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C^0-LIMITS OF LEGENDRIAN KNOTS

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Abstract. Take a sequence of contactomorphisms of a contact three-manifold that C^0-converges to a homeomorphism. If the images of a Legendrian knot limit to a smooth knot under this sequence, we show that it is contactomorphic to the original knot. We prove this by establishing that, on one hand, non–Legendrian knots admit a type of contact-squashing (similar to squeezing) onto transverse knots while, on the other hand, Legendrian knots do not admit such a squashing. The non-trivial input from contact topology that is needed is (a local version of) the Thurston–Bennequin inequality.

1. Introduction and results

A knot K inside a contact 3-manifold (M^3, ξ) is **Legendrian** (resp. **transverse**) if, for *all* points $p \in K$, $T_pK \subset \xi_p$ (resp. $T_pK \not\subset \xi_p$). In this article, all knots are considered to be smooth co-orientable embeddings of S^1 into a contact 3-manifold, where the contact structure of the latter is assumed to be co-orientable; we do not make additional assumptions on the ambient contact manifold, i.e., it can be either closed or open. Generalizing the notion of transverse, the knot K is called **non–Legendrian** if, for *some* $p \in K$, $T_pK \not\subset \xi_p$. Both Legendrian and transverse knots have been widely studied, and each class exhibits various interesting rigidity phenomena. Non–Legendrian knots are somewhat more flexible, especially when considered from a quantitative viewpoint; for example, in the case when there exists a contactomorphism of (M^{2n+1}, ξ) that connects two non–Legendrian n-dimensional submanifolds, Rosen–Zhang [RZ20, Section 1] have shown that there exists such a contactomorphism of arbitrarily small Hofer norm.

General non–Legendrian knots in the contact geometric setting have not received the same amount of attention as transverse and Legendrian knots. This article shows that non–Legendrian knots behave more like transverse knots than Legendrian knots, at least when it comes to quantitative questions. Indeed, the starting point of the results of this article is the following type of flexibility: a non–Legendrian knot can be "squashed" arbitrarily close to some given transverse knot. (See Theorem A for the precise statement.) Non–Lagrangian submanifolds (which fail the Lagrangian tangency condition at at least one point) in symplectic manifolds also demonstrate some flexibility; for example, non–Lagrangians which are C^1-close to Lagrangians and have vanishing Euler characteristic will have vanishing displacement energy [Pol95, Corollary 1.6 and Theorem 1.2].

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In the following we fix an arbitrary Riemannian metric on M inducing a distance function d, and denote by

$$B_r(K) \coloneqq \{x \in M;\ d(K, x) < r\} \subset M$$

the set of points of distance less than r from the subset $K \subset M$.

The results in this paper are closely connected to the concept of (ambient) contact squeezing, see Definition 1.9. We begin with a first proposed (Legendrian) version of this, which we call squashing.

Definition 1.1. Let $K_0, K \subset (M, \xi)$ be submanifolds of a contact manifold. We say that the *contact isotopy* $\varphi_t \colon M \to M$ **finely squashes** K_0 **onto** K if there exists $\epsilon(t)$ with $\lim_{t\to+\infty} \epsilon(t) = 0$ such that for all $t \gg 0$ sufficiently large, $\varphi_t(K_0) \subset B_{\epsilon(t)}(K)$ and $\varphi_t(K_0)$ is smoothly isotopic to K inside $B_{\epsilon(t)}(K)$.

Example 1.2. The squashing property need not be symmetric. Let $K = j^1 0 \subset (J^1 S^1 = T^* S^1 \times \mathbb{R}_z, dz - p d\theta)$ be the zero section, and let K_{stab} be a Legedendrian stabilization of K. (A stabilization in the Lagrangian projection $T^* S^1$ is a Reidemeister-1 move, while in the front projection $S^1 \times \mathbb{R}_z$ it is an added "zigzag." In particular K and K_{stab} are smoothly isotopic.) The fibrewise rescaling $(\theta, p, z) \mapsto (\theta, p/t, z/t)$ for $t \geq 1$ is a contact isotopy that finely squashes K_{stab} onto K.

Now embed $J^1 S^1$ as a standard neighborhood of the standard Legendrian unknot in the standard contact vector space $\mathbb{R}^3 = J^1\mathbb{R}$ so that $j^1 0$ is identified with the standard Legendrian unknot, which we again denote by K, and hence K_{stab} is a stabilization of the unknot. The Chekanov-Eliashberg (or "Legendrian contact homology") DGA of K has an augmentation. So [DRS20, Theorem 1.7] implies there do NOT exist a contact isotopy $\varphi_t \colon \mathbb{R}^3 \to \mathbb{R}^3$ and a one-jet neighborhood V of K_{stab}, such that $\varphi_1(K) \subset V$ and $[\varphi_1(K)] \neq 0 \in H_1(V; \mathbb{Z}_2) = \mathbb{Z}_2$. In particular, this implies that K does not finely squash onto K_{stab} when considered in $\mathbb{R}^3$, and thus that it also cannot be squashed onto K_{stab} when considered inside $J^1 S^1$. This could also be proven by an argument as in the proof of Theorem 1.3, based upon convex surfaces and the Thurston–Bennequin invariant.

This construction-argument generalizes to arbitrary dimensions where $K \subset J^1(M)$ is any closed Legendrian whose Chekanov-Eliashberg DGA has an augmentation, and where the Legendrian K_{stab} is constructed from K by adding a small loose chart.

One of our main results is that non–Legendrian knots are flexible in the sense that they can be squashed onto transverse knots.

Theorem A. *Let $K \subset (M^3, \xi)$ be a non–Legendrian knot. There exists a transverse knot $T \subset (M^3, \xi)$ and a contact isotopy $\varphi_t \colon M \xrightarrow{\cong} M$ that squashes K onto T.*

In particular, by replacing M with a small tubular neighborhood of K, we can assume that the transverse knot T lives in that neighborhood.

Theorem B and part (i) of Lemma 1.7 prove that Legendrians cannot be squashed onto transverse knots, and therefore, by the transitivity of the squashing property provided by part (ii) of Lemma 1.7, they also cannot be squashed onto non–Legendrian knots.

Non-squeezing results are a central theme in symplectic topology, going back to Gromov's famous non-squeezing result in symplectic manifolds [Gro85]. In contact topology, the notion of squeezing a domain into an open subset by a contact isotopy was defined by Eliashberg–Kim–Polterovich [EKP06], who also provided non-trivial obstructions for squeezing. Definition 1.1 of squashing is related to the notion of squeezing. However, squashing allows K_0 to be a closed submanifold (like a knot). In addition, while squeezing places a subset into some fixed open subset, a squashing of K_0 onto K places the former into an arbitrarily small neighborhoods of the latter subset K (typically a transverse or non–Legendrian knot). Our notion of finely squashing (as well as our coarser version of squashing, see Definition 1.5) is strictly stronger than Hausdorff convergence of the involved subsets. So the initial submanifold K_0 can be visualized as "squashing onto" the latter submanifold K by the isotopy, which motivates the choice of terminology. In Section 1.1 we compare our notion of squashing, saying a subset squashes *onto* another one, with the notion of squeezing in [EKP06], where subsets squeeze *into* other subsets. The subsets in [EKP06] are diffeomorphic to open solid tori inside the standard contact prequantization $\mathbb{R}^{2n} \times S^1$ and, as discussed in Section 1.1, there are relations between squeezing solid tori into other solid tori, and squashing knots onto cores of solid tori.

We have previously defined a Legendrian version of squeezing [DRS20, Section 1.2] and proved a non-squeezing result for certain non-loose Legendrians onto loose Legendrians [DRS20, Theorem 1.7], which we use in Example 1.2. This is closely related to the non-squashing phenomena studied here. This result was generalized in [Laz19, Corollary 1.12]. The aforementioned articles established this Legendrian version of non-squeezing in arbitrary dimensions using holomorphic curve technology. The results in this article are based on parts of the theory of convex surfaces that so far only has been thoroughly developed in dimension three.

The local smooth isotopy equivalence requirement in Definition 1.1 is in part motivated by the following Legendrian flexibility: if $K_0 = \Lambda \subset M = \mathbb{R}^{2n+1}$ is any closed Legendrian, then there exists a contact isotopy taking Λ into an arbitrarily small neighborhood of any point in any submanifold K. (The analogous flexibility statement does not hold for symplectomorphisms of Lagrangians in $\mathbb{R}^{2n}$.) To avoid such phenomena in [DRS20, Section 1.2], we impose the weaker $\mathbb{Z}_2$-homological constraint as in Example 1.2.

The classification of contact structures on solid tori by Giroux [Gir00] and Honda [Hon00], based upon the convex surface theory by Giroux [Gir91], implies that Legendrian approximations of transverse knots must be increasingly stabilized. More precisely:

Theorem 1.3 (Giroux [Gir00] and Honda [Hon00])**.** *For a Legendrian knot Λ that lives inside a tubular neighborhood of a transverse knot, with the additional assumption that the two knots are smoothly isotopic inside the given tubular neighborhood, one can give a bound from below on the number of stabilizations that the Legendrian has in terms of the distance from the Legendrian to the transverse knot. Furthermore, this number tends to $+\infty$ as this distance tends to zero.*

Remark 1.4. When the Legendrian knot is null-homologous, and thus has a well-defined Thurston–Bennequin invariant, it immediately follows from the aforementioned result that Legendrians cannot be squashed onto transverse knots. Indeed, Definition 1.1 provides the hypotheses to apply Theorem 1.3, forcing the number of stabilizations of the Legendrian to increase during the squashing isotopy, which contradicts that this number is a Legendrian isotopy invariant. We will refer to results that give obstructions to squashing as **non-squashing results** (in analogy to the classical non-squeezing results that have been proven in both contact and symplectic settings). Section 3.3 is dedicated to extending this non-squashing result from null-homologous to arbitrary Legendrian knots.

To the authors' knowledge, Theorem 1.3 has not been explicitly stated in the literature. Since our work here does not rely on the previous result, but rather use weaker results in the same spirit that concern relative Thurston–Bennequin numbers, we only provide a brief sketch of the ideas that go into the proof.

Sketch of proof of Theorem 1.3. Consider a Legendrian Λ which is close to a transverse knot T in the same isotopy class. By Giroux's theory of convex surfaces [Gir91], one can produce an embedded convex annulus A inside the normal neighborhood of the transverse knot with boundary $\partial A = \Lambda \sqcup \Lambda_k$. Here Λ_k is the standard k-fold stabilized Legendrian approximation of the transverse knot T described in Section 3.2, which is contained on the boundary of a tubular neighborhood of T, while Λ is contained in the interior of the neighborhood. Lemma 3.5 implies that we can take $k \gg 0$ arbitrarily large, if Λ is taken to be sufficiently close to T.

