \bar{y}$ and we have $f(\bar{y}) > f(\bar{x}), \forall \bar{x} < \bar{y} \in K \cap \bar{I}_r$.

Now define $\bar{f} : \bar{I}_r \to \mathbb{R}$ such that $\bar{f} = f$ on $K \cap \bar{I}_r$ and $\bar{f}$ is linear outside of $K \cap \bar{I}_r$. In fact, let $\tilde{x} \in \bar{I}_r$, if $\tilde{x} \notin K$, then $\tilde{x} \in (z_0, z_1)$ with $z_0, z_1 \in K$. Thus $\bar{f}(z_1) \geq \bar{f}(z_0)$, and we can linearly interpolate to define $\bar{f}(\tilde{x})$ as an increasing function. Hence this extension $\bar{f}$ is differentiable almost everywhere on $\bar{I}_r$.

Now we go back to the function f on set K. $\bar{f}$ is a.e. differentiable on K and we can simply suppose here $\bar{f}$ is differentiable and $f = \bar{f}$ on K whose Lebesgue measure is positive. Now choose a point $x_1 \in K$, either x_1 is right-isolated in K which is eliminated from K before or it is a right-accumulation point in K. For the latter case, we would have

$$g(x_1) = \lim_{n\to\infty} \frac{f(y_n) - f(x_1)}{(y_n - x)^p} = \lim_{n\to\infty} \frac{\bar{f}(y_n) - \bar{f}(x_1)}{(y_n - x_1)^p} = 0,$$

for any sequence $\{y_n\}$ in K with $y_n \to x_1$. We then get $g = 0$ a.e. on K, which is a contradiction. Hence $\mathcal{L}(E^+) = 0$. Similarly, $\mathcal{L}(E^-) = 0$. Hence $f^{(p+)} = 0$ a.e. on $\mathbb{R}$.

If $m < \alpha < m+1$ for some integer $m > 0$, since we have $f^{(\alpha+)} = (f^{(m)})^{(\alpha-m+)}$, we may conclude again that $f^{(\alpha+)}(x) = 0$ a.e. on $\mathbb{R}$. □

Remark 1.17. For Proposition 1.16, we actually only need a weaker condition on the function f, namely that

$$\lim_{y\to x, y>x} \frac{f^{(m)}(y) - f^{(m)}(x)}{|y-x|^\beta}$$

exists for $x \in \mathbb{R}$ with $m = \lfloor \alpha \rfloor$ and $\alpha = m + \beta$, which may be thought as classical definition of αth order derivative. We call

$$\lim_{x\to a, x>a} \frac{f^{(m)}(x) - f^{(m)}(a)}{|x-a|^\beta}$$

the 'classical' left fractional derivative of order α if it exists. However, when this limit exists, it is not guaranteed that f admits a local fractional derivative.

We can actually state a stronger result.

Proposition 1.18. *For $f \in C^\alpha$, the Hausdorff dimension of the set*

$$E_f = \left\{ x \in \mathbb{R} : \liminf_{y \to x, y > x} \frac{|f^{(m)}(y) - f^{(m)}(x)|}{|y-x|^\beta} > 0 \right\}$$

is at most $\beta = \alpha - \lfloor \alpha \rfloor$.

Proof. WLOG, we suppose $m = 0$ and $0 < \alpha < 1$, then $\beta = \alpha$. For each ϵ, δ, we define the set $E^\delta_{\epsilon+}$ to be the subset of E such that for all $x \in E^\delta_{\epsilon+}$, we have

$$f(y) > f(x) + \epsilon(y-x)^\alpha, \forall x < y < x + \delta.$$

We only need to show the Hausdorff dimension of the set $E^\delta_{\epsilon+}$ is at most α. Furthermore, we can restrict the set $E^\delta_{\epsilon+}$ on the interval $[0, \delta]$. Hence we will work on $[0, \delta]$ with $E^\delta_{\epsilon+}$.

First of all, suppose the Hausdorff dimension of the set $E^\delta_{\epsilon+}$ is larger than α. In this case, we consider partitions $\{0, \frac{\delta}{k}, \cdots, \frac{(k-1)\delta}{k}, \delta\}$ for $k \in \mathbb{N}$. In each interval $[\frac{i\delta}{k}, \frac{(i+1)\delta}{k}]$, we choose the leftmost and rightmost points of the set $E^\delta_{\epsilon+}$ and denote this interval as I^i_k (if exists). Then $\cup_{i=0}^{k-1} I^i_k$ covers the set $E^\delta_{\epsilon+}$ and by the definition of Hausdorff measure, we know that $\sum_{i=0}^{k-1} |I^i_k|^\alpha \to \infty$ as $k \to \infty$. Now let a^i_k be the left boundary point of I^i_k and b^i_k be the right boundary point of I^i_k. Now for any two points $a < b \in \{a^i_k, b^i_k, i = 0, \cdots, k-1\}$, we have

$$f(b) \geq f(a) + \epsilon(b-a)^\alpha$$

since f is continuous and we can approximate a, b by points in $E^\delta_{\epsilon+}$. Then we have

$$\begin{aligned} f(b^{k-1}_k) - f(a^0_k) &= \sum_{i=1}^{k-1} f(b^i_k) - f(a^i_k) + f(a^i_k) - f(b^{i-1}_k) + f(b^0_k) - f(a^0_k) \\ &\geq \epsilon \sum_{i=0}^{k-1} |I^i_k|^\alpha \to \infty, \end{aligned}$$

which gives a contradiction when $k \to \infty$. □

2. Fractional Itô calculus

We now consider noninteger $p > 2$ and derive Itô–type change of variable formulas for paths with nonzero pth variation along a sequence of partitions $\{\pi_n\}_{n \in \mathbb{N}}$, first for functions (Section 2.1) then for path-dependent functionals (Section 2.2). In each case the focus is the existence of a 'fractional' Itô remainder term: we will see that the existence of a nonzero Itô term depends on the fine structure of the function and its fractional derivative.

2.1. Change of variable formula.

We first derive an Itô–type change of variable formula for functions of paths with pth variation along a sequence of partitions $\{\pi_n\}_{n \in \mathbb{N}}$.

Let $S \in V_p(\pi)$ be a path which admits pth order variation along some sequence of partitions $\{\pi_n\}_{n\in\mathbb{N}}$. We make the following assumptions.

Assumption 2.1. *For any $k \in \mathbb{R}$, we have*

$$\int_0^T \mathbf{1}_{\{S(t)=k\}} d[S]^p_\pi = 0.$$

Let $m = \lfloor p \rfloor$.

Assumption 2.2. *$f \in C^m(\mathbb{R})$ and admits a left local fractional derivative of order p everywhere.*

Assumption 2.3. *The complement of the set*

$$\Gamma_f = \{x \in \mathbb{R} : \exists U \ni x \text{ open, } (a,b) \mapsto (C^p_{a+} f)(b) \text{ is continuous on } \{(\bar{a},\bar{b}) \in U \times U : \bar{a} \le \bar{b}\}\}$$

is locally finite, i.e., for any compact set $K \in \mathbb{R}$, the set $\Gamma^c_f \cap K$ has only finite number of points.

We first give a simple lemma regarding the set Γ_f.

Lemma 2.4. *The left local fractional derivative of order p of f is equal to zero on Γ_f: $\forall x \in \Gamma_f,\ f^{(p+)}(x) = 0$.*

Proof. Let $x \in \Gamma_f$. There exists $x \in U$ open such that $(a,b) \mapsto C^p_{a+} f(b)$ is continuous on $\{(a,b) \in U \times U : a \le b\}$. Hence $f^{(p+)}$ is continuous on U. Since $f^{(p+)}$ is zero a.e., $f^{(p+)}(x) = 0$. $\square$

Assumption 2.1 will be satisfied if the weighted occupation measure γ_S defined by

$$\gamma_S(A) = \int_0^T \mathbf{1}_{\{S(t)\in A\}} d[S]^p_\pi(t)$$

is atomless. Assumption 2.1 is satisfied in particular if γ_S has a Lebesgue density [17], which corresponds to a local time of order p [11]. However, as the following example shows, Assumption 2.1 may fail to be satisfied even if the path has a nonzero pth order variation.

Example 2.5 (A counterexample)**.** We now give an example of path failing to satisfy Assumption 2.1, in the spirit of [12, Example 3.6].[1]

Let $p > 2$ and define the intervals

$$I^1_1 = \left(\frac{1}{3}, \frac{2}{3}\right), \quad I^1_2 = \left(\frac{1}{9}, \frac{2}{9}\right), \quad I^2_2 = \left(\frac{7}{9}, \frac{8}{9}\right), \cdots, I^i_j, j = 1, \cdots, 2^{i-1}, \tag{5}$$

and let $C = [0,1] \setminus \cup_{i=1}^\infty \cup_{j=1}^{2^{i-1}} I^i_j$, which is the Cantor ternary set [3]. Let $c : [0,1] \to \mathbb{R}_+$ be the associated Cantor function, which is defined by

$$c(x) = \begin{cases} \sum_{n=1}^\infty \frac{a_n}{2^n}, & x = \sum_{n=1}^\infty \frac{2a_n}{3^n} \in C, \quad a_n \in \{0,1\} \\ \sup_{y \le x, y \in C} c(y), & x \in [0,1] \setminus C. \end{cases}$$

We can see it is a nondecreasing function increasing only on the Cantor set C.

[1] [12, Example 3.6] aims to construct a path which admits quadratic variation but does not possess local time along some sequence of partitions. There seems to be an issue with the construction in [12] but the underlying idea is still useful for our construction. In fact, the third equality of equation (3.12) in [12] should be $\sum_{i=1}^n (\epsilon_n)^i = \frac{\epsilon_n - \epsilon_n^{n+1}}{1-\epsilon_n} \to 0$ not 1.

Consider the function

$$S(t) = |2^{\log_3(2\cdot \min_{u\in C}|t-u|)}|^{\frac{1}{p}}.$$

We are going to construct a sequence of partitions such that the pth variation of S along this sequence will be the Cantor function c. In this case, we will see

$$\int_0^1 \mathbf{1}_{\{S(t)=0\}} d[S]^p = \int_0^1 \mathbf{1}_{t\in C} dc(t) = c(1) - c(0) = 1,$$

which shows Assumption 2.1 is not satisfied.

We begin with the partition $\{\pi^i_{j,n}\}$, which denotes the nth partition in the interval I^i_j and let $\pi_n = \cup_{i=1}^n \cup_j \pi^i_{j,n}$. Define $t^{i,0}_{j,n} = \inf I^i_j$ and

$$t^{i,k+1}_{j,n} = \inf\left\{ t > t^{i,k}_{j,n} : S(t) \in \left(\frac{1}{k_n}\sup_{t\in I^i_j} S(t)\right)\mathbb{Z}\right\},$$

then $t^{i,2k_n}_{j,n} = \sup I^i_j$. k_n is an integer to be determined. We then do the calculation

$$\sum_{k=0}^{2k_n-1} |S(t^{i,k+1}_{j,n}) - S(t^{i,k}_{j,n})|^p = \sum_{k=0}^{2k_n-1}\left|\frac{1}{k_n}2^{-\frac{i}{p}}\right|^p = 2^{1-i}k_n^{1-p}.$$

Then the sum over the nth partition will be

$$\sum_{i=1}^{n}\sum_{j=1}^{2^{i-1}} 2^{1-i}k_n^{1-p} = n\cdot k_n^{1-p}.$$

Choosing $k_n = \lfloor n^{\frac{1}{p-1}}\rfloor$ we obtain

$$nk_n^{1-p} \geq n\cdot (n^{\frac{1}{p-1}})^{1-p} = 1$$

and

$$nk_n^{1-p} \leq n\cdot (n^{\frac{1}{p-1}} - 1)^{1-p} = \left(1 - n^{\frac{1}{1-p}}\right)^{1-p} \overset{n\to\infty}{\to} 1.$$

Hence we see

$$[S]^p_\pi(1) = 1.$$

Furthermore, since

$$2^{1-i}k_n^{1-p} \overset{n\to\infty}{\to} 0,$$

we see that $[S]^p_\pi$ will not change on the interval I^i_j and by symmetry, we finally can show that $[S]^p = c$, the Cantor function.

Theorem 2.6 extends the result of [11] to the case of functions with fractional regularity.