We use the language of [Hon00]. A sufficiently small tubular neighborhood of the transverse knot is tight. So the dividing curves of the convex tori inside this neighborhood satisfy the minimally twisting property. Consider the dividing curves of the annulus A. The minimally twisting property implies the existence of bypass half-disks in A for the boundary component $\Lambda \subset \partial A$ (unless Λ is isotopic to Λ_k) while no analogous bypass half-disk exists for Λ_k. Unless Λ and Λ_k are Legendrian isotopic, we thus deduce from the existence of the bypass half-disk that Λ is obtained from Λ_k by stabilization. □

In order to deduce that a smooth image of a Legendrian knot under a C^0-converging sequence of contactomorphisms again is Legendrian, we need a stronger type of non-squashing result than the consequence of Theorem 1.3 outlined in Remark 1.4. The main point is that we need a non-squashing result for *contactomorphisms* and not merely contact isotopies. One of the crucial results is that Legendrians also cannot be squashed onto non–Legendrians in this weaker sense; see Theorem B.

Definition 1.5. We say that the *sequence of contactomorphisms* $\varphi_i \colon M \to M$ **coarsely squashes** $K_0 \subset M$ **onto** $K \subset M$, where K_0 and K are submanifolds, if the following holds.

(1) There exists $\epsilon_i > 0$ with $\lim_{i \to +\infty} \epsilon_i = 0$ such that for all $i \gg 0$, $\varphi_i(K_0) \subset B_{\epsilon_i}(K)$ and $\varphi_i(K_0)$ is smoothly isotopic to K inside $B_{\epsilon_i}(K)$.
(2) For any $r > 0$ and $\epsilon > 0$, there exists some $i_{r,\epsilon} \gg 0$ such that
$$d(\varphi_i \circ \varphi_j^{-1}(x), x) < \epsilon$$
for all $i \geq j \geq i_{r,\epsilon}$ and $x \in M \setminus B_r(K)$.

Part (1) of Definition 1.5 is a "discretized" version of the property in Definition 1.1 that the contact isotopy keeps K_0 near K for all times.

Example 1.6. In certain contact manifolds one can find a sequence of contactomorphisms, a Legendrian knot $K_0 = \Lambda_0$, and a transverse knot $K = T$, that satisfies part (1) of Definition 1.5; this is the reason why we want to define squashing as something stronger than merely what is postulated in part (1). For such an example, consider the contact manifold given as the ideal boundary $\partial_\infty(\mathbb{C}^* \times \mathbb{C}) \cong S^1 \times S^2$

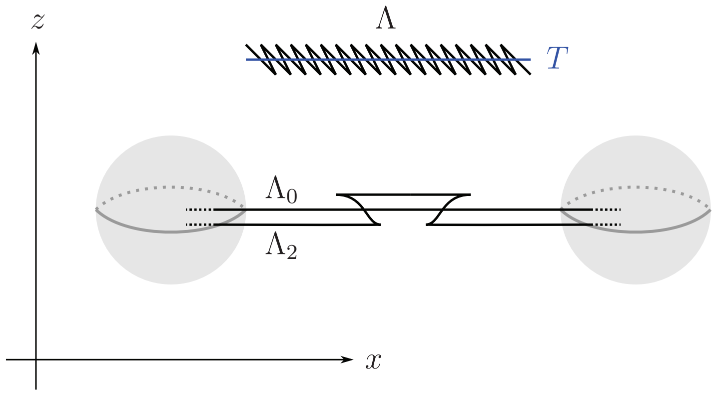

FIGURE 1. Above: A transverse knot $T = \{z = z_0,\ y = -1\}$ can be approximated by a Legendrian knot Λ if the latter is sufficiently stabilized (stabilizations correspond to zig-zags). Below: A homologically essential Legendrian knot Λ_0 inside $\partial_\infty(\mathbb{C}^* \times \mathbb{C})$, depicted as a Kirby diagram with a single Weinstein one-handle attached to S^3. The Legendrian Λ_0 is Legendrian isotopic to its two-fold stabilization Λ_2 and thus, by induction, is Legendrian isotopic to a $2k$-fold stabilization for any $k \geq 0$.

of the Weinstein manifold $\mathbb{C}^* \times \mathbb{C}$, and the Legendrian core given as a connected component

$$\Lambda_0 \subset \partial_\infty(S^1 \times \mathfrak{Re}(\mathbb{C})) \subset \partial_\infty(\mathbb{C}^* \times \mathbb{C})$$

of the Legendrian link at infinity. The Legendrian Λ_0 is shown in the Kirby diagram in Figure 1. It is homologically essential, and Legendrian isotopic to a two-fold stabilization of itself, consisting of one positive and one negative stabilization; see, e.g., [DG09, Figure 19] for more details. Now consider the transverse core given as a connected component

$$T \subset \partial_\infty(\mathbb{C}^* \times \{0\}) \subset \partial_\infty(\mathbb{C}^* \times \mathbb{C})$$

of a transverse two-component link at infinity. It is possible to C^0-approximate T by a sufficiently stabilized Legendrian core in the same smooth isotopy class. For example, the upper figure in Figure 1 depicts a transverse arc that is approximated by a Legendrian with many positive stabilizations. In particular, there is a Legendrian isotopy of the Legendrian core Λ into an arbitrarily small neighborhood of the transverse core, so that the Legendrian is smoothly isotopic to the transverse knot inside the same neighborhood. Note that if a Legendrian has many positive and negative stabilizations, then the negative stabilizations can be shrunk arbitrarily, in order to not interfere with the approximation that is made by using the positive stabilizations.

We are not sure if part (2) is the most natural definition if one wants a notion of squashing that precludes the possibility of squashing a Legendrian onto a transverse knot. However, as we prove in Section 2, one good feature of Definition 1.5 is that the existence of squashing sequences becomes transitive in the following manner.

Lemma 1.7.

(i) *If there exists a contact isotopy $\psi_t\colon M \to M$ that finely squashes a submanifold $K_0 \subset M$ onto a submanifold $K \subset M$ then one can produce a coarsely squashing sequence $\varphi_i\colon M \xrightarrow{\cong} M$ of contactomorphisms of K_0 onto K. The support of φ_i can be assumed to be contained inside the support of ψ_t.*

(ii) *Consider two sequences of contactomorphisms*

$$\varphi_i^{(\nu)}\colon M \to M, \ \ \nu = 1, 2,$$

where $\{\varphi_i^{(\nu)}\}$ coarsely squashes K_ν onto $K_{\nu-1}$. Then there exists a suitable re-indexing $\alpha(i) \geq i$ for which

$$\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)}\colon M \to M$$

is a sequence of contactomorphisms that coarsely squashes K_2 onto K_0.

(iii) *The property of either an isotopy or a sequence of contactomorphisms to squash a submanifold K_0 onto K does not depend on the choice of Riemannian metric on M.*

We establish the non-squashing result for Legendrian knots onto transverse knots.

Theorem B. *Let $\Lambda \subset (M, \xi)$ be a Legendrian knot. If $T \subset (M, \xi)$ is a transverse knot, then there does not exist any sequence of contactomorphisms that coarsely squashes Λ onto T.*

The proof of our non-squashing result Theorem B does not rely on the fact that a Legendrian that is close to a transverse knot in the same isotopy class must be stabilized, as shown in Theorem 1.3; however, the proof establishes that its relative Thurston–Bennequin number admits a bound from above, where this bounds tends to $-\infty$ as the distance to the transverse knot tends to zero. If one would like to deduce the existence of stabilizations for the knot, one could subsequently use the classification result for Legendrian knots by Eliashberg–Fraser [EF09] or Ding–Geiges [DG07].

It turns out that the only ingredient from the classification of contact structures that is needed for Theorem B is the *Thurston–Bennequin inequality for Legendrian unknots in* $\mathbb{R}^3$ as proven by Bennequin in [Ben83]. Of course, this inequality is also highly non-trivial, as it, e.g., implies that the standard contact 3-sphere is tight.

In the case when $H_1(M) = H_2(M) = 0$, so that the ordinary (i.e., non-relative) Thurston–Bennequin number of any Legendrian knot is well-defined, this non-squashing result can be seen to follow directly from Theorem 1.3 as outlined in Remark 1.4. For the general statement, the main ingredient is the Thurston–Bennequin inequality for Legendrian knots in standard $\mathbb{R}^3$ proven by Bennequin [Ben83] (or, more precisely, a relative formulation for unknotted Legendrian cores of the solid torus J^1S^1).

In combination with the existence of squashing of non–Legendrians onto transverse knots proven by Theorem A, we obtain the following non-squashing for Legendrians into a neighborhood of a non–Legendrian.

Corollary C. *Let $\Lambda \subset (M^3, \xi)$ be a Legendrian knot. If $K \subset (M^3, \xi)$ is a non–Legendrian knot, then there does not exist a sequence of contactomorphisms $\varphi_i\colon M \to M$ that coarsely squashes Λ onto K.*

Proof. Assume that there exists a sequence of contactomorphisms that squashes Λ onto K. Apply Theorem A to produce a contact isotopy that squashes K onto a transverse knot T. By Lemma 1.7 we can find a sequence of contactomorphisms that squashes Λ onto T; this is in contradiction with Theorem B. □

Remark 1.8. In contact manifolds of dimension $2n+1 \geq 5$ the result analogous to Corollary C does not hold: there are contact isotopies that squash certain Legendrians onto non–Legendrians. Such examples can be constructed by alluding to Murphy's h-principle for loose Legendrians [Mur]. Namely, by this h-principle we can approximate any n-dimensional non–Legendrian submanifold by a loose Legendrian while keeping control of its formal Legendrian isotopy class. The loose Legendrian approximations are Legendrian isotopic by the same h-principle.

The main difference between high dimensions and dimension $2n+1=3$ in this respect is that, in the low dimensional case, one cannot add stabilizations inside a sufficiently small neighborhood of a transverse knot (or, more generally, non–Legendrian knot) without decreasing the relative Thurston–Bennequin number.

In symplectic geometry the existence of capacities for Lagrangian submanifolds defined by Floer homology has given rise to many rigidity phenomena of a quantitative nature. In particular, in [LS94] Laudenbach–Sikorav showed that Lagrangians cannot be placed inside neighborhoods of non–Lagrangians. This result can be used to show that a smooth limit of Lagrangians under a sequence of symplectomorphisms that converge to a homeomorphism must again be Lagrangian. The analogous result for coisotropic manifolds was shown in codimension one by Opshtein [Ops09]. The full answer was later given by Humilière–Leclercq–Seyfaddini who established the analogous result for arbitrary coisotropic submanifolds in [HLS15]. The analogous questions in the setting of contact topology have only seen partial results [Nak20, RZ20, Ush21]. Using Corollary C, we settle the question in dimension three.

Theorem D. *Let (M^3, ξ) be a three-dimensional contact manifold and $\varphi_i \in$ Cont(M, ξ) a sequence of contactomorphisms that converge in C^0-norm to a homeomorphism φ_∞. Let $\Lambda \subset (M, \xi)$ be a Legendrian knot whose image $\varphi_\infty(\Lambda)$ is a smooth knot. Then $\varphi_\infty(\Lambda)$ is Legendrian as well. In addition, there exists a globally defined smooth contactomorphism of M that maps Λ to $\varphi_\infty(\Lambda)$.*

Nakamura proves the first statement in Theorem D for arbitrary dimension assuming that for some contact form there exists a uniform lower bound on the lengths of the Reeb chords from $\varphi_i(\Lambda)$ to itself [Nak20, Theorem 3.4]. He also assumes some technical conditions that we have since lifted [DRS21, Corollary 1.5]. Rosen and Zhang prove the first part of Theorem D in arbitrary dimensions assuming a uniform convergence of the conformal factors f_i (defined by $\varphi_i^*\alpha = f_i\alpha$ for a contact form α) [RZ20, Theorem 1.4]. Usher generalizes Rosen and Zhang's result assuming certain lower bounds on the f_i [Ush21, Theorem 1.2]. Observe that the latter works do not make any claims about the contactomorphism type of the limit.

Since any tangent vector in the contact plane can be realized as the tangent to a small Legendrian knot, Theorem D is strong enough to settle "C^0-rigidity of contactomorphisms" in this dimension: a smooth C^0-limit of contactomorphisms is itself a contactomorphism. This result was first proven by Eliashberg [Eli87]; see work by Müller–Spaeth for a more recent proof [MS14]. Note that, in the case when the C^0-limit homeomorphism φ_∞ is smooth, the C^0-rigidity of contactomorphisms can itself be used to derive the conclusion of Theorem D.

1.1. **Digression: squashing onto vs. squeezing into.** Here we compare our notion of *squashing onto*, with the notion of *squeezing into* introduced by Eliashberg–Kim–Polterovich [EKP06].

Definition 1.9. The subset U_1 can be **squeezed into** U_2 if there exists a contact isotopy ϕ_t for which $\phi_0 = \mathrm{Id}_M$ and such that $\overline{\phi_1(U_1)} \subset U_2$.

If U_1 is pre-compact like the following examples, ϕ_1 can be assumed to have compact support. Let $B^k_R \subset \mathbb{R}^k$ denote the open ball of radius R centered at the origin. The open subset $B^{2n}_R \times S^1$ of the pre-quantization of the symplectic vector-space $(\mathbb{R}^{2n} \times S^1, d\theta - pdq)$ cannot be squeezed into itself when $\frac{1}{2}R^2 \geq 1$ is an integer [EKP06, Theorem 1.5]; however, for $n > 1$ and $\frac{1}{2}(R_1)^2 < \frac{1}{2}(R_2)^2 < 1$, $B^{2n}_{R_2} \times S^1$ can be squeezed into $B^{2n}_{R_1} \times S^1$ [EKP06, Theorem 1.3]. Chiu proved that $B^{2n}_R \times S^1$ cannot be squeezed into itself for arbitrary $\frac{1}{2}R^2 \geq 1$ [Chi17]. Fraser generalized Chiu's result proving that there exist no contactomorphism ϕ such that $\overline{\phi(B^{2n}_R \times S^1)} \subset \phi(B^{2n}_R \times S^1)$ for all $\frac{1}{2}R^2 \geq 1$ [Fra16]. Since ϕ need not be the time-1 map of a contact isotopy, Fraser's coarser squeezing definition is more analogous to Definition 1.5. (To compare notation, in [EKP06, Fra16, Chi17] $B^{2n}(R) \times S^1 = \{z \in \mathbb{R}^{2n} \mid \pi\|z\|^2 < R\} \times \mathbb{R}/\mathbb{Z}$, while in this article $B^{2n}_R \times S^1 = \{z \in \mathbb{R}^{2n} \mid \|z\|^2 < R^2\} \times \mathbb{R}/(2\pi\mathbb{Z})$.)

The squeezings provided by [EKP06, Theorem 1.3] do not exist when $n = 1$ instead of $n > 1$ [Eli91]. In fact, the obstruction for squeezing the solid torus $B^2_{R_2} \times S^1$ into $B^2_{R_1} \times S^1$ when $R_1 \leq R_2$ that holds in the case when $n = 1$ can be seen to be closely related to the same mechanism that governs our non-squashing result in Theorem B, i.e., the local Thurston–Bennequin inequality. The point is that in the particular case of the contact manifold $\mathbb{R}^2 \times S^1$, an even stronger form of non-squashing holds for Legendrians isotopic to $\{0\} \times S^1$ than the one given by Theorem B; namely, if such a Legendrian is placed inside a small neighborhood of the transverse knot $\{0\} \times S^1$ by a contact isotopy, then it is automatically smoothly isotopic, inside the same neighborhood, to this transverse knot. We give examples of non-squeezing results for open subsets that can be proven by using this mechanism.

- If some fixed $B^2_{\sqrt{2/k}} \times S^1$ squeezes into $B^2_{R_1} \times S^1$ for $R_1 > 0$ arbitrarily small, then that would mean that the Legendrian knots that foliate the boundary of $B^2_{\sqrt{2/k}} \times S^1$ would admit contact isotopies that place them in arbitrarily small neighborhoods of the transverse knot $\{0\} \times S^1 \subset B^2_{R_1} \times S^1$, i.e., the core of the solid torus. This contradicts the local Thurston–Bennequin inequality which is the main technical ingredient of the proof of Theorem B.
- The local Thurston–Bennequin inequality for the Legendrian knots that foliate the boundary of $B^2_{\sqrt{2}} \times S^1$ shows that these Legendrians cannot be placed inside $B^2_{\sqrt{2}} \times S^1$ by a contact isotopy; see Cant's recent result [Can23, Proposition 5] that is based upon techniques from this article. In particular, this provides an alternative proof of the fact that the solid torus $B^2_{\sqrt{2}} \times S^1$ cannot be squeezed into itself.

- The local Thurston–Bennequin inequality also implies the following result about non-squeezing of open subsets in the sense of Eliashberg–Kim–Polterovich. If U_2 is a solid torus neighborhood of a fixed Legendrian knot in $M = \mathbb{R}^2 \times S^1$ in which the knot is smoothly isotopic to the transverse knot $\{0\} \times S^1$, then there exists a solid torus neighborhood $U_1 = B^2_{R_1} \times S^1$ of the latter transverse knot into which U_2 cannot be squeezed.

We emphasize that these three results should be known to experts, and implicitly contained in the low-dimensional classification results by Giroux [Gir00] and Honda [Hon00].

2. Transitivity of squashing (proof of Lemma 1.7)

We prove Lemma 1.7.

Part (i): Consider the contact Hamiltonian $H_t \colon M \to \mathbb{R}$ that generates the contact isotopy ψ_t. We cut off H_t via a sequence of bump functions $\rho_t \cdot H_t$ that have support contained inside $B_{\epsilon(t)}(K)$ for all $t \geq 0$, while $\rho_t \equiv 1$ holds near $\psi_t(K_0)$. The new contact isotopy φ_t obtained restricts to the old isotopy along K_0, and hence squashes K_0 onto K as well.

The corresponding sequence of contactomorphisms φ_i for the integer times $i = 0, 1, 2, 3, \ldots$ is the sought sequence that squashes K_1 onto K. For part (2) of Definition 1.5, we may take

$$i_{r,\epsilon} \coloneqq \min\{i_0;\ \epsilon(i) < r \text{ for all } i \geq i_0\}$$

to be independent of ϵ. In this case, the maps $\varphi_i \circ \varphi_j^{-1}$ with $i \geq j \geq i_{r,\epsilon}$ all have support contained inside $B_r(K)$, i.e., $\varphi_i \circ \varphi_j^{-1}(x) = x$ for $x \notin B_r(K)$.

Part (ii): By the assumption that $\varphi_i^{(\nu)}$ are sequences that squashes K_ν onto $K_{\nu-1}$ we get that, for any $r, \epsilon > 0$, there are $i^{(\nu)}_{r,\epsilon}$ such that

$$d(\varphi_i^{(\nu)} \circ (\varphi_j^{(\nu)})^{-1}(x), x) < \epsilon$$

holds for all $x \notin B_r(K_{\nu-1})$ and $i \geq j \geq i^{(\nu)}_{r,\epsilon}$. In particular,

$$(\varphi_j^{(1)})^{-1}(B_{r-\delta}(K_0)) \subset (\varphi_i^{(1)})^{-1}(B_r(K_0)) \subset (\varphi_j^{(1)})^{-1}(B_{r+\delta}(K_0))$$

may be assumed to hold for all sufficiently small $\delta > 0$ and $i \geq j \geq i^{(1)}_{r/2,\epsilon/2}$. By the definition of squashing, we can assume that $K_1 \subset (\varphi_j^{(1)})^{-1}(B_{r-\delta}(K_0))$ is satisfied after increasing $i^{(1)}_{r/2,\epsilon/2} \gg 0$ further and taking $j \geq i^{(1)}_{r/2,\epsilon/2}$. In other words, all images $(\varphi_i^{(1)})^{-1}(B_r(K_0))$ can be assumed to contain a fixed neighborhood $(\varphi^{(1)}_{i_{r/2,\epsilon/2}})^{-1}(B_{r-\delta}(K_0)) \supset K_1$ whenever $i \geq i^{(1)}_{r/2,\epsilon/2}$.

We claim that the sequence $\varphi_i^{(1)} \circ \varphi^{(2)}_{\alpha(i)}$ squashes K_2 onto K_0 for a suitable increasing re-indexing $\alpha(i) \geq i$ where $\alpha(i) - i \gg 0$ is taken to be sufficiently large.

First we verify that part (1) of the definition is satisfied. Note that we have an inclusion,

$$\varphi^{(2)}_{\alpha(i)}(K_2) \subset B_{\epsilon^{(2)}_{\alpha(i)}}(K_1)$$

where the sequence $\epsilon^{(2)}_{\alpha(i)}$ satisfies $\lim_{i \to +\infty} \epsilon^{(2)}_{\alpha(i)} = 0$. Consequently, $B_{\epsilon^{(2)}_{\alpha(i)}}(K_1) \subset (\varphi_j^{(1)})^{-1}(B_r(K_0))$ may be assumed to hold for any arbitrary $r > 0$ and all $i \gg 0$, whenever $j \gg i^{(1)}_{r,\epsilon}$. In conclusion, for any $r > 0$,

$$\varphi^{(1)}_i \circ \varphi^{(2)}_{\alpha(i)}(K_2) \subset B_r(K_0)$$

is satisfied whenever we take α to satisfy $\alpha(i) - i \gg 0$. The image of K_2 is smoothly isotopic to K_0 inside the same subset.