Theorem 2.6. *Let $S \in V_p(\pi)$ satisfy Assumption* 2.1. *If f is a continuous function satisfying Assumptions* 2.2 *and* 2.3 *for $m = \lfloor p\rfloor$, then*

$$\forall t\in[0,T], \qquad f(S(t)) = f(S(0)) + \int_0^t f'(S(u))dS(u),$$

where the last term is a limit of compensated Riemann sums of order $m = \lfloor p\rfloor$:

$$\int_0^t f'(S(u))dS(u) = \lim_{n\to\infty}\sum_{t_i\in\pi_n}\sum_{j=1}^{m}\frac{f^{(j)}(S(t\wedge t_i))}{j!}(S(t\wedge t_{i+1}) - S(t\wedge t_i))^j.$$

Proof. By classical Taylor's formula with integral remainder of integer order m, we have

$$\begin{aligned} f(S(t)) - f(S(0)) &= \sum_{[t_i,t_{i+1}]\in\pi_n} f(S(t\wedge t_{i+1})) - f(S(t\wedge t_i)) \\ &= \sum_{[t_i,t_{i+1}]\in\pi_n} \Big(f(S(t\wedge t_{i+1})) - f(S(t\wedge t_i)) - \cdots \\ &\qquad\qquad - \frac{f^{(m)}(S(t\wedge t_i))}{m!}(S(t\wedge t_{i+1}) - S(t\wedge t_i))^n \Big) \\ &\quad + \sum_{[t_i,t_{i+1}]\in\pi_n} \sum_{j=1}^{m} \frac{f^{(j)}(S(t\wedge t_i))}{j!}(S(t\wedge t_{i+1}) - S(t\wedge t_i))^j \\ &= \sum_{[t_i,t_{i+1}]\in\pi_n} \frac{1}{\Gamma(m)} \int_{S(t\wedge t_i)}^{S(t\wedge t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t\wedge t_i)))(S(t\wedge t_{i+1}) - r)^{m-1} dr + L^n, \end{aligned}$$

where $L^n = \sum_{[t_i,t_{i+1}]\in\pi_n} \sum_{j=1}^{m} \frac{f^{(j)}(S(t\wedge t_i))}{j!}(S(t\wedge t_{i+1}) - S(t\wedge t_i))^j$. Our aim is to estimate the limit of the quantity

$$R^n := \sum_{[t_i,t_{i+1}]\in\pi_n} \frac{1}{\Gamma(m)} \int_{S(t\wedge t_i)}^{S(t\wedge t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t\wedge t_i)))(S(t\wedge t_{i+1}) - r)^{m-1} dr,$$

and show this quantity converges to 0 so our result holds. In order to obtain an estimate, we divided the partitions into three parts: $C^n_\epsilon = \{[t_i,t_{i+1}] \in \pi_n : |S(t\wedge t_i) - k| \le \epsilon \text{ for some } k \in \Gamma^c_f\}$, $C^n_{1,\epsilon} = \{[t_i,t_{i+1}] \in \pi_n \setminus C^n_\epsilon : S(t\wedge t_i) \le S(t\wedge t_{i+1})\}$ and $C^n_{2,\epsilon} = \{[t_i,t_{i+1}] \in \pi_n \setminus C^n_\epsilon : S(t\wedge t_{i+1}) < S(t\wedge t_i)\}$ for arbitrary $\epsilon > 0$.

(1) On C^n_ϵ: Denote

$$R^n_\epsilon := \sum_{[t_i,t_{i+1}]\in C^n_\epsilon} \frac{1}{\Gamma(m)} \int_{S(t\wedge t_i)}^{S(t\wedge t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t\wedge t_i)))(S(t\wedge t_{i+1}) - r)^{m-1} dr.$$

By Hölder continuity of f and continuity of S, there exists a constant $M > 0$ such that

$$\begin{aligned} |R^n_\epsilon| &\le \sum_{[t_i,t_{i+1}]\in C^n_\epsilon} M|S(t\wedge t_{i+1}) - S(t\wedge t_i)|^p \\ &\le \sum_{[t_i,t_{i+1}]\in\pi_n} M g_\epsilon(S(t\wedge t_i))|S(t\wedge t_{i+1}) - S(t\wedge t_i)|^p. \end{aligned}$$

For example $M = \frac{\Gamma(p-m+1)\|f\|_{C^p}}{\Gamma(p+1)}$. Here g_ϵ is a continuous function taking value 1 on $[k-\epsilon, k+\epsilon]$ for each $k \in \Gamma^c_f$ and value 0 outside of $\cup_{k\in\Gamma^c_f}[k-2\epsilon, k+2\epsilon]$ with $\|g_\epsilon\|_\infty \le 1$. When ϵ is small enough, we may assume $[k-2\epsilon, k+2\epsilon]$ are disjoint. Then by definition of the pth order variation, we see that

$$\limsup_{n\to\infty} |R^n_\epsilon| \le \int_0^t M g_\epsilon(S(r)) d[S]^p(r).$$

Letting $\epsilon \to 0$, we obtain from Assumption 2.1 that

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} |R^n_\epsilon| \le \sum_{k\in\Gamma^c_f} \int_0^t M \mathbf{1}_{\{S(r)=k\}} d[S]^p(r) = 0.$$

(2) Outside C_ϵ^n: Since $S \in V^p(\pi)$, there exists $N_\epsilon \in \mathbb{N}$ such that for all $n > N_\epsilon$, we have for all $[t_i, t_{i+1}] \in \pi_n$

$$|S(t_{i+1}) - S(t_i)| \le \frac{\epsilon}{2}.$$

Thus for $[t_i, t_{i+1}] \notin C_\epsilon^n$, we have $(S(t_{i+1}) - k)(S(t_i) - k) > 0$ for all $k \in \Gamma_f^c$ and $S(t_i), S(t_{i+1})$ is away from Γ_f^c with at least $\frac{\epsilon}{2}$ distance, which means we have the following estimate

$$|C^p_{a_1^+} f(b_1) - C^p_{a_2^+} f(b_2)| \le \omega_\epsilon(\sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}) \tag{6}$$

for some modulus of continuity ω_ϵ, $a_1, a_2, b_1, b_2 \in [S(t_i), S(t_{i+1})]$ or $[S(t_{i+1}), S(t_i)]$ and $a_1 \le b_1, a_2 \le b_2$ by Assumptions 2.2 and 2.3.

(3) We now consider the set $C_{1,\epsilon}^n$. Let $\alpha = p - m$ and denote

$$R_{1,\epsilon}^n := \sum_{[t_i,t_{i+1}] \in C_{1,\epsilon}^n} \frac{1}{\Gamma(m)} \int_{S(t\wedge t_i)}^{S(t\wedge t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t\wedge t_i)))(S(t\wedge t_{i+1}) - r)^{m-1} dr.$$

We have by Proposition 1.10,

$$\begin{aligned} R_{1,\epsilon}^n &= \sum_{[t_i,t_{i+1}] \in C_{1,\epsilon}^n} \frac{1}{\Gamma(m)} \int_{S(t\wedge t_i)}^{S(t\wedge t_{i+1})} (S(t\wedge t_{i+1}) - r)^{m-1} \frac{1}{\Gamma(\alpha)} \int_{S_{t\wedge t_i}}^{r} \frac{C^p_{S(t\wedge t_i)^+} f(s)}{(r-s)^{1-\alpha}} ds dr \\ &= \sum_{[t_i,t_{i+1}] \in C_{1,\epsilon}^n} \frac{1}{\Gamma(m)\Gamma(\alpha)} \int_{S(t\wedge t_i)}^{S(t\wedge t_{i+1})} \int_s^{S_{t\wedge t_{i+1}}} (S(t\wedge t_{i+1}) - r)^{m-1} \frac{C^p_{S(t\wedge t_i)^+} f(s)}{(r-s)^{1-\alpha}} dr ds \\ &= \sum_{[t_i,t_{i+1}] \in C_{1,\epsilon}^n} \frac{1}{\Gamma(p)} \int_{S(t\wedge t_i)}^{S(t\wedge t_{i+1})} (S(t\wedge t_{i+1}) - s)^{p-1} C^p_{S(t\wedge t_i)^+} f(s) ds. \end{aligned}$$

As $S(t_i) \in \Gamma_f$ by Lemma 2.4 we have $f^{(p+)}(S(t_i)) = C^p_{S(t_i)^+} f(S(t_i)) = 0$. Then by inequality (6) we have

$$\begin{aligned} &\left| \int_{S(t\wedge t_i)}^{S(t\wedge t_{i+1})} (S(t\wedge t_{i+1}) - s)^{p-1} C^p_{S(t\wedge t_i)^+} f(s) ds \right| \\ &\qquad \le \int_{S(t\wedge t_i)}^{S(t\wedge t_{i+1})} (S(t\wedge t_{i+1}) - s)^{p-1} \omega_\epsilon(s - S(t\wedge t_i)) ds \\ &\qquad \le \frac{1}{p} \omega_\epsilon(S(t_{i+1}) - S(t_i)) |S(t_{i+1}) - S(t_i)|^p. \end{aligned}$$

Thus we obtain

$$|R_{1,\epsilon}^n| \le \sum_{[t_i,t_{i+1}] \in \pi_n} \frac{\omega_\epsilon(S(t_{i+1}) - S(t_i))}{\Gamma(p+1)} |S(t_{i+1}) - S(t_i)|^p.$$

Since $osc(S, \pi_n) \to 0$ as $n \to \infty$, we have $\lim_{n\to\infty} |R_{1,\epsilon}^n| = 0$.

(4) On the set $C_{2,\epsilon}^n$: Denote

$$R_{2,\epsilon}^n = \sum_{[t_i,t_{i+1}] \in C_{2,\epsilon}^n} \frac{1}{\Gamma(m)} \int_{S(t\wedge t_i)}^{S(t\wedge t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t\wedge t_i)))(S(t\wedge t_{i+1}) - r)^{m-1} dr.$$

By Proposition 1.10, $R^n_{2,\epsilon}$ can be expressed as

$$\sum_{[t_i,t_{i+1}]\in C_{2,\epsilon}} \frac{1}{\Gamma(m)\Gamma(\alpha)} \int_{S(t\wedge t_{i+1})}^{S(t\wedge t_i)} \int_{S(t\wedge t_{i+1})}^{s} (S(t\wedge t_{i+1})-r)^{m-1} \frac{C^p_{r+}f(s)}{(S(t\wedge t_i)-s)^{1-\alpha}} drds.$$

Again by inequality (6), we will have

$$\begin{aligned}|C^p_{r+}f(s)| &\le \omega_\epsilon(\sqrt{(r-S(t\wedge t_{i+1}))^2+(s-S(t\wedge t_{i+1}))^2}) \\ &\le \omega_\epsilon(\sqrt{2}|S(t\wedge t_{i+1})-S(t\wedge t_i)|).\end{aligned}$$

Hence, we can obtain the estimate

$$\begin{aligned}&\left|\int_{S(t\wedge t_{i+1})}^{S_{t\wedge t_i}} \int_{S(t\wedge t_{i+1})}^{s} (S(t\wedge t_{i+1})-r)^{m-1} \frac{C^p_{r+}f(s)}{(S(t\wedge t_i)-s)^{1-\alpha}} drds\right| \\ &\le \int_{S(t\wedge t_{i+1})}^{S_{t\wedge t_i}} \int_{S(t\wedge t_{i+1})}^{s} (S(t\wedge t_{i+1})-r)^{m-1} \frac{\omega_\epsilon(\sqrt{2}|S(t\wedge t_{i+1})-S(t\wedge t_i)|)}{(S(t\wedge t_i)-s)^{1-\alpha}} drds \\ &\le \frac{1}{m\alpha}\omega_\epsilon(\sqrt{2}|S(t\wedge t_{i+1})-S(t\wedge t_i)|)|S(t_{i+1})-S(t_i)|^p.\end{aligned}$$

Thus we obtain that $\lim_{n\to\infty}|R^n_{2,\epsilon}|=0$.

Thus, we have $R^n = R^n_\epsilon + R^n_{1,\epsilon} + R^n_{2,\epsilon}$ and

$$\lim_{n\to\infty}|R^n| \le \lim_{\epsilon\to 0}\limsup_{n\to\infty}|R^n_\epsilon| + |R^n_{1,\epsilon}| + |R_{2,\epsilon^n}| = 0.$$

Hence we see $\lim_{n\to\infty} L^n$ exists and we denote it as $\int_0^t (f(S(u)))dS(u)$, which gives the result. □

Remark 2.7. We cannot expect the Itô formula in the fractional case to have the same form as in the integer case, i.e., there might be no Itô term in the fractional case even if Assumption 2.1 does not hold. For Example 2.5, we can show that for $2<p<3$ and $f(x)=|x|^p$, the Itô term still vanishes even though we have

$$\int_0^1 f^{(p+)}(S(t))d[S]^p(t) = \frac{1}{\Gamma(p+1)}\int_0^1 \mathbf{1}_{\{S(t)=0\}}dc(t) = \frac{c(1)-c(0)}{\Gamma(p+1)} = \frac{1}{\Gamma(p+1)},$$

since $\{S(t)=0\}=C$ is the support of the function c. In this case, we will show that

$$0 = f(S(1)) - f(S(0)) = \int_0^1 f'(S(u))dS(u).$$

In fact, we will need to calculate

$$\lim_{n\to\infty}\sum_{[t_i,t_{i+1}]\in\pi_n} f'(S(t_i))(S(t_{i+1})-S(t_i)) + \frac{1}{2}f''(S(t_i))(S(t_{i+1})-S(t_i))^2,$$

which is in fact the 'rough integral' of f' along the reduced order-p Itô rough path associated with S [11]. Splitting the terms across each I^i_j we have

$$\begin{aligned}&\sum_{k=0}^{k_n-1} p(S(t^{i,k}_{j,n}))^{p-1}\frac{1}{k_n}2^{-\frac{i}{p}} - \sum_{k=k_n}^{2k_n-1} p(S(t^{i,k}_{j,n}))^{p-1}\frac{1}{k_n}2^{-\frac{i}{p}} \\ &\quad = -p(S(t^{i,k_n}_{j,n}))^{p-1}\frac{1}{k_n}2^{-\frac{i}{p}} = -p\cdot(2^{-\frac{i}{p}})^{p-1}\cdot\frac{2^{-\frac{i}{p}}}{k_n}.\end{aligned}$$

Thus

$$\sum_{[t_i,t_{i+1}]\in\pi_n} f'(S(t_i))(S(t_{i+1})-S(t_i)) = \sum_{i=1}^{n}\sum_{j=1}^{2^{i-1}} -p\cdot\frac{2^{-i}}{k_n} = -\frac{np}{2k_n}.$$

Now we consider the term involving the second derivative. On I_j^i, we have

$$\begin{aligned}
&\sum_{[t_{j,n}^{i,k},t_{j,n}^{i,k+1}]\in\pi_{j,n}^i} \frac{1}{2}f''(S(t_{j,n}^{i,k}))(S(t_{j,n}^{i,k+1})-S(t_{j,n}^{i,k}))^2\\
&\quad= \sum_{[t_{j,n}^{i,k},t_{j,n}^{i,k+1}]\in\pi_{j,n}^i} \frac{p(p-1)}{2}(S(t_{j,n}^{i,k}))^{p-2}\frac{2^{-\frac{2i}{p}}}{k_n^2}\\
&\quad= \frac{1}{2}\left(\sum_{k=0}^{k_n-1} p(p-1)k^{p-2}\frac{2^{-i}}{k_n^p} + \sum_{k=1}^{k_n} p(p-1)k^{p-2}\frac{2^{-i}}{k_n^p}\right).
\end{aligned}$$

Summing the second-order terms over π_n we obtain:

$$\frac{np(p-1)}{4}\frac{2\sum_{k=1}^{k_n}k^{p-2}-k_n^{p-2}}{k_n^p}.$$

Adding together the first-order term $-\frac{np}{2k_n}$, we need to calculate

$$\lim_{n\to\infty}\frac{np}{4}\left((p-1)\frac{2\sum_{k=1}^{k_n}k^{p-2}-k_n^{p-2}}{k_n^p}-\frac{2}{k_n}\right).$$

We thus observe that

$$\lim_{n\to\infty}\frac{np}{4}(p-1)\frac{1}{k_n^2} = \lim_{n\to\infty}\frac{np}{4}(p-1)\frac{1}{n^{\frac{2}{p-1}}}.$$

Since $\frac{2}{p-1}>1$ this implies

$$\lim_{n\to\infty}\frac{np}{4}(p-1)\frac{1}{k_n^2}=0.$$

Hence it remains to consider

$$\lim_{n\to\infty}\frac{np}{2}\left((p-1)\frac{\sum_{k=1}^{k_n}k^{p-2}}{k_n^p}-\frac{1}{k_n}\right).$$

Using the inequality

$$\frac{k_n^{p-1}}{p-1}=\int_0^{k_n}x^{p-2}dx\le\sum_{k=1}^{k_n}k^{p-2}\le\int_1^{k_n+1}x^{p-2}dx=\frac{(k_n+1)^{p-1}-1}{p-1},$$

we obtain first

$$\lim_{n\to\infty}\frac{np}{2}\left((p-1)\frac{\sum_{k=1}^{k_n}k^{p-2}}{k_n^p}-\frac{1}{k_n}\right)\ge\lim_{n\to\infty}\frac{np}{2}\left((p-1)\frac{k_n^{p-1}}{(p-1)k_n^p}-\frac{1}{k_n}\right)=0,$$

and second

$$\lim_{n\to\infty}\frac{np}{2}\left((p-1)\frac{\sum_{k=1}^{k_n}k^{p-2}}{k_n^p}-\frac{1}{k_n}\right)\le\lim_{n\to\infty}\frac{np}{2}\left(\frac{(k_n+1)^{p-1}-1-k_n^{p-1}}{k_n^p}\right). \tag{7}$$

Noting that $\lim_{n\to\infty} nk_n^{1-p} = 1$ and $k_n \to \infty$, we see the right-hand side of inequality (7) equals to

$$\lim_{k_n\to\infty} p\frac{k_n^{p-1}}{2}\left(\frac{(k_n+1)^{p-1} - 1 - k_n^{p-1}}{k_n^p}\right) = \frac{p}{2}\lim_{x\to\infty}\frac{(x+1)^{p-1} - 1 - x^{p-1}}{x}.$$

By L'Hôpital's rule, above limit equals to

$$\begin{aligned}&\lim_{x\to\infty}\frac{p(p-1)}{2}((x+1)^{p-2} - x^{p-2})\\ &\quad= \frac{p(p-1)}{2}\lim_{x\to\infty} x^{p-2}\left(\left(1+\frac{1}{x}\right)^{p-2} - 1\right) = \frac{p(p-1)}{2}\lim_{x\to\infty} x^{p-2}\frac{p-2}{x} = 0\end{aligned}$$

since $p - 2 < 1$. Thus in this case we see the remainder term is zero, which is different from the integer case. Hence we see that **the pathwise change of variable formula in Theorem 2.6 may hold even if Assumption 2.1 does not hold.**

We now give an example of path with the same pth variation as above but leading to a nonzero remainder term in the change of variable formula.

Example 2.8. Define the intervals I_j^i as in (5). Then we define $S|_{I_j^i}$ by induction on i. Let $g(t) = 2\min\{t, 1-t\}$ on $[0,1]$ and 0 otherwise. For $a < b$, we define $g(t,[a,b]) = g\left(\frac{t-a}{b-a}\right)$.

First for $i = 1$, we divide the interval I_j^i into r_i smaller intervals $I_{j,k}^i, k = 1,\cdots,r_i$ such that $|I_{j,k}^i| = \frac{|I_j^i|}{r_i}$ and two of each are nonintersecting. On each interval $I_{j,k}^i$, we define $S(t) = 2^{-i}g(t, I_{j,k}^i)$ for $k = 1,\cdots,r_i, j = 1,\cdots,2^{i-1}$. In other words, S is defined as the limit

$$S(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{2^{i-1}}\sum_{k=1}^{r_i} 2^{-i}g(t, I_{j,k}^i).$$

Let $\pi_n = (\tau_l^n)$ be the dyadic Lebesgue partition associated with S:

$$\tau_0^n = 0, \qquad \tau_{l+1}^n = \inf\{t > \tau_l^n,\ |S(t) - S(\tau_l^n)| > 2^{-n}\}.$$

Hence, we have the calculation

$$\begin{aligned}[S]^p(1) &= \lim_{n\to\infty}\sum_{\pi_n}|S(\tau_{l+1}^n) - S(\tau_l^n)|^p = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{2^{i-1}}\sum_{k=1}^{r_i} 2\times 2^{-np}\times 2^{n-i}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n} r_i 2^{-np+n}.\end{aligned}$$

By choosing

$$r_i = \lfloor 2^{(i-1)(p-1)}(2^{p-1} - 1)\rfloor,$$

so that

$$1 = \lim_{n\to\infty}\sum_{i=1}^{n} r_i 2^{-np+n},$$

together with similar to what was discussed in Remark 2.7, we have $[S]^p(t) = c(t)$, the Cantor function. Let $f(x) = |x|^p, 2 < p < \frac{5}{2}$ and $T = 1$, we are going to calculate the Itô remainder term for $f(S(1)) - f(S(0))$. We calculate the limit

$$\lim_{n\to\infty} \sum_{[t_i,t_{i+1}]\in\pi_n} p(S(t_i))^{p-1}(S(t_{i+1}) - S(t_i)) + \frac{p(p-1)}{2}(S(t_i))^{p-2}(S(t_{i+1}) - S(t_i))^2$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{2^{i-1}} r_i \left(-p2^{-n}(2^{-i})^{p-1} + \frac{p(p-1)}{2} 2^{-2n} c_n^i\right),$$

where

$$c_n^i = 2 \sum_{k=1}^{2^{n-i}-1} 2^{-n(p-2)} k^{p-2} + 2^{-n(p-2)}(2^{n-i})^{p-2}.$$

We see directly

$$c_n^i < 2^{1-n(p-2)} \int_0^{2^{n-i}} x^{p-2} dx + 2^{-n(p-2)}(2^{n-i})^{p-2}.$$

Thus,

$$\frac{p(p-1)}{2} 2^{-2n} c_n^i < p2^{-n-ip+i} + \frac{p(p-1)}{2} 2^{-np+(n-i)(p-2)}.$$

This leads to

$$\begin{aligned}
&\sum_{i=1}^{n} \sum_{j=1}^{2^{i-1}} r_i \left(-p2^{-n}(2^{-i})^{p-1} + \frac{p(p-1)}{2} 2^{-2n} c_n^i\right) \\
&\quad< \sum_{i=1}^{n-1} 2^{i-1} r_i \left(\frac{p(p-1)}{2} 2^{-2n-ip+2i}\right) + 2^{n-1} r_n \left(-p2^{-np} + \frac{p(p-1)}{2} 2^{-np}\right) \\
&\quad< \sum_{i=1}^{n-1} 2^{-p}(2^{p-1}-1) \left(\frac{p(p-1)}{2} 2^{-2n+2i}\right) + 2^{-p}(2^{p-1}-1)\left(-p + \frac{p(p-1)}{2}\right) \\
&\quad= p2^{-p}(2^{p-1}-1)\left(\frac{p-1}{2} \frac{1-(\frac{1}{4})^{n-1}}{3} + -1 + \frac{p-1}{2}\right).
\end{aligned}$$

Let $n \to \infty$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{2^{i-1}} r_i \left(-p2^{-n}(2^{-i})^{p-1} + \frac{p(p-1)}{2} 2^{-2n} c_n^i\right) \le p2^{-p}(2^{p-1}-1)\left(\frac{2(p-1)}{3} - 1\right).$$

So if $2 < p < \frac{5}{2}$, then $\frac{2(p-1)}{3} - 1 < 0$ and

$$\int_0^1 f'(S(t))dS(t) < 0.$$

The equality $f(S(1)) - f(S(0)) = 0$ shows there should be a nonzero remainder. However, in this case, we still have

$$\int_0^1 \mathbf{1}_{\{S(t)=0\}} d[S]^p_\pi(t) = 1.$$

So Assumption 2.1 is not satisfied.

In fact, we can provide a formula for the Itô remainder term for this path and function $f = |x|^p$. Take $T = 1$, $m = \lfloor p \rfloor$ and $\alpha = p - m$. Let

$$G_f^p(a,b) = \frac{1}{(m-1)!|b-a|^p} \int_a^b (f^{(m)}(x) - f^{(m)}(a))(b-x)^{m-1} dx$$

for $a \neq b$ and take the limit value whenever it exists for $a = b$. For the function $f(x) = |x|^p$, we can see G_f^p is defined on $\mathbb{R}^2/\{(0,0)\}$ and for $k > 0$,

$$G_f^p(ka, kb) = G_f^p(a,b),$$

which suggests to consider G_f^p as a function on the unit circle S^1. We define the projection map $P : \mathbb{R}^2/\{(0,0)\} \to S^1$ by $p(x) = \frac{x}{\|x\|}$ and define a sequence of measures by

$$\tilde{\nu}_n := \frac{1}{N_n} \sum_{i=0}^{N_n - 1} \delta_{P(S(t_i), S(t_{i+1}))},$$

where N_n is the number of intervals in the partition π_n. Furthermore, we define

$$\hat{G}_f^p(\theta) = G_f^p(\cos(\theta), \sin(\theta))$$

for $\theta \in [0, 2\pi)$. Since the distance between two successive points in partition π_n is 2^{-n}, the remainder term $\sum_{[t_i, t_{i+1}] \in \pi_n} G_f^p(S_{t_{i+1}}, S_{t_i}) |S(t_{i+1}) - S(t_i)|^p$ for the partition π_n can be rewritten as

$$N_n \cdot 2^{-np} \int \hat{G}_f^p(x) \tilde{\nu}_n(dx)$$

and we have $N_n = \sum_{i=1}^n \sum_{j=1}^{2^{i-1}} \sum_{k=1}^{r_i} 2^{n-i+1} = 2^n \sum_{i=1}^n r_i$. Hence the remainder term is

$$2^{n-np} \sum_{i=1}^n r_i \int \hat{G}_f^p(x) \tilde{\nu}_n(dx).$$

We have $\lim_{n\to\infty} 2^{n-np} \sum_{i=1}^n r_i = 1$. It is easy to see that if we have the weak convergence of $\tilde{\nu}_n$, we can obtain the limit expression.

Let us now compute the limit of the sequence $\tilde{\nu}_n$. Since $S(t_i) = k2^{-n}$ for some positive number k and $t_i \in \pi_n$, the result $\tan(P(S(t_i), S(t_{i+1})))$ will be equal to $\frac{k+1}{k}$, which means the limit $\tilde{\nu}$ will be supported on these points. And

$$\tilde{\nu}\left(\arctan\left(\frac{k+1}{k}\right)\right) = \lim_{n\to\infty} \tilde{\nu}_n \left(\arctan\left(\frac{k+1}{k}\right)\right) = \frac{1}{2^{\lceil \log_2(k+1) \rceil p}} \frac{2^{p-1}-1}{2^p - 1}.$$

By symmetry, we have

$$\tilde{\nu}\left(\arctan\left(\frac{k}{k+1}\right)\right) = \tilde{\nu}\left(\arctan\left(\frac{k+1}{k}\right)\right) = \frac{1}{2^{\lceil \log_2(k+1) \rceil p}} \frac{2^{p-1}-1}{2^p - 1}.$$

Hence we have a change of variable formula for the path S defined in Example 2.8 and $f(x) = |x|^p$:

$$f(S(1)) - f(S(0)) = \int_0^1 f'(S(u)) dS(u) + \int_{S^1} \hat{G}_f^p(x) \tilde{\nu}(dx). \tag{8}$$

The above calculations show that the remainder term may be nonstandard when Assumption 2.1 is not satisfied.