What remains is to verify part (2) of the definition. Take

$$i_{r,\epsilon} := \max(i^{(1)}_{r,\epsilon/4}, i^{(2)}_{\rho(\epsilon),\epsilon/4}),$$

for $\rho(\epsilon) > 0$ sufficiently small so that the inclusion

$$B_{\rho(\epsilon)}(K_0) \subset (\varphi^{(1)}_i)^{-1}(B_r(K_0))$$

is satisfied for all $i \geq i^{(1)}_{r,\epsilon/4}$. It is then readily checked that part (2) is satisfied for the sequence $\{\varphi^{(1)}_i \circ \varphi^{(2)}_{\alpha(i)}\}$ of contactomorphisms.

Part (iii): This is obvious since the property of convergence is independent of the metric, as it only depends on the topology. □

3. Some tb prerequisites (proof of Theorem B)

The material in this section concerns a type of non-squashing behavior for Legendrians that can roughly be described as follows: a Legendrian that approximates a transverse knot sufficiently well (in a certain technical sense) can be destabilized. This matches well with the intuition that one needs to add zig-zags in order to approximate non–Legendrian knots by Legendrians; see Figure 1. As said in Section 1, this result is implicitly contained in the proofs of the classification of contact structures on solid tori from [Gir00], [Hon00]. However, we choose a different path here, and instead prove the result by directly relying only on the Thurston–Bennequin inequality for Legendrian knots in tight three-manifolds. Recall that the Thurston–Bennequin inequality [Ben83] for Legendrian unknots $\Lambda \subset (S^3, \xi_{st})$ in the standard contact sphere states that

$$\mathtt{tb}(\Lambda) \leq -1.$$

This is a strong result that, e.g., implies the tightness of the standard sphere. We start by recalling certain topological notions in contact manifolds, such as the Thurston–Bennequin number.

3.1. **Twisting and Thurston–Bennequin.** Define the **linking number** of two disjoint oriented null-homologous knots $K_0 \sqcup K_1 \subset M^3$ by the algebraic intersection number

$$\mathtt{lk}(K_0, K_1) := K_0 \bullet \Sigma$$

where Σ is a choice of two-chain with boundary $\partial\Sigma = K_1$. When the ambient manifold satisfies $H_2(M) = 0$ this linking number does not depend on the choice of null-homology.

A **framing** of a knot $K \subset M^3$ inside an orientable three-dimensional manifold can be defined either as a non-vanishing normal vector field, or as a small piece of an embedded orientable surface Σ whose boundary contains the knot. Recall that two different framings of an oriented knot have a well-defined winding number in $\mathbb{Z}$, which vanishes if and only if the two framings are homotopic. This winding number can be interpreted as the "difference of framings" via the formula

$$d(\mathrm{Fr}_{\Sigma_0}, \mathrm{Fr}_{\Sigma_1}) := K_{\Sigma_0} \bullet \Sigma_1 \in \mathbb{Z},$$

where K_{Σ_0} is a sufficiently small push-off of K along a non-vanishing normal vector field that is tangent to the surface Σ_0. Here Σ_i are given orientations that agree on the boundary component K; it thus follows that this number only depends on the orientation of the ambient three-manifold. In the case when Σ_1 is embedded and $K = \partial\Sigma_1$ is its entire boundary, we get the identity $d(\mathrm{Fr}_{\Sigma_0}, \mathrm{Fr}_{\Sigma_1}) = \mathtt{lk}(K_{\Sigma_0}, K)$.

Recall that a contact structure on a three-dimensional manifold induces a canonical orientation via the locally defined volume form $\alpha \wedge d\alpha$. A Legendrian knot Λ has the canonical framing $\mathrm{Fr}_{\mathrm{Reeb}}$ given by push-off in the Reeb direction. In the case when $\Lambda \subset \mathbb{R}^3$ we have the canonical Seifert framing induced by a bounding surface Σ_Λ. We define the **Thurston–Bennequin** number via

$$\mathtt{tb}(\Lambda) := d(\mathrm{Fr}_{\mathrm{Reeb}}, \mathrm{Fr}_{\mathrm{Seifert}}) = \mathtt{lk}(\Lambda_{\mathrm{Reeb}}, \Lambda),$$

where Λ_{Reeb} denotes a small push-off in the Reeb direction. In arbitrary contact manifolds one can define the Thurston–Bennequin number by a similar formula when the knot is null-homologous; in general, this number depends on a choice of null-homology. In addition, given a fixed knot $K \subset M$, we can define a **relative Thurston–Bennequin number** for any Legendrian knot $\Lambda \subset M \setminus K$ that satisfies $\{\pm[\Lambda]\} = \{\pm[K]\} \subset H_1(M)$. Again, this number depends on the choice of a chain Σ with $\partial\Sigma = \Lambda \cup K$ in general; we denote it by

$$\mathtt{tb}_{K,\Sigma}(\Lambda) := \Lambda_{\mathrm{Reeb}} \bullet \Sigma.$$

This number is invariant under contactomorphisms ϕ in the sense that

$$\mathtt{tb}_{K,\Sigma}(\Lambda) = \mathtt{tb}_{\phi(K),\phi(\Sigma)}(\phi(\Lambda)).$$

When $H_2(M) = 0$ it immediately follows that $\mathtt{tb}_{K,\Sigma}(\Lambda)$ is independent of the choice of chain Σ, in which case we will simply write $\mathtt{tb}_K(\Lambda)$.

When Λ is either contained in a surface Σ, or equal to one of its boundary components, one can define the following quantity related to the Thurston–Bennequin number. The **twisting number** is given by

$$\mathtt{tw}(\Lambda, \mathrm{Fr}_\Sigma) := d(\mathrm{Fr}_{\mathrm{Reeb}}, \mathrm{Fr}_\Sigma) \in \mathbb{Z}$$

where Fr_Σ is the framing induced by the surface. When $\partial\Sigma = \Lambda$ we immediately get

$$\mathtt{tw}(\Lambda, \mathrm{Fr}_\Sigma) = \mathtt{tb}_\Sigma(\Lambda) := \mathtt{tb}_{\emptyset,\Sigma}(\Lambda)$$

where the right-hand side is the non-relative Thurston–Bennequin number.

The following results are standard.

Lemma 3.1.

(1) *Suppose $K \subset (M^3, \xi)$ is a smooth knot, $\Lambda \subset M \setminus K$ is a Legendrian knot, Σ' is a (possibly) singular chain with $\partial\Sigma' = \Lambda \sqcup K$ an oriented link, and Σ'' is a singular chain with $K = -\partial\Sigma''$. Then*

$$\mathtt{tb}_{\Sigma'\cup\Sigma''}(\Lambda) = \mathtt{tb}_{\Sigma',K}(\Lambda) + \Lambda \bullet \Sigma'' = \mathtt{tb}_{\Sigma',K}(\Lambda) - \mathtt{lk}(\Lambda, K).$$

(2) *Let $\Sigma \subset (M^3, \xi)$ be an oriented embedded surface with boundary*

$$\partial\Sigma = \Lambda_0 \sqcup -\Lambda_1$$

an oriented Legendrian link. Then

$$\mathtt{tb}_{\Sigma',K}(\Lambda_1) - \mathtt{tb}_{\Sigma\cup\Sigma',K}(\Lambda_0) = \mathtt{tw}(\Lambda_1, \mathrm{Fr}_\Sigma) - \mathtt{tw}(\Lambda_0, \mathrm{Fr}_\Sigma)$$

where $K \subset M \setminus \Sigma$ is either a knot or the empty set $K = \emptyset$, and Σ' is a singular chain that satisfies $\partial\Sigma' = \Lambda_1 \cup K$.

Proof. Part (1): This is a straightforward computation of algebraic intersection numbers.

Part (2): First we use the fact that Σ is embedded in order to compute

$$\mathtt{tb}_{\Sigma\cup\Sigma',K}(\Lambda_0) = \mathtt{tw}(\Lambda_0, \mathrm{Fr}_\Sigma) + (\Lambda_0)_{\mathrm{Reeb}} \bullet \Sigma' \tag{3.1}$$

where the second term counts intersections of $(\Lambda_0)_{\mathrm{Reeb}}$ and Σ'.

Note that the push-off Σ_{Reeb} in the Reeb-direction is an embedded homology between $(\Lambda_0)_{\mathrm{Reeb}}$ and $(\Lambda_1)_{\mathrm{Reeb}}$. We will analyze the intersection locus $\Sigma_{\mathrm{Reeb}} \cap \Sigma'$. For simplicity we consider the case when the chain Σ' is an immersed surface. For Σ_{Reeb} a sufficiently small push-off, followed by a small generic perturbation, the intersections consist of a union of oriented paths in Σ_{Reeb} whose boundary points transversely intersect the boundary

$$\partial\Sigma_{\mathrm{Reeb}} = (\Lambda_0)_{\mathrm{Reeb}} - (\Lambda_1)_{\mathrm{Reeb}},$$

except for a number of boundary components that are in bijection with the finite number of transverse intersection points

$$\partial\Sigma' \cap \Sigma_{\mathrm{Reeb}} = \Lambda_1 \cap \Sigma_{\mathrm{Reeb}} \subset \Sigma_{\mathrm{Reeb}} \setminus \partial\Sigma_{\mathrm{Reeb}}$$

in the interior of Σ_{Reeb}. A signed count of these different boundary points gives rise to the identity

$$(\Lambda_0)_{\mathrm{Reeb}} \bullet \Sigma' = (\Lambda_1)_{\mathrm{Reeb}} \bullet \Sigma' - \Lambda_1 \bullet \Sigma_{\mathrm{Reeb}}$$

of algebraic intersection numbers.

In the latter equation, the first term on the right-hand side is equal to $\mathtt{tb}_{\Sigma',K}(\Lambda_1)$, while the second term is equal to

$$-(\Lambda_1)_{\mathrm{Reeb}} \bullet \Sigma = -\mathtt{tw}(\Lambda_1, \mathrm{Fr}_\Sigma),$$

where we again have used the fact that Σ is embedded. To conclude:

$$(\Lambda_0)_{\mathrm{Reeb}} \bullet \Sigma' = \mathtt{tb}_{\Sigma',K}(\Lambda_1) - \mathtt{tw}(\Lambda_1, \mathrm{Fr}_\Sigma)$$

which gives the sought equality between Thurston–Bennequin and twisting numbers when combined with equation (3.1). □

From part (2) of Lemma 3.1 we immediately deduce the following.