We can give in fact a general method for calculating the remainder term for paths satisfying certain assumptions.

For a function f and $p = m + \alpha$, where $m = \lfloor p \rfloor$, we consider the map G_f^p : $\{(x, y) \in \mathbb{R}^2,\ x \neq y\} \mapsto \mathbb{R}$ defined by

$$G_f^p(a, b) = \frac{1}{(m-1)!|b-a|^p} \int_a^b (f^{(m)}(x) - f^{(m)}(a))(b-x)^{m-1} dx. \tag{9}$$

If $f^{(m)}$ is continuous, it is easy to see G_f^p is a continuous function on $\{(x, y) \in \mathbb{R}^2,\ x \neq y\}$.

In order to compute the remainder term in the fractional Itô formula, we 'stratify' the increments across the partition by the values of G_f^p, to build an auxiliary quotient space X which is required to have certain properties.

Assumption 2.9 (Auxiliary quotient space). *There exist a space X and a map*

$$P_f^p : \{(x, y) \in \mathbb{R}^2,\ x \neq y\} \to X$$

such that

(i) *For all $x \in X$, the map G_f^p defined by* (9) *is constant on $(P_f^p)^{-1}(x)$;*

(ii) *The sequence of measures on $[0, T] \times X$ defined by*

$$\nu_n(dt, dx) := \sum_{[t_i, t_{i+1}] \in \pi_n} |S(t_{i+1}) - S(t_i)|^p \delta_{t_i}(t) \delta_{P_f^p(S(t_i), S(t_{i+1}))}(x)$$

converges weakly to a measure ν on $[0, T] \times X$.

Remark 2.10. The measure $\tilde{\nu}$ in Example 2.8 is actually the measure $\nu([0, 1], dx)$ on X defined in Assumption 2.9. In fact, we have

$$\tilde{\nu}_n(dx) = \frac{1}{[S]_n^p(1)} \nu_n([0, 1], dx),$$

where $[S]_n^p(1) = \sum_{[t_i, t_{i+1}] \in \pi_n, t_i \leq 1} |S(t_{i+1}) - S(t_i)|^p$. Letting $n \to \infty$, weak convergence implies

$$\tilde{\nu}(dx) = \frac{1}{[S]^p(1)} \nu([0, 1], dx) = \nu([0, 1], dx).$$

Hence Assumption 2.9 is a reasonable condition which covers Example 2.8.

Furthermore, as $S \in V_p(\pi)$ is equivalent to the weak convergence of

$$\nu_n(dt, X) = \sum_{[t_i, t_{i+1}] \in \pi_n} |S(t_{i+1}) - S(t_i)|^p \delta_{t_i}(t),$$

Assumption 2.9 is in fact stronger than the existence of pth variation along the sequence $\{\pi_n\}$.

One can always choose $X = \mathbb{R}$ but this choice may not be the most tractable for the calculation of the remainder term.

Theorem 2.11. *Let $S \in V_p(\pi)$ such that there exists (X, P_f^p) satisfying Assumption* 2.9. *There exists a unique map $\hat{G}_f^p : X \to \mathbb{R}$ such that $\hat{G}_f^p \circ P_f^p = G_f^p$ and*

$$f(S(t)) - f(S(0)) = \int_0^t f'(S(u)) dS(u) + \int_0^t \int_X \hat{G}_f^p(x) \nu(ds, dx),$$

where $\int_0^t f'(S(u)) dS(u)$ is defined as in Theorem 2.6.

Proof. The existence of $\hat{G}_f^p$ is given by universality of the quotient map construction. Using a Taylor expansion, we have actually

$$\begin{aligned} &f(S(t)) - f(S(0)) \\ &\quad = \sum_{[t_i,t_{i+1}]\in\pi_n, t_i\le t} \sum_{j=1}^{m} \frac{f^{(j)}(S(t\wedge t_i))}{j!}(S(t\wedge t_{i+1}) - S(t\wedge t_i))^j \\ &\qquad + \sum_{[t_i,t_{i+1}]\in\pi_n, t_i\le t} G_f^p(S(t\wedge t_i), S(t\wedge t_{i+1}))|S(t\wedge t_{i+1}) - S(t\wedge t_i)|^p. \end{aligned}$$

By definition of ν_n, we have

$$\begin{aligned} f(S(t)) - f(S(0)) = &\sum_{[t_i,t_{i+1}]\in\pi_n, t_i\le t} \sum_{j=1}^{m} \frac{f^{(j)}(S(t\wedge t_i))}{j!}(S(t\wedge t_{i+1}) - S(t\wedge t_i))^j \\ &+ \int_0^t \int_X \hat{G}_f^p(x)\tilde{\nu}_n(dsdx). \end{aligned}$$

Using the weak convergence of ν_n in Assumption 2.9 yields the desired result. $\square$

We can see from Theorem 2.11 that when the Itô remainder term is nonzero, the formula is not in the classical form. Hence we will need more information to describe the Itô remainder term in the fractional case.

Let us go back to zero remainder case. Consider the set $(E^p)'$ of all functions satisfying Assumptions 2.2 and 2.3. Then $(E^p)'$ is a subset of $C^p(\mathbb{R})$. Let E^p be the closure of $(E^p)'$ in $C^p(\mathbb{R})$ equipped with the semi-norms

$$\sup_{x\in K}\{f^{(k)}(x) : k = 0, \cdots, m\} + \sup_{x,y\in K}\left\{\frac{|f^{(m)}(y) - f^{(m)}(x)|}{|y-x|^\alpha}\right\},$$

where $p = m + \alpha$ with $m < p < m + 1$. Then we have the following theorem.

Theorem 2.12. *Let $S \in V_p(\pi)$ satisfying Assumption 2.1 and $f \in E^p$. Then for any $t \in [0, T]$*

$$f(S(t)) = f(S(0)) + \int f'(S(u))dS(u),$$

where the integral is a limit of compensated Riemann sums of order $m = \lfloor p \rfloor$:

$$\int f'(S(u))dS(u) = \lim_{n\to\infty} \sum_{t_i\in\pi_n} \sum_{j=1}^{m} \frac{f^{(j)}(S(t\wedge t_i))}{j!}(S(t\wedge t_{j+1}) - S(t\wedge t_j))^j.$$

Proof. We prove the result for $t = T$ for convenience. The case $t \neq T$ can be obtained by changing all t_i to $t \wedge t_i$. Suppose $f_k \in (E^p)'$ and $f_k \to f$ in C^p. We have

$$f(S(T)) - f(S(0)) = L^n + \sum_{t_i\in\pi_n} \frac{1}{\Gamma(m)} \int_{S(t_i)}^{S(t_{i+1})} (f^{(m)}(r) - f^{(m)}(S(t_i)))(S(t_{i+1}) - r)^{m-1}dr.$$

Let K be a compact set containing the path of $S([0, T])$. Since $f_k \to f$ in C^p, we have for every $\epsilon > 0$, there exists $N > 0$ and for all $k > N$,

$$\sup_{x,y\in K} \frac{|(f^{(m)} - f_k^{(m)})(y) - (f^{(m)} - f_k^{(m)})(x)|}{|y-x|^\alpha} \le \epsilon.$$

Hence

$$\left|\sum_{[t_i,t_{i+1}]\in\pi_n}\frac{1}{\Gamma(m)}\int_{S(t_i)}^{S(t_{i+1})}((f^{(m)}-f_k^{(m)})(r)-(f^{(m)}-f_k^{(m)})(S(t_i)))(S(t_{i+1})-r)^{m-1}dr\right|$$
$$\le \epsilon\sum_{[t_i,t_{i+1}]\in\pi_n}\frac{1}{\Gamma(m)}\int_{S(t_i)}^{S(t_{i+1})}|r-S(t_i)|^{\alpha}|S(t_{i+1})-r|^{m-1}dr$$
$$\le \epsilon\sum_{[t_i,t_{i+1}]\in\pi_n}\frac{1}{m!}|S(t_{i+1})-S(t_i)|^{p}.$$

Hence we have

$$\limsup_{n\to\infty}\left|\sum_{[t_i,t_{i+1}]\in\pi_n}\frac{1}{\Gamma(m)}\int_{S(t_i)}^{S(t_{i+1})}(f^{(m)}(r)-f^{(m)}(S(t_i)))(S(t_{i+1})-r)^{m-1}dr\right|$$
$$\le \limsup_{n\to\infty}\left|\sum_{[t_i,t_{i+1}]\in\pi_n}\frac{1}{\Gamma(m)}\int_{S(t_i)}^{S(t_{i+1})}(f_k^{(m)}(r)-f_k^{(m)}(S(t_i)))(S(t_{i+1})-r)^{m-1}dr\right|$$
$$+\frac{\epsilon}{m!}[S]^p(t)$$

for all $k>N$. Since $f_k\in E$, we then have

$$\limsup_{n\to\infty}\left|\sum_{[t_i,t_{i+1}]\in\pi_n}\frac{1}{\Gamma(m)}\int_{S(t_i)}^{S(t_{i+1})}(f^{(m)}(r)-f^{(m)}(S(t_i)))(S(t_{i+1})-r)^{m-1}dr\right|\le\frac{\epsilon}{m!}[S]^p(t).$$

Finally letting ϵ tends to 0, we have

$$\lim_{n\to\infty}\left|\sum_{[t_i,t_{i+1}]\in\pi_n}\frac{1}{\Gamma(m)}\int_{S(t_i)}^{S(t_{i+1})}(f^{(m)}(r)-f^{(m)}(S(t_i)))(S(t_{i+1})-r)^{m-1}dr\right|=0.$$

Hence L^n converges when n tends to infinity and we have the desired result. $\square$

Remark 2.13. It can be seen in the proof of Theorem 2.6 that we only need

$$\int_0^T \mathbf{1}_{\{S(t)=k\}}d[S]^p_\pi(t)=0 \tag{10}$$

for $k\in\Gamma_f^c$. Denote P_f the set of paths satisfying (10) on Γ_f^c.

Given a continuous path $S\in V_p(\pi)$, then by Fubini's theorem, we actually have that

$$\int_{\mathbb{R}}\int_0^T \mathbf{1}_{\{S(t)=k\}}d[S]^p_\pi(t)dk=0,$$

which means (10) is satisfied by almost all $k\in\mathbb{R}$. Hence we can consider the set $(E_S^p)'$ of all functions f such that Γ_f contains all points that do not satisfy condition (10). We can then construct the closure E_S^p for a given path S.

Example 2.14 (Examples of functions belonging to E^p)**.**

(1) All functions $f\in C^{m+1}(\mathbb{R})$.

(2) The function $f(x)=|x-k|^p$ for some $k\in\mathbb{R}$. It can be seen in Example 1.12 that $C^p_{a^+}f(x)$ is continuous on $\{(\bar a,\bar x)\in U\times U:\bar a\le\bar x\}$ for any compact set U contained in (k,∞) or $(-\infty,k)$, which means $\Gamma_f=\mathbb{R}\setminus\{k\}$ and hence is a function in $E_p'\subset E_p$.

(3) The linear combinations of $|x-k|^p$. For example $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}|x-x_n|^p$ with $\{x_n\}$ an ordered set of rationals in $[0,1]$.

Remark 2.15. Smooth functions belong to E^p. Denote by $C^{p+}(\mathbb{R}) \subset E^p$ the completion of the smooth function under Hölder norm, i.e., the set of functions $f \in C^p$ such that on every compact set K:

$$\lim_{\delta\to 0} \sup_{\substack{x,y\in K \\ |x-y|\le\delta}} \left\{ \frac{|f^{(m)}(x) - f^{(m)}(y)|}{|x-y|^\alpha} \right\} = 0.$$

Then for any $q \notin \mathbb{N} > p$, we have $E^q \subset C^q \subset C^{p+} \subset E_S$ for any path $S \in V_p(\pi) \cap C([0,T],\mathbb{R})$.