Corollary 3.2. *Let $\Lambda_0, \Lambda_1 \subset (M, \xi)$ be two Legendrian knots inside a contact manifold M that satisfies $H_2(M) = 0$, where $\Lambda_1 \sqcup \Lambda_2 = \partial\Sigma$ is the boundary of an embedded orientable surface $\Sigma \subset M$. For any knot*

$$K \subset M \setminus \Sigma$$

in the same homology class (we allow $K = \emptyset$), the difference

$$\mathtt{tb}_K(\Lambda_0) - \mathtt{tb}_K(\Lambda_1)$$

of relative Thurston–Bennequin numbers is independent of the choice of such K.

Recall that for any manifold X and any $f \in C^\infty(X)$ (including the constant- and zero-functions $f \equiv c, f \equiv 0$), the one-jet

$$j^1 f := \{(x, d_x f, f(x) | x \in X\}$$

is a Legendrian in $J^1(X)$ with its canonical contact structure. The crucial technical result that we rely on is the following relative version of the Thurston–Bennequin inequality:

Lemma 3.3 (Bennequin [Ben83])**.** *Consider a Legendrian knot $\Lambda \subset J^1S^1$ which is smoothly isotopic to the zero section j^10, and fix a reference Legendrian $K = j^1c$ for $c \gg 0$. It follows that the relative Thurston–Bennequin invariants satisfy*

$$\mathtt{tb}_K(\Lambda) \leq \mathtt{tb}_K(j^10) = 0,$$

i.e., the zero-section has maximal relative Thurston–Bennequin invariant.

Proof. Construct a contact embedding

$$F\colon (J^1S^1, \xi_{st}) \hookrightarrow (\mathbb{R}^3, \xi_{st})$$

that takes the one-jet j^1C of a constant function to a standard Legendrian unknot, i.e., a knot which is Legendrian isotopic to $\Lambda_{st} \subset (\mathbb{R}^3, \xi_{st})$ with $\mathtt{tb}(\Lambda_{st}) = -1$. Using this we immediately compute

$$\mathtt{lk}(F(j^1C), F(j^10)) = \mathtt{tb}(\Lambda_{st}) = -1$$

for any $C > 0$. For $c \gg 0$ we thus get $\mathtt{lk}(F(j^1c), F(\Lambda)) = -1$ as well, since Λ and j^10 can be assumed to be smoothly isotopic inside $J^1S^1 \setminus j^1c$.

The image $F(\Lambda)$ is also a Legendrian unknot. Consider an embedded annulus $\Sigma' \subset F(J^1S^1)$ with boundary $\partial\Sigma' = F(K) \cup F(\Lambda)$, and let $\Sigma'' \subset \mathbb{R}^3$ be a null-homology of $F(K) = F(j^1c)$.

Alluding to part (1) of Lemma 3.1 with $F(K) = F(j^1c)$, $c \gg 0$, and Σ' and Σ'' as previously defined, we conclude

$$\mathtt{tb}_K(\Lambda) = \mathtt{tb}(F(\Lambda)) - \mathtt{lk}(F(j^1c), F(\Lambda)) = \mathtt{tb}(F(\Lambda)) + 1.$$

In particular, we get

$$\mathtt{tb}_K(j^10) = \mathtt{tb}(\Lambda_{st}) + 1 = 0.$$

Finally, the Thurston–Bennequin inequality [Ben83] gives

$$\mathtt{tb}(F(\Lambda)) \leq \mathtt{tb}(\Lambda_{st}) = -1$$

from which the sought inequality follows. □

3.2. **Standard Legendrians near a transverse knot.** In this subsection we analyze the standard contact solid tori

$$B^2_{\sqrt{2/k}} \times S^1 \subset \left(\mathbb{R}^2_{(x,y)} \times S^1_\theta, \ker\left(d\theta - (1/2)\left(y\,dx - x\,dy\right)\right)\right)$$

For any integer k, the boundary torus $\partial B^2_{\sqrt{2/k}}$ is foliated by the Legendrian knots

$$\Lambda_k := \left\{ \left(\sqrt{2/k}e^{ik\theta}, \theta + \theta_0\right);\ \theta \in S^1 \right\} \subset \mathbb{R}^2 \times S^1.$$

These Legendrian knots are smoothly isotopic to the core $T = \{0\} \times S^1$ of the solid torus $B^2_{\sqrt{2/k}} \times S^1$, which is a transverse knot.

Lemma 3.4. *There is a contact-form-preserving contact embedding of*

$$(B^2_{\sqrt{2}} \times S^1_\theta, d\theta - (1/2)(xdy - ydx))$$

into $(S^3, x\,dy - y\,dx)$, with image being the complement of a standard transverse unknot. This embedding takes Λ_2 to the standard Legendrian unknot with $\mathtt{tb} = -1$.

It follows that, for any fixed $K \subset B^2_{\sqrt{2}} \times S^1$ that satisfies $[K] = [\{0\} \times S^1] \in H_1(B^2_{\sqrt{2}} \times S^1)$, the relative Thurston–Bennequin invariant

$$\mathtt{tb}_K(\Lambda), \quad \textit{for} \quad \Lambda \subset (B^2_{\sqrt{2}} \times S^1) \setminus K,\ [\Lambda] = [K] \in H_1(B^2_{\sqrt{2}} \times S^1),$$

satisfies the bound

$$\mathtt{tb}_K(\Lambda) \leq -1 + \mathtt{lk}(\Lambda, K)$$

whenever Λ *is smoothly isotopic to* $\{0\} \times S^1$.

Proof. Recall that $(S^3, x\,dy - y\,dx)$ is foliated by periodic Reeb orbits of length 2π, which gives it the structure of the prequantization S^1-bundle over $\mathbb{C}P^1$ with curvature 2π. The complement of a single fibre of this prequantum bundle can thus be identified with the trivial prequantum bundle

$$\left(B^2_{\sqrt{2}} \times S^1, \ker\left(d\theta - (1/2)\left(y\,dx - x\,dy\right)\right)\right) \to B^2_{\sqrt{2}}.$$

The standard Legendrian unknot in S^3 can be realized as the intersection $S^3 \cap \mathfrak{Re}\mathbb{C}^2$, and can thus be seen to be the two-fold cover of the equator in the prequantum bundle projection $S^3 \to \mathbb{C}P^1$. Since Λ_2 lives over a disc of total area π, it can be identified with the unknot in this chart $B^2_{\sqrt{2}} \times S^1$.

Similarly to the proof of Lemma 3.3, the uniform upper bound then follows from the Thurston–Bennequin inequality for Legendrian unknots in S^3 together with part (1) of Lemma 3.1. Note that the linking number $\mathtt{lk}(\Lambda, K)$ computed in S^3 does not depend on the choice of Λ as above. □

Lemma 3.5. *Take any reference knot* $K \subset \left(\mathbb{R}^2 \setminus B^2_{\sqrt{2/m}}\right) \times S^1$ *which is homologous to* $\{0\} \times S^1$. *For* $m \leq k$, *the Legendrian knots*

$$\Lambda_k \subset \partial B^2_{\sqrt{2/k}} \times S^1 \quad \text{and} \quad \Lambda_m \subset \partial B^2_{\sqrt{2/m}} \times S^1$$

satisfy

$$\mathtt{tb}_K(\Lambda_m) - \mathtt{tb}_K(\Lambda_k) = -(m-k).$$

Hence, it follows that

$$\mathtt{tb}_K(\Lambda_m) - \mathtt{tb}_K(\Lambda) \geq -(m-k)$$

for any Legendrian Λ *which is contained inside a standard neighborhood of* Λ_k *while being smoothly isotopic to* Λ_k *inside the same neighborhood.*

Proof. We begin by establishing the relation

$$\mathtt{tb}_K(\Lambda_m) - \mathtt{tb}_K(\Lambda_k) = -(m-k)$$

between relative Thurston–Bennequin numbers. For this we use the contact embedding

$$B^2_{\sqrt{2}} \times S^1 \hookrightarrow (S^3, \ker(xdy - ydx))$$

provided by Lemma 3.4. Since the core $\{0\} \times S^1$ of the solid torus bounds a disc in the prequantization bundle $S^3 \to \mathbb{C}P^1$ that has intersection number $-k+1$ with Λ_k (the disc can be taken to intersect the torus $\partial B^2_{\sqrt{2/k}} \times S^1$ transversely in a curve of slope 1 in a framing for which Λ_k has slope k), one readily computes $\mathtt{tb}(\Lambda_k) = -k+1$ inside S^3; to see this, use a null-homology of Λ_k that consists of an annulus in the solid torus with boundary $\Lambda_k \cup (\{0\} \times S^1)$ together with the aforementioned disc. (Note that, in particular, Λ_2 is the standard unknot.) The sought relation for the relative Thurston–Bennequin numbers then follows from Corollary 3.2.

We continue with the inequality

$$\mathtt{tb}_K(\Lambda_m) - \mathtt{tb}_K(\Lambda) \geq -(m-k).$$

Note that there exists a smoothly embedded cylinder $\Sigma \subset B^2_{\sqrt{2/m}} \times S^1$ with boundary $\partial\Sigma = \Lambda_m \sqcup \Lambda_k$; hence such a cylinder with boundary equal to $\Lambda_m \sqcup \Lambda$ also exists. Corollary 3.2 now implies that each of the differences

$$\mathtt{tb}_K(\Lambda_m) - \mathtt{tb}_K(\Lambda) \quad \text{and} \quad \mathtt{tb}_K(\Lambda_m) - \mathtt{tb}_K(\Lambda_k)$$

are independent on the choice of reference knot

$$K \subset \left(\mathbb{R}^2 \setminus B^2_{\sqrt{2/m}}\right) \times S^1$$

In particular, Lemma 3.3 shows that

$$\mathtt{tb}_K(\Lambda_k) - \mathtt{tb}_K(\Lambda) \geq 0.$$

One can now compute

$$\begin{aligned} &\mathtt{tb}_K(\Lambda_m) - \mathtt{tb}_K(\Lambda) \\ &\quad = \mathtt{tb}_K(\Lambda_m) - \mathtt{tb}_K(\Lambda_k) + (\mathtt{tb}_K(\Lambda_k) - \mathtt{tb}_K(\Lambda)) \\ &\quad \geq \mathtt{tb}_K(\Lambda_m) - \mathtt{tb}_K(\Lambda_k) = -(m-k) \end{aligned}$$

as sought. □

3.3. **Non-squashing results for Legendrian knots into neighborhoods of transverse knots.** In this subsection we can finally prove Theorem B.

We argue by contradiction and assume that there exists $\varphi_i \colon M \to M$, such that in the language of Definition 1.5, the sequence

$$\varphi_i^{(1)} \coloneqq \varphi_i \circ \varphi_{j_0}^{-1} \colon M \to M, \;\; i \geq j_0,$$

of contactomorphisms for $j_0 \coloneqq i_{r/3,\epsilon/3}$ squashes the Legendrian $\Lambda' \coloneqq \varphi_{j_0}(\Lambda)$ onto T. By the definition of $i_{r/3,\epsilon/3}$ it follows that $d(\varphi_i^{(1)}(x), x) < \epsilon/3$ holds on the subset $M \setminus B_{r/3}(T)$ whenever $i \geq j_0$. After increasing $j_0 \gg 0, j_0 \geq i_{r/3,\epsilon/3}$ even further, we may also assume that $\Lambda' \subset B_{r/3}(T)$ is satisfied for the same choice of $r > 0$.

By part (iii) of Lemma 1.7 the property of being a squashing sequence does not depend on the choice of metric. After choosing an appropriate metric on M, and taking the choice of $r > 0$ made in Definition 1.5 to be sufficiently small, the transverse neighborhood theorem implies that one can find a neighborhood $U \subset M$ of the transverse knot $T \subset M$ that is contactomorphic to

$$\left(B^2_{2r} \times S^1_\theta, \ker\left(d\theta - (1/2)\left(y\,dx - x\,dy\right)\right)\right),$$

under which T is identified with $\{0\} \times S^1$ and $B_s(T)$ is identified with $B^2_s \times S^1$ for all $s \leq 2r$. Note that, by the above, we may assume that

$$B_{r/2}(T) \subset \varphi_i^{(1)}(B_r(T)) \subset B_{2r}(T) \hookrightarrow \mathbb{R}^2 \times S^1$$

whenever $0 < \epsilon \ll r$ is taken to be sufficiently small.

There is a compactly supported contact isotopy of $B^2_r \times S^1$ that squashes the transverse knot $\{0\} \times S^1$ onto any of the Legendrian knots Λ_k inside the same neighborhood, where the knots $\Lambda_k \subset \partial B^2_{\sqrt{2/k}} \times S^1$ were described in Section 3.2. Namely, one can use the explicitly constructed isotopy

$$\Lambda_{k,t} \coloneqq \left\{\left(t\sqrt{2/k}e^{ik\theta}, \theta + \theta_0\right);\, \theta \in S^1\right\} \subset \mathbb{R}^2 \times S^1$$

which is through transverse knots for all $t \in [0,1)$ (at $t = 1$ the embedding becomes equal to the Legendrian knot Λ_k). Here we need to use the standard fact that transverse isotopies are generated by an ambient contact isotopy ψ_t; see Corollary 4.2. Note that ψ_t can be assumed to be supported inside $B_{\sqrt{2/k}}(T)$. Below we will take $k \gg 0$.

Consider the sequence $\varphi_i^{(2)}$ of contactomorphisms that is produced by part (i) of Lemma 1.7 applied to the above contact isotopy ψ_t that squashes T onto Λ_k. Part (ii) of Lemma 1.7 applied to the sequences $\varphi_i^{(1)}$ and $\varphi_i^{(2)}$, i.e., the transitivity of the existence of squashing sequences, implies that there is a sequence of contactomorphisms $\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)} \colon M \to M$ that squashes $\Lambda' \subset B_{r/3}(T)$ onto Λ_k. Note that, by part (i) of Lemma 1.7, after choosing $k \gg 0$ in order for $\sqrt{2/k} < r/4$ to hold, we can assume that $\varphi_{\alpha(i)}^{(2)}|_{M \setminus B_{r/4}(T)} = \mathrm{Id}_M$.

The remainder of the proof consists of computations and estimates of relative Thurston–Bennequin numbers $\mathtt{tb}_{K''}(\Lambda'')$ for Legendrians $\Lambda'' \subset B_{r/2}(T)$ and smooth knots $K'' \subset B_{2r}(T) \setminus B_{r/2}(T)$, where $[K''] = [\Lambda''] \in H_1(B_{2r}(T)) = \mathbb{Z}$ are generators of the first homology. The relative Thurston–Bennequin number in general depends on a choice of two-chain. However, we will always consider these relative Thurston–Bennequin numbers as defined inside the contact manifold $B_{2r}(T) \cong B^2_{2r} \times S^1$; since $H_2(B_{2r}(T)) = 0$ these numbers are well-defined (depending only on K'').

Fix an arbitrary smooth knot $K \subset \partial\overline{B_{r/2}(T)}$ for which $[K] = [T] \in H_1(\mathbb{R}^2 \times S^1)$. We start by finding an estimate for the relative Thurston–Bennequin number $\mathtt{tb}_K(\Lambda')$ (where this invariant is computed inside the contact manifold $B_{2r}(T)$). Since $\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)}$ squashes Λ' onto Λ_k, Lemma 3.5 implies that

$$\mathtt{tb}_K(\Lambda_m) - \mathtt{tb}_K(\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)}(\Lambda')) \geq -(m-k)$$

whenever $\Lambda_k, \Lambda_m \subset B_{r/2}(T)$ and $i \gg 0$ is sufficiently large, and $m \leq k$; to that end we note that, for large i, $\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)}(\Lambda')$ is contained inside a standard contact neighborhood $J^1\Lambda_k \hookrightarrow \mathbb{R}^2 \times S^1$ of Λ_k, in which $\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)}(\Lambda')$ is isotopic to $j^1 0 = \Lambda_k$.

It now follows that

$$\mathtt{tb}_{(\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)})^{-1}(K)}((\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)})^{-1}(\Lambda_m)) - \mathtt{tb}_{(\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)})^{-1}(K)}(\Lambda') \geq -(m-k)$$

for $i \gg 0$ large. Note that

$$(\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)})(B_r(T)) \supset B_{r-\epsilon}(T)$$

which means that the latter inequality is between relative Thurston–Bennequin numbers computed in $B_{2r}(T)$, and where $(\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)})^{-1}(\Lambda_m) \subset B_{r-\epsilon}(T)$. (Recall that $0 < \epsilon \ll r$ is sufficiently small.)

Corollary 3.2 implies that

$$\mathtt{tb}_{K'}(\Lambda') - (m-k) \leq \mathtt{tb}_{K'}((\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)})^{-1}(\Lambda_m))$$

holds for all $K' \subset B_{2r}(T) \setminus B_{r-\epsilon}(T)$ in the homology class $[K'] = [T]$. Taking $k \to +\infty$ while keeping $m > 0$ and K' fixed implies that the right-hand side tends to $+\infty$. In other words, the Legendrian $(\varphi_i^{(1)} \circ \varphi_{\alpha(i)}^{(2)})^{-1}(\Lambda_m)$ that is isotopic to T can be assumed to have a relative Thurston–Bennequin number that is greater than the upper bound from Lemma 3.4, which is a contradiction on the Thurston–Bennequin numbers in $B^2_{2r} \times S^1$ for Legendrians in the same smooth isotopy class as T.

4. Normal neighborhood for non–Legendrians (proof of Theorem A)

Here we establish a normal neighborhood theorem for non–Legendrian knots. The goal is to use the standard neighborhood for transverse knots and segments (as analyzed, for example, in Section 3.2) for proving the existence of squashing for non–Legendrians onto some transverse knot as stated in Theorem A. Throughout this section, (M^3, ξ) is a contact 3-manifold, possibly non-compact, with co-oriented contact structure $\xi = \ker \alpha$.

Theorem 4.1. *Let $K \subset (M^3, \xi = \ker \alpha)$ be a smooth co-oriented knot inside a contact three-manifold with a co-oriented contact structure $\xi = \ker \alpha$, and choose a parametrization $\gamma(\theta) \in K$. Then there exists a neighborhood $U \supset K$ that admits a contact embedding*

$$\phi \colon (U, \xi) \hookrightarrow (J^1 S^1, \xi_{st} = \ker(dz - p d\theta))$$

that extends the map

$$\gamma(\theta) \mapsto (\theta, p, z) = (\theta, -\alpha(\dot\gamma(\theta)), 0)$$

where the value of the p-coordinate measures the failure of the Legendrian property.

Proof. We start by choosing a contact form α on M. Then we pick a generic smooth family $P_\theta \subset T_{\gamma(\theta)}M$ of tangent two-plane fields along K that are transverse to both the line field TK and the contact planes ξ (the latter condition just means that the plane does not coincide with ξ); in particular, the intersection $P_\theta \cap \xi_{\gamma(\theta)}$ is one-dimensional. Since generic one-parameter families of two-planes inside a three-dimensional vector space are everywhere transverse, this can be achieved simply by choosing a generic family of two-planes that are transverse to TK. Then we choose a pair of smooth non-vanishing vector fields V_1, V_2 of the rank-2 vector bundle $P \to K$, where $V_1 \in P \cap \xi$. Note that ξ is orientable along K since the contact structure is co-orientable, while P is orientable along K since the knot is co-orientable; hence V_1 is a trivial real line-bundle. We then choose V_2 so that $(V_1(\theta), V_2(\theta))$ form a basis of P_θ at every point. The condition that K is co-orientable is used in the last step. After renormalizing, we may require that $\alpha(V_2) = 1$ is satisfied.

Using these two vector fields and the exponential map, we can construct a smooth embedding

$$\psi \colon U \hookrightarrow J^1 S^1$$

of a neighborhood $U \supset K$ that extends the map

$$\gamma(\theta) \mapsto \{(\theta, p, z) = (\theta, -\alpha(\dot\gamma(\theta)), 0)\}$$

and whose differential maps the vector field V_1 to ∂_p and V_2 to ∂_z. It follows that ψ pulls back $\alpha_{st} = dz - p\,d\theta$ to a contact form

$$\beta := \psi^* \alpha_{st}$$

that satisfies $\beta|_{T_{\gamma(\theta)}M} = \alpha|_{T_{\gamma(\theta)}M}$ along the knot K.

Since the contact manifold M is three-dimensional and $\ker \alpha = \ker \beta$ along K, the convex interpolation $\beta_t = (1-t)\beta + t\alpha$ is a family of contact forms along K.

Since being a contact form is an open condition, β_t are all contact forms in some small neighborhood of K.