Example 2.16 (Example for P_f with $f(x) = |x|^p$). Let B^H be a fractional Brownian motion with Hurst index H on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $p = 1/H$. Then

$$\mathbb{E}\left[\int_0^T \mathbf{1}_{\{B^H(t)=0\}} dt\right] = 0 \qquad \text{so} \qquad \mathbb{P}(B^H \in P_f^p) = 1.$$

Remark 2.17 (Time-dependent case). Along the lines of the proofs of Theorems 2.6 and 2.12, we can generalize these formulas to the time-dependent case with a modified definition of the set Γ_f. For $f \in C^{1,p}([0,T]\times\mathbb{R})$, define the open set

$$\Gamma_f = \{(t,x) \in [0,T]\times\mathbb{R} : \exists \text{ open set } U \ni (t,x) \\ \text{such that } C^p_{a+}f(s,b) \text{ is continuous on } U\tilde{\times}U\}$$

where $U\tilde{\times}U = \{(s,a,b) : (s,a),(s,b) \in U, a \le b\}$. Then under the assumption

$$\int_0^T \mathbf{1}_{\{(t,S(t))\notin\Gamma_f\}} d[S]^p_\pi(t) = 0,$$

we have the change of variable formula

$$f(t,S(t)) - f(0,S(0)) = \int_0^t \partial_t f(u,S(u))du + \int_0^t D_x f(u,S(u))dS(u),$$

where $m = \lfloor p \rfloor$ and

$$\int_0^t D_x f(u,S(u))dS(u) = \lim_{n\to\infty} \sum_{t_i\in\pi_n} \sum_{j=1}^m \frac{f^{(j)}(t\wedge t_i, S(t\wedge t_i))}{j!}(S(t\wedge t_{i+1}) - S(t\wedge t_i))^j.$$

Remark 2.18. Our argument also applies to the integer case. For example, when $p=2$, we have $m = \lfloor p \rfloor = 2$. For any function $f \in C^2$, $\Gamma_f = \mathbb{R}$. Hence we have

$$\int_0^T \mathbf{1}_{\{S(t)\in\Gamma_f^c\}} d[S]^p_\pi = 0.$$

Thus we have the result

$$f(S(t)) - f(S(0)) = \int_0^t f'(S(u))dS(u),$$

where

$$\int_0^t f'(S(u))dS(u) := \lim_{n\to\infty} \sum_{[t_i,t_{i+1}]\in\pi_n} f'(S(t_i))(S(t_{i+1})-S(t_i)) + \frac{1}{2}f''(S(t_i))(S(t_{i+1})-S(t_i))^2.$$

However, by changing the definition of $\int_0^t f'(S(u))dS(u)$ to

$$\lim_{n\to\infty} \sum_{[t_i,t_{i+1}]\in\pi_n} f'(S(t_i))(S(t_{i+1})-S(t_i)),$$

we recover the result in [15]. Similarly, for even integers p, a slight change of definition of $\int_0^t f'(S(u))dS(u)$ recovers the result in [11]. Additionally, we can also explore the odd integer by our argument. A similar change of definition of $\int_0^t f'(S(u))dS(u)$ gives the remainder term

$$\lim_{n\to\infty} \sum_{[t_i,t_{i+1}]\in\pi_n} \frac{1}{p!}f^{(p)}(S(t_i))(S(t_{i+1})-S(t_i))^p.$$

The condition $S \in V_p(\pi)$ means that this expression converges absolutely. However, as indicated in the appendix in [11], the limit would be typically zero.

2.2. **Extension to path-dependent functionals and ϕ-variation.** The above results may be extended to path functionals using the Dupire derivative [14] and the associated nonanticipative functional calculus [6, 7, 9, 10]. We recall here some concepts from the nonanticipative functional calculus [6, 7, 10].

Denote by $S_t(s) = S(s \wedge t)$ the path stopped at t. We consider the space $D([0,T],\mathbb{R})$ of càdlàg paths from $[0,T]$ to $\mathbb{R}$. Let

$$\Lambda_T = \{(t,\omega_t) : (t,\omega) \in [0,T] \times D([0,T],\mathbb{R})\}$$

be the space of stopped paths. This is a complete metric space equipped with

$$d_\infty((t,\omega),(t',\omega')) := \sup_{s\in[0,T]} |\omega(s\wedge t) - \omega'(s\wedge t')| + |t-t'|.$$

We will also need to stop paths 'right before' a given time, and set for $t > 0$

$$\omega_{t-}(s) := \begin{cases} \omega(s), & s < t, \\ \lim_{r\uparrow t} \omega(r), & s \geq t, \end{cases}$$

while $\omega_{0-} = \omega_0$.

Definition 2.19. A nonanticipative functional is a map $F : \Lambda_T \to \mathbb{R}$. Let F be a nonanticipative functional.

- We write $F \in \mathbb{C}_l^{0,0}$ if for all $t \in [0,T]$ the map $F(t,\cdot) : D([0,T],\mathbb{R}) \to \mathbb{R}$ is continuous and if for all $(t,\omega) \in \Lambda_T$ and all $\epsilon > 0$, there exists $\delta > 0$ such that for all $(t',\omega') \in \Lambda_T$ with $t' < t$ and $d_\infty((t,\omega),(t',\omega')) < \delta$, we have $|F(t,\omega) - F(t',\omega')| < \epsilon$.
- We write $F \in \mathbb{B}(\Lambda_T)$ if for every $t_0 \in [0,T)$ and every $K > 0$ there exists $C_{K,t_0} > 0$ such that for all $t \in [0,t_0]$ and all $\omega \in D([0,T],\mathbb{R})$ with $\sup_{s\in[0,t]} |\omega(s)| \leq K$, we have $|F(t,\omega)| \leq C_{K,t_0}$.

- F is horizontally differentiable at $(t,\omega)\in\Lambda_T$ if its horizontal derivative
$$\mathcal{D}F(t,\omega) := \lim_{h\downarrow 0}\frac{F(t+h,\omega_t)-F(t,\omega_t)}{h}$$
exists. If it exists for all $(t,\omega)\in\Lambda_T$, then $\mathcal{D}F$ is a nonanticipative functional.
- F is vertically differentiable at $(t,\omega)\in\Lambda_T$ if its vertical derivative
$$\nabla_\omega F(t,\omega) := \lim_{h\to 0}\frac{F(t,\omega_t+h\mathbf{1}_{[t,T]})-F(t,\omega_t)}{h}$$
exists. If it exists for all $(t,\omega)\in\Lambda_T$, then $\nabla_\omega F$ is a nonanticipative functional. In particular, we define recursively $\nabla_\omega^{k+1}F := \nabla\nabla_\omega^k F$ whenever this is well-defined.
- For $p\in\mathbb{N}_0$, we say that $F\in\mathbb{C}_b^{1,p}(\Lambda_T)$ if F is horizontally differentiable and p times vertically differentiable in every $(t,\omega)\in\Lambda_T$, and if $F,\mathcal{D}F,\nabla_\omega^k F\in\mathbb{C}_l^{0,0}(\Lambda_T)\cap\mathbb{B}(\Lambda_T)$ for $k=1,\cdots,p$.

There is no straightforward extension of fractional Caputo derivative for nonanticipative functional. We choose an alternative approach to obtain extension of our change of variable formulas to the path-dependent case.

Definition 2.20 (Vertical Hölder continuity)**.** Let $0<\alpha<1$.

- F is vertically α–Hölder continuous on $E\subset\Lambda_T$ if
$$L := \lim_{\epsilon\to 0}\sup_{(t,\omega)\in E}\sup_{|h|\le\epsilon}\frac{|F(t,\omega_t+h\mathbf{1}_{[t,T]})-F(t,\omega_t)|}{|h|^\alpha}<\infty.$$
We call L the vertical Hölder coefficient of F over E, and we denote $F\in\mathbb{C}^{0,\alpha}(E)$.
- Let $E_n=\{(t,\omega)\in\Lambda : d((t,\omega),(0,0))<n\}$. F is locally vertically α–Hölder continuous on $E\subset\Lambda_T$ if F is vertically α–Hölder continuous on E_n for each n. We denote $F\in\mathbb{C}^{0,\alpha}_{loc}(E)$.
- For $m<p<m+1$, we say $F\in\mathbb{C}^{0,p}_{loc}$ if $\nabla_\omega^m F\in\mathbb{C}^{0,p-m}_{loc}$. For $E\subset\Lambda_T$, we say $F\in\mathbb{C}^{0,p}(E)$ if $\nabla_\omega^m F\in\mathbb{C}^{0,p-m}(E)$.

We now give assumptions on the functional $F\in\mathbb{C}_b^{1,m}\cap\mathbb{C}^{0,p}_{loc}$, where $m=\lfloor p\rfloor$.

Assumption 2.21. *There exists a sequence of continuous functions $g_k\in\mathbb{C}(\Lambda_T,\mathbb{R})$ with support $E_k := supp(g_k)$ such that*

(1) $E_{k+1}\subset E_k$*,* $g_k|_{E_{k+1}}=1, g_k\le 1$ *and* $F\in\mathbb{C}^{0,p}_{loc}(\Lambda_T\setminus E_k)$ *for each k with vertical Hölder coefficient $L=0$.*
(2) *There exists a bounded function g on Λ_T such that $g_k\to g$ pointwise and*
$$\int_0^T g((t,S_t))d[S]_t^p=0.$$

Define the piecewise-constant approximation S^n to S along the partition π_n:
$$S^n(t) := \sum_{[t_i,t_{i+1}]\in\pi_n} S(t_{j+1})\mathbf{1}_{[t_j,t_{j+1})}(t)+S(T)\mathbf{1}_{\{T\}}(t).$$
Then $\lim_{n\to\infty}\|S^n-S\|_\infty=0$ whenever $osc(S,\pi_n)\to 0$.

Theorem 2.22. *Let $p \notin \mathbb{N}$, $m = \lfloor p \rfloor$ and $S \in V_p(\pi)$. If $F \in \mathbb{C}_b^{1,m} \cap \mathbb{C}_{loc}^{0,p}$ satisfies Assumption* 2.21 *then the limit*

$$\int_0^T \nabla_\omega F(t, S_{t-}) d^\pi S(t) := \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\pi_n} \sum_{k=1}^m \frac{1}{k!} \nabla_\omega^k F(t_j, S_{t_j-}^n)(S(t_{j+1}\wedge t) - S(t_j \wedge t))^k$$

exists and we have

$$F(t, S_t) = F(0, S_0) + \int_0^t \mathcal{D}F(s, S_s)ds + \int_0^t \nabla_\omega F(s, S_s)dS(s).$$

Proof. We write

$$\begin{aligned} F(t, S_t^n) - F(0, S_0^n) &= \sum_{[t_j,t_{j+1}]\in\pi_n} (F(t_{j+1} \wedge t, S_{t_{j+1}\wedge t-}^n) - F(t_j \wedge t, S_{t_j\wedge t-}^n)) \\ &\quad + F(t, S_t^n) - F(t, S_{t-}^n) \\ &= \sum_{[t_j,t_{j+1}]\in\pi_n} (F(t_{j+1} \wedge t, S_{t_{j+1}\wedge t-}^n) - F(t_j \wedge t, S_{t_j\wedge t-}^n)) + o(1). \end{aligned}$$

We only need to consider j with $t_{j+1} \le t$ since the remainder is $o(1)$ as $n \to \infty$ by left continuity of F. We split the difference into two parts:

$$\begin{aligned} &F(t_{j+1}, S_{t_{j+1}-}^n) - F(t_j, S_{t_j-}^n) \\ &\quad = (F(t_{j+1}, S_{t_{j+1}-}^n) - F(t_j, S_{t_j}^n)) + (F(t_j, S_{t_j}^n) - F(t_j, S_{t_j-}^n)). \end{aligned}$$

Now since $S_{t_j}^n = S_{t_{j+1}-}^n$ on $[0, t_{j+1}]$, we have

$$F(t_{j+1}, S_{t_{j+1}-}^n) - F(t_j, S_{t_j}^n) = \int_{t_j}^{t_{j+1}} \mathcal{D}F(u, S_{t_j}^n)du = \int_{t_j}^{t_{j+1}} \mathcal{D}F(u, S^n)du,$$

since $\mathcal{D}F$ is a nonanticipative functional. Hence we have

$$\lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} (F(t_{j+1}, S_{t_{j+1}-}^n) - F(t_j, S_{t_j}^n)) = \int_0^t \mathcal{D}F(u, S_u)du$$

by continuity. It remains to consider the term

$$F(t_j, S_{t_j}^n) - F(t_j, S_{t_j-}^n) = F(t_j, S_{t_j-}^{n,S_{t_j,t_{j+1}}}) - F(t_j, S_{t_j-}^n),$$

where $S_{t_j,t_{j+1}} = S(t_{j+1}) - S(t_j)$ and $S_{t_j-}^{n,x}(s) := S_{t_j-}^n(s) + \mathbf{1}_{[t_j,T]}(s)x$. Now from Taylor's formula and definition of vertical derivative, we obtain

$$\begin{aligned} F(t_j, S_{t_j-}^{n,S_{t_j,t_{j+1}}}) &= F(t_j, S_{t_j-}^n) + \sum_{k=1}^m \frac{\nabla_\omega^k F(t_j, S_{t_j-}^n)}{k!}(S(t_{j+1}) - S(t_j))^k \\ &\quad + \frac{1}{(m-1)!} \int_0^{S_{t_j,t_{j+1}}} (\nabla_\omega^m F(t_j, S_{t_j-}^{n,u}) - \nabla_\omega^m F(t_j, S_{t_j-}^n))(S_{t_j,t_{j+1}} - u)^{m-1}du. \end{aligned}$$

From the definition of $\mathbb{C}^{0,p}$, we see that there exists $M > 0$ such that

$$|\nabla_\omega^m F(t_j, S_{t_j-}^{n,u}) - \nabla_\omega^m F(t_j, S_{t_j-}^n)| \le M|u|^{p-m},$$

for each t_j when n is large enough. This shows that

$$\left|\frac{1}{(m-1)!}\int_0^{S_{t_j,t_{j+1}}}(\nabla_\omega^m F(t_j,S^{n,u}_{t_j-})-\nabla_\omega^m F(t_j,S^n_{t_j-}))(S_{t_j,t_{j+1}}-u)^{m-1}du\right| \\ \leq \tilde{M}|S(t_{j+1})-S(t_j)|^p.$$

for some constant $\tilde{M}$. Fix $k\in\mathbb{N}$, and divide the partition intervals into two parts. The first part π_n^1 contains $[t_j,t_{j+1}]$ such that $(t_j,S^n_{t_j-})\in E_k$ and the second part π_n^2 contains $[t_j,t_{j+1}]$ such that $(t_j,S^n_{t_j-})\notin E_k$. Then we have

$$\sum_{[t_j,t_{j+1}]\in\pi_n}\left|\frac{1}{(m-1)!}\int_0^{S_{t_j,t_{j+1}}}(\nabla_\omega^m F(t_j,S^{n,u}_{t_j-})-\nabla_\omega^m F(t_j,S^n_{t_j-}))(S_{t_j,t_{j+1}}-u)^{m-1}du\right| \\ \leq \sum_{[t_j,t_{j+1}]\in\pi_n^1}\tilde{M}|S(t_{j+1})-S(t_j)|^p+\sum_{[t_j,t_{j+1}]\in\pi_n^2}\tilde{M}L(S^n_{t_j-},osc(S,\pi_n))|S(t_{j+1})-S(t_j)|^p,$$

where

$$L(\omega,\epsilon):=\sup_{|h|\leq\epsilon}\frac{|\nabla_\omega^m F(t,\omega_t+h\mathbf{1}_{[t,T]})-\nabla_\omega^m F(t,\omega_t)|}{|h|^{p-m}}.$$

From the definition of vertical Hölder continuity, we have

$$\lim_{\epsilon\to0}\sup_{\omega\in E\setminus E_k}L(\omega,\epsilon)=0$$

for each k with E a bounded subset of Λ_T containing all stopped paths occurring in this proof. Hence

$$\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n^2}\tilde{M}L(S^n_{t_j-},osc(S,\pi_n))|S(t_{j+1})-S(t_j)|^p=0.$$

As for the first part, we have

$$\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n^1}\tilde{M}|S(t_{j+1})-S(t_j)|^p\leq\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n}\tilde{M}g_k((t_j,S^n_{t_j-})|S(t_{j+1})-S(t_j)|^p \\ =\int_o^t\tilde{M}g_k((t,S_t))d[S]_t,$$

by Assumption 2.21. Letting k tend to infinity yields the result. □

The notion of pth order variation can be extended to the more general concept of ϕ-variation [19] for more general functions $\phi:\mathbb{R}\mapsto\mathbb{R}$:

Definition 2.23 (ϕ-variation along a partition sequence)**.** Let $\phi:\mathbb{R}\mapsto\mathbb{R}$ be an even function. A continuous path $S\in\mathbb{C}([0,T],\mathbb{R})$ is said to have a ϕ-variation along a sequence of partitions $\pi=(\pi_n)_{n\geq1}$ if $osc(S,\pi_n)\to0$ and the sequence of measures

$$\mu^n:=\sum_{t_i\in\pi_n}|\phi(S(t_{i+1})-S(t_i))|\delta(\cdot-t_i)$$

converges weakly to a measure μ without atoms. In this case we write $S\in V_\phi(\pi)$ and $[S]^\phi(t)=\mu([0,t])$ for $t\in[0,T]$, and we call $[S]^\phi$ the ϕ-variation of S.

Remark 2.24. If we consider an expansion of f of the form

$$f(y)=f(x)+\sum_{k=1}^m\frac{f^{(k)}}{k!}(y-x)^k+g(x)\phi(y-x)+o(\phi(y-x)),$$

then following the steps in the proof of Proposition 1.16, we can show that $g = 0$ almost everywhere. Hence a regular remainder should not appear in this case.

Under the following condition on the function f we can obtain an extension of Theorem 2.22 to the case of paths with finite ϕ-variation.

Assumption 2.25. *There exists a sequence of open sets U_i such that $\bar{U}_{i+1} \subset U_i$ for $i = 1, 2, \cdots$ and $\lim_{\delta\to 0} \sup_{|x-y|\le\delta, x,y\in K} \frac{f^{(m)}(y)-f^{(m)}(x)}{|\phi^{(m)}(y-x)|} = 0$ for all compact sets $K \in \mathbb{R} \setminus U_i$ and, denoting $C = \cap_i^\infty \bar{U}_i$, we have*

$$\int_0^T \mathbf{1}_{\{S(t)\in C\}} d[S]_\pi^\phi(t) = 0.$$

We state the following result without proof, the proof being similar to that of Theorem 2.22.

Theorem 2.26. *Let $m \in \mathbb{N}$ and $\phi \in C^m(\mathbb{R}^+, \mathbb{R}^+)$ such that*

$$\lim_{x\to 0} \frac{\phi(x)}{x^m} = 0, \qquad \lim_{x\to 0} \frac{\phi(x)}{x^{m+1}} = \infty.$$

If $f \in C^m(\mathbb{R})$ is such that Assumption 2.25 is satisfied, we have

$$f(S(t)) - f(S(0)) = \int_0^t f'(S(u))dS(u),$$

where the integral is defined as a (pointwise) limit of compensated Riemann sums:

$$\int_0^t f'(S(u))dS(u) = \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^m \frac{f^{(k)}(S(t_j))}{k!}(S(t_{j+1}\wedge t) - S(t_j\wedge t))^k.$$

3. An isometry formula for the pathwise integral

Ananova and Cont [1] proved a pathwise analogue of the Itô isometry for the Föllmer integral with respect to paths with finite quadratic variation. This relation was extended to $p \ge 1$ in [11]. In the same flavor we derive here an isometry relation for the pathwise integral in terms of the ϕ-variation, where $\phi : \mathbb{R} \mapsto \mathbb{R}$ is an even function satisfying the following assumptions.

Assumption 3.1.

(1) *ϕ is strictly increasing, continuous and $\phi(0) = 0$.*
(2) *ϕ is convex on $[0, \infty)$ and $x \mapsto \log\phi(x)$ is a convex function of $\log x$ on $(0, \infty)$.*
(3) *For $x > 0$, the limit*

$$\varphi(x) := \lim_{y\to 0} \frac{\phi(xy)}{\phi(y)}$$

exists and the convergence is uniform on bounded sets.

(4) $\infty > p(\phi) = \sup\left\{p : \lim_{x\to 0} \frac{\phi(x)}{|x|^p} = 0\right\} > 0.$

We first introduce a generalized version of Minkowski's inequality here [23].

Lemma 3.2. *If ϕ satisfies first two conditions in Assumption 3.1, then for any positive sequences $\{a_n\}$ and $\{b_n\}$ we have*

$$\phi^{-1}\left(\sum \phi(a_n + b_n)\right) \le \phi^{-1}\left(\sum \phi(a_n)\right) + \phi^{-1}\left(\sum \phi(b_n)\right).$$

Theorem 3.3. *Let ϕ be an even function satisfying Assumption 3.1 and π_n a sequence of partitions of $[0,T]$ with vanishing mesh. Let $S \in V_\phi(\pi) \cap C^\alpha([0,T],\mathbb{R})$ for some $\alpha > \frac{\sqrt{1+\frac{4}{p(\phi)}}-1}{2}$. Let $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$ be Lipschitz-continuous with respect to d_∞ with $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T)$. Then $F(\cdot,S) \in V_\phi(\pi)$ and*

$$[F(\cdot,S)]_\pi^\phi(t) = \int_0^t \varphi(|\nabla_\omega F(s,S_s)|)d[S]_\pi^\phi(s).$$

Proof. We have the same

$$R_F(s,t) := F(t,S_t) - F(s,S_t) - \nabla_\omega F(s,S_s)(S(t)-S(s)), \quad |R_F(s,t)| \le C|t-s|^{\alpha+\alpha^2}.$$

Let $\gamma_F(s,t) := \nabla_\omega F(s,S_s)(S(t)-S(s))$, we have from Lemma 3.2 that

$$\begin{aligned}
&\phi^{-1}\left(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n\\ t_{j+1}\le t}} \phi(|F(t_{j+1},S_{t_{j+1}}) - F(t_j,S_{t_j})|)\right) \\
&\quad \le \phi^{-1}\left(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n\\ t_{j+1}\le t}} \phi(|R_F(t_j,t_{j+1})|)\right) + \phi^{-1}\left(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n\\ t_{j+1}\le t}} \phi(|\gamma_F(t_j,t_{j+1})|)\right) \\
&\quad \le 2\phi^{-1}\left(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n\\ t_{j+1}\le t}} \phi(|R_F(t_j,t_{j+1})|)\right) \\
&\qquad + \phi^{-1}\left(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n\\ t_{j+1}\le t}} \phi(|F(t_{j+1},S_{t_{j+1}}) - F(t_j,S_{t_j})|)\right)
\end{aligned} \tag{11}$$

and since $R_F(s,t) \le C|t-s|^{\alpha^2+\alpha}$, we have

$$\phi^{-1}\left(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n\\ t_{j+1}\le t}} \phi(|R_F(t_j,t_{j+1})|)\right) \le \phi^{-1}\left(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n\\ t_{j+1}\le t}} \phi(C|t_{j+1}-t_j|^{\alpha+\alpha^2})\right).$$

Furthermore due to $\alpha > \frac{\sqrt{1+\frac{4}{p(\phi)}}-1}{2}$, we know $\alpha+\alpha^2 > \frac{1}{p(\phi)}$. Hence there exists $\epsilon > 0$ such that

$$\phi(C|t_{j+1}-t_j|^{\alpha+\alpha^2}) \le \phi(C|t_{j+1}-t_j|^{\frac{1}{p(\phi)-\epsilon}}).$$

Then by the definition of $p(\phi)$, we see that

$$\phi(C|t_{j+1}-t_j|^{\frac{1}{p(\phi)-\epsilon}}) \le \omega(|t_{j+1}-t_j|)C^{p(\phi)-\epsilon/2}|t_{j+1}-t_j|^{\frac{p(\phi)-\epsilon/2}{p(\phi)-\epsilon}},$$

where ω is continuous on $\mathbb{R}^+$ and $\omega(0)=0$. Combine all above, we get

$$\lim_{n\to\infty}\phi^{-1}\left(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n\\ t_{j+1}\le t}}\phi(|R_F(t_j,t_{j+1})|)\right)=0$$

and we also have

$$\lim_{n\to\infty}\phi^{-1}\left(\sum_{\substack{[t_j,t_{j+1}]\in\pi_n\\ t_{j+1}\le t}}\phi(|\gamma_F(t_j,t_{j+1})|)\right)=\phi^{-1}\left(\int_0^t\varphi(|\nabla_\omega F(s,S_s)|)d[S]^\phi(s)\right).$$

In fact, since $\nabla_\omega F(t,S_t)\in\mathbb{B}(\Lambda_T)$, there exists $M>0$ such that $|\nabla_\omega F(u,S_u)|\le M$. For each $\epsilon>0$, there exists $\delta>0$ such that

$$\left|\frac{\phi(xy)}{\phi(y)}-\varphi(x)\right|\le\epsilon.$$

for all $|x|\le M, |y|\le\delta$. Then we have for n large enough (i.e., $osc(S,\pi_n)\le\delta$)

$$\begin{aligned}&\sum_{[t_j,t_{j+1}]\in\pi_n,t_{j+1}\le t}\phi(|\nabla_\omega F(t_j,S_{t_j})(S(t_{j+1})-S(t_j))|)\\ &\quad\le\sum_{[t_j,t_{j+1}]\in\pi_n,t_{j+1}\le t}\varphi(|\nabla_\omega F(t_j,S_{t_j})|)\phi(|S(t_{j+1})-S(t_j)|)+\epsilon\phi(|S(t_{j+1})-S(t_j)|).\end{aligned}$$

Then let n tend to infinity and ϵ tend to 0, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n,t_{j+1}\le t}\phi(|\nabla_\omega F(t_j,S_{t_j})(S(t_{j+1})-S(t_j))|)\\ &\quad=\int_0^t\varphi(|\nabla_\omega F(s,S_s)|)d[S]^\phi(s).\end{aligned}$$

Then taking limit in (11), we obtain the result. □

In particular, if we set $\phi(x)=|x|^p$ for $p>0$, we have $\varphi(x)=\lim_{y\to0}\frac{|xy|^p}{|y|^p}=|x|^p$. Hence, we have the corollary.

Corollary 3.4. *Let $p\in\mathbb{R}^+$, and $\alpha>((1+\frac{4}{p})^{\frac12}-1)/2$, let π_n be a sequence of partitions with vanishing mesh, and let $S\in V_p(\pi)\cap C^\alpha([0,T],\mathbb{R})$. Let $F\in\mathbb{C}_b^{1,2}(\Lambda_T)$ such that $\nabla_\omega F\in\mathbb{C}_b^{1,1}(\Lambda_T))$. Assume furthermore that F is Lipschitz-continuous with respect to d_∞. Then $F(\cdot,S)\in V_p(\pi)$ and*

$$[F(\cdot,S)]^p(t)=\int_0^t|\nabla_\omega F(s,S_s)|^p d[S]^p(s).$$

Remark 3.5. Notice that there always exists $\alpha>\frac{\sqrt{1+\frac{4}{p(\phi)}}-1}{2}$ such that $V_\phi\cap C^\alpha([0,T],\mathbb{R})$ is not empty.