A standard application of Moser's trick, see, e.g., the proof of [Gei08, Theorem 2.5.22], produces a smooth isotopy ψ_t with $\psi_0 = \mathrm{Id}_M$ defined in some small neighborhood of K, where $\psi_t|_K = \mathrm{Id}_K$ and $(\psi \circ \psi_t)^* \alpha_{st} = e^{F_t}\beta_t$ for some $F_t \colon M \to \mathbb{R}$. In other words, $\psi \circ \psi_1$ is the sought contact embedding. □

Corollary 4.2. *Consider a smooth isotopy*

$$\gamma_t \colon A \hookrightarrow (M^3, \xi = \ker \alpha)$$

of a union of knots and arcs A that is fixed near the boundary ∂A, and which satisfies $e^{F_t}\gamma_t^*\alpha = \eta_t^*(\gamma_0^*\alpha)$ for some smooth path of reparametrizations $\eta_t \colon A \to A$, $\eta_0 = \mathrm{Id}_A$, that fixes a neighborhood of the boundary, where $F_t \colon A \to \mathbb{R}$ is a smooth path of smooth functions that satisfy $F_0 \equiv 0$. Then the path of embeddings $\gamma_t \circ \eta_t^{-1}$ is induced by an ambient contact isotopy that can be taken to fix a neighborhood of the boundary.

Proof. The pull-back of $e^{-F_t}\alpha$ is constant under the path of embeddings $\gamma_t \circ \eta_t^{-1}$. The proof of Theorem 4.1 can be extended to produce a smooth family of contact embeddings $\psi_t \colon U_t \hookrightarrow J^1 A$ of neighborhoods $U_t \supset \gamma_t \circ \eta_t^{-1}(A)$, where the images $\psi_t(\gamma_t \circ \eta_t^{-1}(A))$ remain fixed in the family. In addition we may assume that this family of embeddings is fixed near the boundary of A.

Considering the inverses ψ_t^{-1}, we obtain a family of contact embeddings whose domain is fixed and contains $\psi_0(\gamma_0)$. Since contact isotopies are generated by Hamiltonians, there exists a global contact isotopy φ_t of M for which $\psi_t^{-1} = \varphi_t \circ \psi_0^{-1}$. In particular,

$$\varphi_t \circ \gamma_0 = \gamma_t \circ \eta_t^{-1}$$

holds as sought. □

Lemma 4.3. *Let K be a non–Legendrian knot inside a contact manifold (M^3, ξ). Then in any neighborhood of K there exists a non–Legendrian knot K_1 which can be identified with*

$$\{p = g(\theta), z = 0\} \subset (J^1 S^1 = S^1_\theta \times \mathbb{R}_p \times \mathbb{R}_z, \ker(dz - p\, d\theta))$$

under a locally defined contactomorphism, where $g^{-1}(0) \subsetneq S^1$ is a finite union of closed path-connected sets (intervals) with non-empty interior, such that there exists a contact isotopy that squashes K onto K_1.

Proof. According to Theorem 4.1, there exists a contact embedding of a neighborhood $U \supset K$ inside M into an open subset of

$$(J^1 S^1 = S^1_\theta \times \mathbb{R}_p \times \mathbb{R}_z, \ker(dz - p\, d\theta)),$$

under which K is identified with a curve of the form $C = \{p = f(\theta), z = 0\}$ and U is identified with a neighborhood $U_C \supset C$ in $J^1 S^1$. Since K is non–Legendrian by assumption, the function f is not everywhere zero.

One can find a finite number of pairwise disjoint neighborhoods of the form

$$O_{r_i,[a_i,b_i]} \coloneqq \{\theta \in [a_i, b_i],\ z^2 + p^2 \le r_i^2\} \subset U_C,\ \ i = 1, \ldots, N,$$

where we have used the identification $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, such that

$$C \setminus \bigcup_{i=1}^{N} O_{r_i,[a_i,b_i]}$$

consists of a finite number of transverse arcs. (Note that the transverse part of C is equal to $C_{tr} = C \setminus \{p = 0\}$.) We can assume that C intersects each $\partial O_{r_i,[a_i,b_i]}$ transversely in the boundary stratum $\{z^2 + p^2 = r_i^2\} \subset \partial O_{r_i,[a_i,b_i]}$.

Consider a family $f_t(\theta)$ of smooth functions for which $f_0 = f$ and such that $C_t := \{p = f_t(\theta), z = 0\}$ coincides with C outside of $\bigcup_{i=1}^N O_{r_i,[a_i,b_i]}$, while $f_t(\theta) = e^{-\rho(t)} f(\theta)$ for $\theta \in [a_i, b_i]$ where:

- $\rho(t) \geq 0$;
- $\rho(t) = 0$ holds in a neighborhood of $\{a_i, b_i\}$; and
- $\rho(t) = t$ holds in the subset $f^{-1}(0) \subset \cup(a_i, b_i) \subset \cup[a_i, b_i]$ (i.e., the non-transverse part of C).

Corollary 4.2 can now readily be applied to produce the corresponding ambient contact isotopy that squashes C onto some knot $K_1 = \{p = g(\theta), z = 0\}$ for which $g^{-1}(0) \subsetneq S^1$ consists of a finite number of closed intervals with non-empty interior. □

Lemma 4.4. *Let K_1 be a non–Legendrian knot that is contactomorphic to*

$$\{z = 0, p = g(\theta)\} \subset (J^1S^1 = S^1_\theta \times \mathbb{R}_p \times \mathbb{R}_z, dz - p\, d\theta)$$

where $g^{-1}(0) \subsetneq S^1$ is a finite union of closed intervals with non-empty interior. Then there exists a contact isotopy that squashes K_1 onto a knot K_2 that satisfies the following.

- *K_2 is contained in an arbitrarily small neighborhood of K_1.*
- *K_2 is nowhere negatively transverse (for some choice of orientation).*
- *The non-transverse part of K_2 again consists of a finite union of closed intervals with non-empty interior.*

In Lemmas 4.4 and 4.6 and their proofs, we recycle the coordinate notation (r, θ, z) and (x, y, z) but hopefully in a way made clear with context.

Proof. We will construct a contact isotopy that fixes the positively transverse portion of K_1, that is $g^{-1}\mathbb{R}_{>0} \subset K_1$, while the remaining parts are squashed onto suitable Legendrian arcs that will be constructed explicitly. The knot K_2 will be taken to consist of these Legendrian arcs adjoined to $g^{-1}\mathbb{R}_{>0} \subset K_1$. Hence, K_2 will have no negatively transverse parts, as sought.

It will be useful to use polar coordinates (r, θ) on $\mathbb{R}^2_{(x,y)}$ in which the standard Liouville form can be expressed as

$$\frac{r^2}{2} d\theta = \frac{1}{2}(x\, dy - y\, dx).$$

One now immediately verifies that

$$\left(\mathbb{R}^2_{(x,y)} \times \mathbb{R}_z, dz - y\, dx\right) \to \left(\mathbb{R}^2_{(x,y)} \times \mathbb{R}_z, dz - \frac{r^2}{2} d\theta\right),$$

$$(x, y, z) \mapsto \left(x, y, z - \frac{1}{2}xy\right),$$

is a strict contactomorphism.

Step I. *Constructing Darboux balls that contain $\partial(g^{-1}\mathbb{R}_{<0})$.*

Using this change of coordinates, the standard neighborhood from Theorem 4.1 gives us the following coordinates near each boundary point $\mathrm{pt} \in \partial(g^{-1}\mathbb{R}_{<0}) \subset K_1$ at which K_1 changes behavior from Legendrian to negatively transverse (resp.

negatively transverse to Legendrian) when following the direction specified by the orientation of the knot. Here and below, denote $B^2_\epsilon := (B^2_\epsilon)_{(x,y)}$. Around each such pt $\in \partial(g^{-1}\mathbb{R}_{<0})$ we can find contact structure preserving coordinates $\big(B^2_\epsilon \times [-\epsilon,\epsilon]_z, dz - \frac{r^2}{2}d\theta\big)$ such that

- K_1 is contained inside the surface $\{z = \frac{1}{2}xy\}$;
- the Legendrian locus of K_1 is contained inside $\{y = z = 0,\ x \leq 0\}$ (resp. $\{y = z = 0,\ x \geq 0\}$); and
- the negatively transverse locus of K_1 is contained inside $\{x > 0,\ y < 0,\ z < 0\}$ (resp. $\{x < 0,\ y < 0,\ z > 0\}$).

Step II. *Normalizing K_1 near the boundary of the Darboux balls from Step* I.

We want to deform K_1 by a contact isotopy supported in the negatively transverse part of K_1 (or, equivalently, produce new coordinates while keeping K_1 fixed) after which K_1, in addition to the three previous conditions, satisfies the following:

- the negatively transverse part coincides with $\{(0,0)\} \times [\epsilon - \eta, \epsilon]$ (resp. $\{(0,0)\} \times [-\epsilon, -(\epsilon - \eta)]$) near $\partial(B^2_\epsilon \times [-\epsilon,\epsilon])$ for some small $\eta > 0$.

Note that, by the last bullet point in Step I, the outgoing negatively transverse part is a curve that can be taken to be normal to the southern hemisphere of the Darboux ball $B^3_\epsilon \subset B^2_\epsilon \times [-\epsilon,\epsilon]$, whereas the incoming negatively transverse part is normal to the northern hemisphere. We now want to deform these arcs so that they pass through the south and north pole, respectively. We claim that this can easily be done by, first, perturbing the transverse arcs near the boundary of B^3_ϵ so that, near the boundary, they are contained inside planes of the form $(\mathbb{R}_r \cdot e^{ic}) \times \mathbb{R}_z$ for some fixed $c \in [0, 2\pi)$ and then deforming the transverse arcs inside the latter planes. Note that here is why, at the start of the proof, we introduce the polar coordinates (r,θ) on $\mathbb{R}^2_{(x,y)}$.

We prove this claim. The characteristic distribution of these planes is the horizontal line field spanned by ∂_r, while the boundary of the Darboux ball B^3_ϵ intersects the plane in the round circle S^1_ϵ. Thus, it is easy to deform K_1 locally near the boundary of the Darboux ball avoiding the horizontal direction and hence staying negatively transverse. We finally produce the sought contact isotopy by alluding to Corollary 4.2.

Step III. *Shrinking the uncontrolled part of K_1 inside the Darboux ball from the previous steps.*

Note that the contact isotopy

$$(r,\theta,z) \mapsto \big(e^{-t}r, \theta, e^{-t}z\big)$$

fixes both the transverse curve $\{r = 0\} = \{x = y = 0\}$ and any Legendrian $\{\theta = \theta_0, z = 0\}$ setwise. In view of the second bullet point in Step I and fourth bullet point in Step II, one can thus apply this rescaling to the part of K_1 that is contained in the previously defined neighborhood $B^2_\epsilon \times [-\epsilon,\epsilon]$ in order to achieve the following:

- outside of an arbitrarily small neighborhood of the origin $(x,y,z) = 0$, the properly embedded unknotted arc $K_1 \cap (B^2_\epsilon \times [-\epsilon,\epsilon])$ is contained inside the union
$$\{z = y = 0\} \ \cup\ \{x = y = 0\}$$
consisting of a Legendrian and a negatively transverse arc.