Example 3.6. We can see $\phi(x) = \frac{x}{\sqrt{-\log x}}$ satisfies Assumption 3.1. In fact, it is strictly increasing and continuous and $\phi(0) = 0$. Furthermore,

$$\phi'(x) = \frac{1}{\sqrt{-\log x}} + \frac{1}{2(\sqrt{-\log x})^3} > 0, \text{ and}$$
$$\phi''(x) = \frac{1}{2x(\sqrt{-\log x})^3} + \frac{3}{4x(\sqrt{-\log x})^5} > 0.$$

Hence ϕ is convex. And we have

$$\log \phi(x) = \log x - \frac{\log(-\log x)}{2}.$$

It is easy to check that $x \to x - \frac{\log(-x)}{2}$ is convex. And

$$\varphi(x) = \lim_{y\to 0} \frac{\phi(|x||y|)}{\phi(|y|)} = \lim_{y\to 0} |x| \sqrt{\frac{-\log y}{-\log x - \log y}} = |x|.$$

Hence we have

$$[F(\cdot, S)]^{\phi}(t) = \int_0^t |\nabla_\omega F(s, S_s)| d[S]^{\phi}_{\pi}(s).$$

4. Multidimensional extensions

The extension of the above results to the multidimensional case is not entirely straightforward, as the space $V_p(\pi)$ is not a vector space [28].

In the case of integer p, some definitions may be extended to the vector case by considering symmetric tensor-valued measures as in [11] but this is not convenient for the fractional case. Our definition below is equivalent to the definition in [11] when p is an integer.

Definition 4.1. Let $p \geq 1$ and $S = (S^1, \cdots, S^d) \in C([0,T], \mathbb{R}^d)$ be a continuous path. Let $\{\pi_n\}$ be a sequence of partition of $[0,T]$. We say that S has a pth order variation along $\pi = \{\pi_n\}$ if $osc(S, \pi_n) \to 0$ and for any linear combination $S_a := \sum_{i=1}^d a_i S^i$, we have $S_a \in V_p(\pi)$. Furthermore, we denote μ_{S_a} to be the weak limit of measures

$$\mu^n_{S_a} := \sum_{[t_j, t_{j+1}] \in \pi_n} \delta(\cdot - t_j) |S_a(t_{j+1}) - S_a(t_j)|^p.$$

We denote $S \in V_p(\pi)$ if S satisfies this property.

Remark 4.2. It can be easily seen [13] that multidimensional fractional Brownian motion satisfies this property along sequences of partitions with fine enough mesh.

Theorem 4.3. *Let $p = m + \alpha$ with $m = \lfloor p \rfloor$ and $S \in V_p(\pi)$. Assume $f : \mathbb{R}^d \to \mathbb{R}$ satisfies*

(1) $\nabla^m f(x) \in C^{\alpha}_{loc}(Sym_m(\mathbb{R}^d))$ *and there exists a sequence of open sets U_k such that $\bar{U}_{k+1} \subset U_k$ and*

$$\lim_{y \to x} \frac{\|\nabla^m f(y) - \nabla^m f(x)\|}{\|y - x\|^{\alpha}} = 0,$$

locally uniformly on U_k^c.

(2) *Setting $C = \cap_k U_k$, for all $a = (a_1, \cdots, a_d) \in \mathbb{R}^d$ we have*

$$\int_0^T \mathbf{1}_{\{S(t)\in C\}} d\mu_{S_a} = 0,$$

where μ_{S_a} is defined as in Definition 4.1.

Then we have

$$f(S(t)) - f(S(0)) = \int_0^t \nabla f(S(u)) dS(u),$$

where

$$\int_0^t \nabla f(S(u)) dS(u) := \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^m \frac{1}{k!} < \nabla^k f(S(t_j)), (S(t_{j+1}) - S(t_j)^{\otimes k} > .$$

Before we prove the theorem, we give a lemma here.

Lemma 4.4. *Let $\alpha_1, \cdots, \alpha_d$ be positive numbers such that $p = \sum_{i=1}^d \alpha_i > 1$. Suppose $S \in V_p(\pi)$ in the sense of Definition 4.1. Then the limit of*

$$\sum_{[t_j,t_{j+1}]\in\pi_n} \Pi_{i=1}^d |S^i(t_{j+1}) - S^i(t_j)|^{\alpha_i}$$

is bounded by a sum of pth variations of 2^d different linear combinations of components of S.

Proof. For any positive number $\alpha_1, \cdots, \alpha_d$ such that $p = \sum_{i=1}^d \alpha_i > 1$, Young's inequality implies

$$\Pi_{i=1}^d |S^i(t_{j+1}) - S^i(t_j)|^{\frac{\alpha_i}{p}} \le \sum_{i=1}^d \frac{\alpha_i}{p} |S^i(t_{j+1}) - S^i(t_j)|,$$

which shows that

$$\Pi_{i=1}^d |S^i(t_{j+1}) - S^i(t_j)|^{\alpha_i} \le \sum_{\epsilon_i=\pm 1} |\sum_{i=1}^d \epsilon_i \frac{\alpha_i}{p} (S^i(t_{j+1}) - S^i(t_j))|^p.$$

By taking the sum over the partition π_n and taking the limit $n \to \infty$, we then obtain the desired result. $\square$

Proof of Theorem 4.3. The technique for the proof is the same as the previous theorems. We only consider the situation $t = T$, the case $t < T$ is similar with an $o(1)$ additional term. Using a Taylor expansion,

$$\begin{aligned} &f(S(T)) - f(S(0)) \\ &\quad = \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^m \frac{1}{k!} < \nabla^m f(S(t_j)), (S(t_{j+1}) - S(t_j))^{\otimes k} > \\ &\qquad + \frac{1}{(m-1)!} \int_0^1 < \nabla^m f(S(t_j) + \lambda(S(t_{j+1}) - S(t_j))) - \nabla^m f(S(t_j)), \\ &\qquad\qquad (S(t_{j+1}) - S(t_j))^{\otimes m} > (1-\lambda)^{m-1} d\lambda. \end{aligned}$$

We denote last integral by $R(t_j, t_{j+1})$. By Hölder continuity of $\nabla^m f$, there exists $M > 0$ such that

$$\frac{\|\nabla^m f(S(t_j) + \lambda(S(t_{j+1}) - S(t_j))) - \nabla^m f(S(t_j))\|}{\lambda^\alpha \|S(t_{j+1}) - S(t_j)\|^\alpha} \le M.$$

This leads to

$$|R(t_j,t_{j+1})|\le\frac{M}{(m-1)!}\int_0^1 \lambda^\alpha\|S(t_{j+1})-S(t_j)\|^\alpha\|(S(t_{j+1})-S(t_j))^{\otimes m}\|(1-\lambda)^{m-1}d\lambda.$$

For $k\in\mathbb{N}$, we divide the partition π_n into two components $\pi_n^1:=\{[t_j,t_{j+1}]\in\pi_n : S(t_j)\in U_k\}$ and $\pi_n^2:=\pi_n\setminus\pi_n^1$. On π_n^1, we have

$$\sum_{[t_j,t_{j+1}]\in\pi_n^1}|R(t_j,t_{j+1})|\le\sum_{[t_j,t_{j+1}]\in\pi_n^1}\tilde{M}\|S(t_{j+1})-S(t_j)\|^\alpha\|(S(t_{j+1})-S(t_j))^{\otimes m}\|$$

with $\tilde{M}=\frac{M}{(m-1)!}\int_0^1\lambda^\alpha(1-\lambda)^{m-1}d\lambda$ and on π_n^2, for each $\epsilon>0$, when n is large enough,

$$\sum_{[t_j,t_{j+1}]\in\pi_n^2}|R(t_j,t_{j+1})|\le\sum_{[t_j,t_{j+1}]\in\pi_n^2}\epsilon\|S(t_{j+1})-S(t_j)\|^\alpha\|(S(t_{j+1})-S(t_j))^{\otimes m}\|.$$

In order to give our result, we need to give a bound for

$$\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n}\|S(t_{j+1})-S(t_j)\|^\alpha\|(S(t_{j+1})-S(t_j))^{\otimes m}\|.$$

First we have

$$\|S(t_{j+1})-S(t_j)\|^\alpha\le C\sum_{i=1}^d|S^i(t_{j+1})-S^i(t_j)|^\alpha$$

for some constant C. By Lemma 4.4, we know that each component of

$$\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n}\|S(t_{j+1})-S(t_j)\|^\alpha(S(t_{j+1})-S(t_j))^{\otimes m}$$

can be bounded by the sum of 2^d different pth varaitions of linear combinations of path S. Hence

$$\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n}\|S(t_{j+1})-S(t_j)\|^\alpha\|(S(t_{j+1})-S(t_j))^{\otimes m}\|$$

is bounded by some constant $C(S,\pi)$ related to S. This measure is dominated by a sum of measures μ_r with enough terms $r=1,\cdots,R$. Then we have

$$\begin{aligned}&\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n}|R(t_j,t_{j+1})|\\&\quad\le\lim_{n\to\infty}\sum_{[t_j,t_{j+1}]\in\pi_n}\tilde{M}g_k(S(t_j))\|S(t_{j+1})-S(t_j)\|^\alpha\|(S(t_{j+1})-S(t_j))^{\otimes m}\|+\epsilon C(S,\pi)\\&\quad\le\sum_{r=1}^R\int_0^T g_k(S(t))d\mu_r+\epsilon C(S,\pi).\end{aligned}$$

Here $g_k\le 1$ is a positive smooth function with support in U_{k-1} and $g_k|_{U_k}=1$. Letting $k\to\infty$ and $\epsilon\to 0$, from the assumption on f we obtain the result. □

Remark 4.5. This result may be extended to time-dependent functions as discussed above.

Remark 4.6. One cannot expect to obtain a multivariate Taylor–type expansion up to order $p \notin \mathbb{N}$ in general. The simplest function $f(x) = |x|^\alpha$ with $0 < \alpha < 1$ does not admit a Taylor–type expansion up to order α at 0 since the limit

$$\lim_{x\to 0} \frac{\|x\|^\alpha}{|x_1|^\alpha + \cdots |x_d|^\alpha}$$

does not exist.

Appendix A

A.1. **Properties of fractional derivatives.** We recall here some basic properties of fractional derivatives used in our proofs. Proofs may be found in [27].

Proposition A.1. *Suppose f is a real function, α, β are two real numbers. We will also use the convention that $I^\alpha_{a+} = D^{-\alpha}_{a+}$. They are well-defined since there is always one nonnegative superscript. Then we have*

(1) *Fractional integration operators $\{I^\alpha_{a+}, \alpha \geq 0\}$($\{I^\alpha_{b-}, \alpha \geq 0\}$) form a semi-group in $L^p(a,b)$ for every $p \geq 1$. It is continuous in uniform topology for $\alpha > 0$ and strongly continuous for $\alpha \geq 0$.*
(2) *In each of the following cases:*
   - $\beta \geq 0, \alpha + \beta \geq 0, f \in L^1([a,b])$,
   - $\beta \leq 0, \alpha \geq 0, f \in I^{-\beta}_{a+}(L^1)(I^{-\beta}_{b-}(L^1))$,
   - $\alpha \leq 0, \alpha + \beta \leq 0, f \in I^{-\alpha-\beta}_{a+}(L_1)(I^{-\alpha-\beta}_{b-}(L_1))$,

   we have
$$I^\alpha_{a+} \circ I^\beta_{a+} f = I^{\alpha+\beta}_{a+} f, \qquad I^\alpha_{b-} \circ I^\beta_{b-} f = I^{\alpha+\beta}_{b-} f.$$
(3) *If $n \leq \alpha < n+1$ and $f \in L^1(a,y)$ and $D^\alpha_{a+} f$ exists on (a,y) for some $y > x \in [a,b]$, then we have*
$$I^\alpha_{a^+} D^\alpha_{a^+} f(x) = f(x) - \sum_{j=1}^{n+1} \frac{D^{\alpha-j}_{a^+} f(a)}{\Gamma(\alpha - j - 1)} (x-a)^{\alpha-j}.$$
   Similarly,
$$I^\alpha_{b^-} D^\alpha_{b^-} f(x) = f(x) - \sum_{j=1}^{n+1} \frac{(-1)^j D^{\alpha-j}_{b^-} f(b)}{\Gamma(\alpha - j - 1)} (b-x)^{\alpha-j}.$$
   Here $D^\kappa_{a^+} f(a) = \lim_{y\to a, y>a} D^\kappa_{a^+} f(y)$ for any $\kappa \in \mathbb{R}$ and similar for the right Riemann–Liouville fractional derivative. Blow up at $(x-a)^{\alpha-n-1}$ is allowed since it is integrable near a and $I^\alpha_{a^+} D^\alpha_{a^+} f$ lives in the space $L^1([a,y])$.

Next, we enumerate below some basic properties of the Caputo derivative.

Proposition A.2. *Let f be a real function and $\alpha, \beta > 0$.*

(1) *If $\alpha = n \in \mathbb{N}$ and f is n times differentiable, then $C^\alpha_{a^+} f = f^{(n)}$ and $C^\alpha_{b^-} f = (-1)^n f^{(n)}$*
(2) *Let $0 < \alpha \leq \beta$ and $f \in L^1[a,b]$, then we have $C^\alpha_{a^+} I^\beta_{a^+} f = I^{\beta-\alpha}_{a^+} f$ and $C^\alpha_{b^-} I^\beta_{b^-} f = I^{\beta-\alpha}_{b^-} f$*
(3) *$C^\alpha_{a^+} C^n_{a^+} = C^{\alpha+n}_{a^+}$ and $C^\alpha_{b^-} C^n_{b^-} = C^{\alpha+n}_{b^-}$ for all $n \in \mathbb{N}$.*

We provide a proof of Proposition 1.10 here. Let us start with a simple but useful lemma, which corresponds to Theorem 3.3 in [20] for the case $\rho = 1$.

Lemma A.3. *Suppose $f \in C^n([a,b])$, then for any $0 < \alpha < n$ and $\alpha \notin \mathbb{N}$, we have actually $C^{\alpha}_{a^+} f(a) = C^{\alpha}_{b^-} f(b) = 0$.*

Proof of Proposition 1.10. Define

$$g(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

From Proposition A.1, we have

$$I^{\alpha}_{a^+} D^{\alpha}_{a^+} g(x) = g(x) - \sum_{j=1}^{n+1} \frac{D^{\alpha-j}_{a^+} g(a)}{\Gamma(\alpha - j + 1)} (x-a)^{\alpha - j}.$$

Now we have by linearity of Riemann–Liouville fractional derivative operator and definition of Caputo fractional derivative that

$$D^{\alpha-j}_{a^+} g(a) = C^{\alpha-j}_{a^+} f(a) - \sum_{k=n-j+1}^{n} \frac{f^{(k)}(a)}{k!} D^{\alpha-j}_{a^+}((x-a)^k)(a).$$

Furthermore, for $k \geq n - j + 1 > \alpha - j$, $D^{\alpha-j}_{a^+}((x-a)^k)(a) = C^{\alpha-j}_{a^+}((x-a)^k)(a)$ by definition. Hence we have

$$D^{\alpha-j}_{a^+} g(a) = C^{\alpha-j}_{a^+} f(a) - \sum_{k=n-j+1}^{n} \frac{f^{(k)}(a)}{k!} C^{\alpha-j}_{a^+}((x-a)^k)(a).$$

Now since $f \in C^n$ and $(\cdot - a)^k \in C^{\infty}$, we see that actually $D^{\alpha-j}_{a^+} g(a) = 0$ for every $j = 1, \cdots, n$ by Lemma A.3. For the case $j = n+1$ and $\alpha < n+1$, we actually have that

$$\begin{aligned}
\left|D^{\alpha-n-1}_{a^+} g(a)\right| &= \lim_{y \to a, y > a} \left|I^{n+1-\alpha}_{a^+} g(y)\right| \\
&= \lim_{y \to a, y > a} \frac{1}{\Gamma(n+1-\alpha)} \left| \int_a^y \frac{g(t)}{(y-t)^{\alpha-n}} dt \right| \\
&\leq \lim_{y \to a, y > a} \frac{1}{\Gamma(n+1-\alpha)} \left| \int_a^y \frac{\max_{t \in [a,y]}\{|g(t)|\}}{(y-t)^{\alpha-n}} dt \right| \\
&= \lim_{y \to a, y > a} \frac{1}{\Gamma(n+1-\alpha)} \max_{t \in [a,y]}\{|g(t)|\}(y-a)^{n+1-\alpha} = 0.
\end{aligned}$$

When $j = n + 1 = \alpha$, then $D^{\alpha-n-1}_{a^+} g(a) = g(a) = 0$. So actually we have

$$g(x) = I^{\alpha}_{a^+} D^{\alpha}_{a^+} g(x) = I^{\alpha}_{a^+} C^{\alpha}_{a^+} f(x),$$

hence the result. The derivation of the other formula is similar. □

A.2. **Proof of Corollary 1.14.** We only prove for the left Caputo derivative case. WLOG, we assume $C^{\alpha}_{a^+} f$ exists on the interval $[a,x]$ since $o(|x-a|^{\alpha})$ only concerns with the x close to a.

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)} \int_a^x \frac{C^{\alpha}_{a^+} f(t)}{(x-t)^{1-\alpha}} dt &= \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f^{(\alpha+)}(a)}{(x-t)^{1-\alpha}} dt + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{C^{\alpha}_{a^+} f(t) - f^{(\alpha+)}(a)}{(x-t)^{1-\alpha}} dt \\
&= \frac{f^{(\alpha+)}(a)}{\Gamma(\alpha+1)}(x-a)^{\alpha} + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{C^{\alpha}_{a^+} f(t) - f^{(\alpha+)}(a)}{(x-t)^{1-\alpha}} dt.
\end{aligned}$$

Furthermore,

$$\left|\frac{1}{\Gamma(\alpha)}\int_a^x \frac{C_{a^+}^{\alpha} f(t) - f^{(\alpha+)}(a)}{(x-t)^{1-\alpha}}dt\right| \leq \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\max_{t\in[a,x]}\{|C_{a^+}^{\alpha} f(t) - f^{(\alpha+)}(a)|\}}{(x-t)^{1-\alpha}}dt$$
$$= \frac{\max_{t\in[a,x]}\{|C_{a^+}^{\alpha} f(t) - f^{(\alpha+)}(a)|\}}{\Gamma(\alpha+1)}(x-a)^{\alpha}.$$

Hence

$$\lim_{x\to a, x\geq a}\frac{\left|\frac{1}{\Gamma(\alpha)}\int_a^x \frac{C_{a^+}^{\alpha} f(t) - f^{(\alpha+)}(a)}{(x-t)^{1-\alpha}}dt\right|}{(x-a)^{\alpha}} \leq \lim_{x\to a, x\geq a}\frac{\max_{t\in[a,x]}\{|C_{a^+}^{\alpha} f(t) - f^{(\alpha+)}(a)|\}}{\Gamma(\alpha+1)} = 0.$$

□

References

[1] A. Ananova and R. Cont, *Pathwise integration with respect to paths of finite quadratic variation* (English, with English and French summaries), J. Math. Pures Appl. (9) **107** (2017), no. 6, 737–757, DOI 10.1016/j.matpur.2016.10.004. MR3650323

[2] J. Bertoin, *Temps locaux et intégration stochastique pour les processus de Dirichlet* (French), Séminaire de Probabilités, XXI, Lecture Notes in Math., vol. 1247, Springer, Berlin, 1987, pp. 191–205, DOI 10.1007/BFb0077634. MR941983

[3] G. Cantor, *De la puissance des ensembles parfaits de points* (French), Acta Math. **4** (1884), no. 1, 381–392, DOI 10.1007/BF02418423. Extrait d'une lettre adressée à l'éditeur. MR1554642

[4] Y. Chen, Y. Yan, and K. Zhang, *On the local fractional derivative*, J. Math. Anal. Appl. **362** (2010), no. 1, 17–33, DOI 10.1016/j.jmaa.2009.08.014. MR2557664

[5] H. Chiu and R. Cont, *On pathwise quadratic variation for càdlàg functions*, Electron. Commun. Probab. **23** (2018), Paper No. 85, 12, DOI 10.1214/18-ECP186. MR3882226

[6] H. Chiu and R. Cont, *Causal functional calculus*, Trans. London Math. Soc. **9** (2022), no. 1, 237–269. MR4535662

[7] R. Cont and D. A. Fournié, *Functional Itô calculus and functional Kolmogorov equations*, Stochastic integration by parts and functional Itô calculus, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, [Cham], 2016, pp. 115–207. MR3497715

[8] R. Cont and P. Das, *Quadratic variation and quadratic roughness*, Bernoulli **29** (2023), no. 1, 496–522, DOI 10.3150/22-bej1466. MR4497256

[9] R. Cont and D. Fournie, *A functional extension of the Ito formula* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **348** (2010), no. 1-2, 57–61, DOI 10.1016/j.crma.2009.11.013. MR2586744

[10] R. Cont and D.-A. Fournié, *Change of variable formulas for non-anticipative functionals on path space*, J. Funct. Anal. **259** (2010), no. 4, 1043–1072, DOI 10.1016/j.jfa.2010.04.017. MR2652181

[11] R. Cont and N. Perkowski, *Pathwise integration and change of variable formulas for continuous paths with arbitrary regularity*, Trans. Amer. Math. Soc. Ser. B **6** (2019), 161–186, DOI 10.1090/btran/34. MR3937343

[12] M. Davis, J. Obłój, and P. Siorpaes, *Pathwise stochastic calculus with local times* (English, with English and French summaries), Ann. Inst. Henri Poincaré Probab. Stat. **54** (2018), no. 1, 1–21, DOI 10.1214/16-AIHP792. MR3765878

[13] R. M. Dudley and R. Norvaiša, *Concrete functional calculus*, Springer Monographs in Mathematics, Springer, New York, 2011, DOI 10.1007/978-1-4419-6950-7. MR2732563

[14] B. Dupire, *Functional Itô calculus*, Quant. Finance **19** (2019), no. 5, 721–729, DOI 10.1080/14697688.2019.1575974. MR3939653

[15] H. Föllmer, *Calcul d'Itô sans probabilités* (French), Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980), Lecture Notes in Math., vol. 850, Springer, Berlin, 1981, pp. 143–150. MR622559

[16] P. K. Friz and M. Hairer, *A course on rough paths*, Universitext, Springer, Cham, 2014. With an introduction to regularity structures, DOI 10.1007/978-3-319-08332-2. MR3289027

[17] D. Geman and J. Horowitz, *Occupation densities*, Ann. Probab. **8** (1980), no. 1, 1–67. MR556414
[18] M. Gubinelli, *Controlling rough paths*, J. Funct. Anal. **216** (2004), no. 1, 86–140, DOI 10.1016/j.jfa.2004.01.002. MR2091358
[19] X. Han, A. Schied, and Z. Zhang, *A probabilistic approach to the* Φ*-variation of classical fractal functions with critical roughness*, Statist. Probab. Lett. **168** (2021), Paper No. 108920, 6, DOI 10.1016/j.spl.2020.108920. MR4149320
[20] F. Jarad, T. Abdeljawad, and D. Baleanu, *On the generalized fractional derivatives and their Caputo modification*, J. Nonlinear Sci. Appl. **10** (2017), no. 5, 2607–2619, DOI 10.22436/jnsa.010.05.27. MR3654527
[21] J. Liouville, *Mémoire sur l'usage que l'on peut faire de la formule de Fourier, dans le calcul des différentielles à indices quelconques* (French), J. Reine Angew. Math. **13** (1835), 219–232, DOI 10.1515/crll.1835.13.219. MR1578043
[22] Rafał M. Łochowski, J. Obłój, D. J. Prömel, and P. Siorpaes, *Local times and Tanaka-Meyer formulae for càdlàg paths*, Electron. J. Probab. **26** (2021), Paper No. 77, 29, DOI 10.1214/21-ejp638. MR4269207
[23] H. P. Mulholland, *On generalizations of Minkowski's inequality in the form of a triangle inequality*, Proc. London Math. Soc. (2) **51** (1950), 294–307, DOI 10.1112/plms/s2-51.4.294. MR33865
[24] M. Pratelli, *A remark on the* $1/H$*-variation of the fractional Brownian motion*, Séminaire de Probabilités XLIII, Lecture Notes in Math., vol. 2006, Springer, Berlin, 2011, pp. 215–219, DOI 10.1007/978-3-642-15217-7_8. MR2790374
[25] B. Riemann, *Gesammelte mathematische Werke, wissenschaftlicher Nachlass und Nachträge* (German), Springer-Verlag, Berlin; BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1990. Based on the edition by Heinrich Weber and Richard Dedekind; Edited and with a preface by Raghavan Narasimhan, DOI 10.1007/978-3-663-10149-9. MR1084595
[26] L. C. G. Rogers, *Arbitrage with fractional Brownian motion*, Math. Finance **7** (1997), no. 1, 95–105, DOI 10.1111/1467-9965.00025. MR1434408
[27] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Integraly i proizvodnye drobnogo poryadka i nekotorye ikh prilozheniya* (Russian), "Nauka i Tekhnika", Minsk, 1987. Edited and with a preface by S. M. Nikol′skiĭ. MR915556
[28] A. Schied, *On a class of generalized Takagi functions with linear pathwise quadratic variation*, J. Math. Anal. Appl. **433** (2016), no. 2, 974–990, DOI 10.1016/j.jmaa.2015.08.022. MR3398747
[29] S. J. Taylor, *Exact asymptotic estimates of Brownian path variation*, Duke Math. J. **39** (1972), 219–241. MR295434
[30] M. Zähle, *Integration with respect to fractal functions and stochastic calculus. I*, Probab. Theory Related Fields **111** (1998), no. 3, 333–374, DOI 10.1007/s004400050171. MR1640795

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