Here we allude to Corollary 4.2 in order to extend the isotopy of the knot to an ambient contact isotopy.

Step IV. *Constructing a standard neighborhood of each component of* $g^{-1}\mathbb{R}_{<0}$.

Recall the well-known fact that a transverse arc has a standard contact neighborhood of the form

$$\{(0,0)\} \times I \subset \left(B^2_{\epsilon'} \times I, dz - \frac{r^2}{2}d\theta\right)$$

for $\epsilon' > 0$ sufficiently small. Using this model, together with the local coordinates previously produced, we can now patch them together to yield a contact neighborhood of an entire component of $g^{-1}\mathbb{R}_{<0}$ that is contactomorphic to

$$\left(B^2_\epsilon \times [-1-\epsilon, \epsilon], dz - \frac{r^2}{2}d\theta\right).$$

Because the Legendrian components have non-empty interiors, K_1 intersected with the subset

$$\{r \in [\delta/2, \epsilon]\} \cup \{z \in \{-1-\epsilon, \epsilon\}\} \subset B^2_\epsilon \times [-1-\epsilon, \epsilon]$$

can be made to coincide with the two Legendrian arcs

$$A_0 := \{r \in (\delta/2, \epsilon),\ \theta = \theta_0,\ z = 0\} \ \cup\ A_1 := \{r \in (\delta/2, \epsilon),\ \theta = \theta_1,\ z = -1\}.$$

Here $\epsilon > 0$ is sufficiently small, and $0 < \delta \ll \epsilon$ can be taken to be arbitrarily small.

For each of these arcs, we will independently make use of the last bullet point in Step III, which shrinks the arcs $\partial(g^{-1}\mathbb{R}_{<0})$ where the knot changes behavior form Legendrian to negatively transverse (and vice versa).

Since one can perform a contact-form-preserving rotation of the neighborhood in the domain, we can assume that $\theta_0 = \pi$ is satisfied for the arc A_0 without loss of generality.

Step V. *Finalizing the constructing of* K_2.

The next step is to deform $A_0 \cup A_1 \subset K_1$ in order to make it coincide with the Legendrian $\{z = y = 0\}$ near the boundary of some subset of the form

$$B^2_\delta \times [-1-\epsilon, \epsilon] \subset \left(B^2_\epsilon \times [-1-\epsilon, \epsilon], dz - \frac{r^2}{2}d\theta\right),$$

where $0 < \delta \ll \epsilon$. This we do with Lemma 4.5. After this has been achieved, the sought squashing onto a knot K_2 is now easy to construct. First we take K_2 to be equal to the deformed version of K_1 outside of the neighborhoods $B^2_\delta \times [-1-\epsilon, \epsilon]$, while it is given by the Legendrian arc $\{z = y = 0\}$ inside of the latter neighborhood. Note that K_1 now can be squashed onto K_2 by simply keeping it fixed outside of the neighborhoods $B^2_\delta \times [-1-\epsilon, \epsilon]$, while we apply a rescaling as in Step III inside the neighborhoods. Again, we rely on Corollary 4.2 for producing an extension to a global contact isotopy. □

Lemma 4.5. *For any* $\epsilon > 0$ *and* $0 < \delta \ll \epsilon$ *sufficiently small, there exists a compactly supported contact isotopy of*

$$\{r \in (\delta/2, \epsilon),\ z \in (-1-\epsilon, \epsilon)\} \subset \left(\mathbb{R}^2_{(x,y)} \times \mathbb{R}_z, dz - (1/2)(y\,dx - x\,dy)\right)$$

that takes the Legendrian arcs

$$A_0 := \{\theta = \pi,\ z = 0\} \ \cup\ A_1 := \{\theta = \theta_1,\ z = -1\}$$

to arcs that coincide with the Legendrian $\{z = y = 0\}$ *near* $\partial(B^2_\delta \times (-1-\epsilon, \epsilon))$.

Proof. We will perform an explicit construct of the Legendrian isotopy. We begin with the basic observation that a curve

$$s \mapsto (r, \theta, z) = (r(s), \theta(s), z(s)) \in \left(B^2 \times \mathbb{R}_z, dz - \frac{r^2}{2} d\theta\right)$$

is Legendrian for this contact form precisely when

$$z(s) = z(0) + \int_0^s \frac{r(\sigma)^2}{2} \theta'(\sigma) d\sigma$$

is satisfied. In particular, we obtain a path of Legendrian embeddings

$$\left\{\theta = \theta_1 + t f(r),\ z = -1 + t \int_{\epsilon'/2}^{r} \frac{\rho^2}{2} f'(\rho) d\rho\right\}$$

for any smooth function $f(r)$. The goal is to produce a Legendrian isotopy of the second arc $\{r \in [\delta/2, \epsilon],\ \theta = \theta_1,\ z = -1\}$ of this form, which is disjoint from the first arc $\{r \in [\delta/2, \epsilon],\ \theta = \pi,\ z = 0\}$.

We will describe a piecewise smooth curve $\gamma(s) = (r(s), \theta(s)) \in B^2_\epsilon$ which has a piecewise smooth embedded Legendrian lift to $B^2_\epsilon \times \mathbb{R}$, and then let $f(r)$ be a suitable function whose graph approximates the curve $\gamma(s)$ in the C^0-sense. It follows that the Legendrian lift of the graph of $f(r)$ can be taken to be C^0-close to the former piecewise smooth Legendrian.

We take the curve γ to be given by the union of

$$\{r \in (0, a), \theta = 0\} \cup \{r = a, \theta \in [0, 2/a^2]\} \cup \{r \in [a, b], \theta = 2/a^2\} \cup \{r = b, \theta \in [0, 2/a^2]\}$$

joined with

$$\{r = b,\ \theta \in [-\delta, 0]\} \cup \{r \in [b, c],\ \theta \in \theta = -\delta\} \cup \{r = c,\ \theta \in [-\delta, 0]\} \cup \{r \in [c, \epsilon]\ \theta = 0\}$$

as depicted in Figure 2.

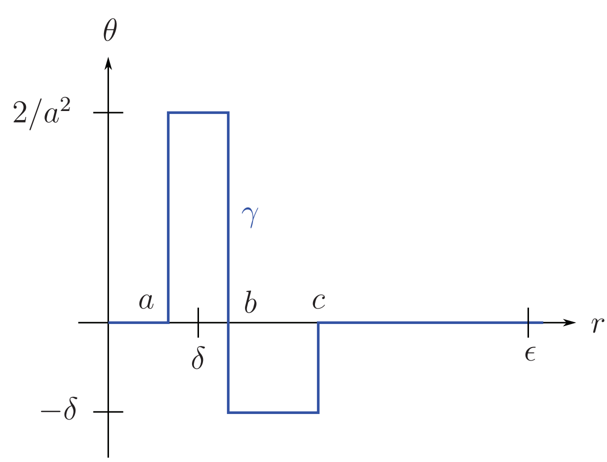

FIGURE 2. The curve γ, which depicts the Lagrangian projection to B^2_ϵ of a piecewise smooth Legendrian in $B^2_\epsilon \times \mathbb{R}$ expressed in polar coordinates.

One easily computes

$$\int_{\gamma\cap\{r\leq t\}}\frac{r^2}{2}d\theta=1$$

for any $t\in(a,b)$. We choose the constants a,b,c to satisfy:

- $c=\sqrt{2\frac{b^2-a^2}{a^2\delta}+b^2}<\epsilon$,
- $a=\sqrt{2}/\sqrt{k2\pi-\theta_1}$ for some integer $k\gg 0$.
- $a<\delta$ and $b>\delta$, with $b^2-a^2>0$ and $\frac{b^2}{a^2}-1>0$ both being sufficiently small.

Here we need to use the assumption that $\delta>0$ can be taken arbitrarily small.

The first bullet point implies

$$\int_\gamma\frac{r^2}{2}d\theta=\frac{a^2}{2}\frac{2}{a^2}-\frac{b^2}{2}\left(\frac{2}{a^2}+\delta\right)+\frac{c^2}{2}\delta=0.$$

The second one implies that $\theta=-\theta_1 \mod 2\pi$ in the region $r\in(a,b)$. Furthermore, one can check that there exists a piecewise smooth Legendrian lift of γ whose z–coordinate satisfies

$$1-\left(\frac{2}{a^2}+\delta\right)\frac{b^2}{2}=1-\frac{b^2}{a^2}+\frac{\delta b^2}{2}\leq z\leq 1 \tag{4.1}$$

where the equality $z=1$ holds precisely in the set $\{r\in(a,b)\}$. By assumption, both quantities $b^2/a^2-1>0$ and $\delta>0$ can be taken to be sufficiently small, which makes the z-coordinate bounded from below by a number just slightly smaller than zero.

A suitable function $f(r)$ whose graph approximates γ can now be used to produce the sought Legendrian isotopy. More precisely, we take $f(r)$ to satisfy the property that the Legendrian lift of its graph $\{\theta=f(r)\}$ still has a z-coordinate that satisfies the same bound as in Equation (4.1), and such that it coincides with the Legendrian lift of γ in the subset where $z=1$. □

Lemma 4.6. *Inside any neighborhood of a non–Legendrian knot K_2 that is positively transverse except at finite number of Legendrian arcs (i.e., satisfies the conclusion of the previous lemma) there exists a neighborhood that is contactomorphic to*

$$U\subset\left(\mathbf{R}^2_{(x,y)}\times S^1_\theta,\ker\left(d\theta-(1/2)\left(y\,dx-x\,dy\right)\right)\right)$$

where

- $(\{(0,0)\}\times S^1)\cup B^3_\epsilon(\{((0,0),\theta_0)\})\cup\ldots\cup B^3_\epsilon(\{((0,0),\theta_N)\})\subset U$ *where* $\theta_0,\ldots,\theta_N\in S^1$ *are cyclically ordered points more than 2ϵ-apart; and*
- *the contactomorphism takes K_2 to a knot that coincides with $\{(0,0)\}\times S^1$ outside of the balls $B^3_\epsilon(\{((0,0),\theta_i)\})$, while its image inside each of these balls is a smoothly unknotted arc with two boundary points contained in the boundary of the ball.*

Proof. Use Theorem 4.1 to map K_2 to the graph

$$\{z=0,p=g(\theta)\}\subset(J^1S^1=S^1_\theta\times\mathbb{R}_p\times\mathbb{R}_z,\ker(dz-p\,d\theta))$$

under a contactomorphic embedding of a neighborhood $U\supset K_2$. Recall that a smooth family of knots of the form

$$\{z=0,p=g_t(\theta)\}\subset(J^1S^1=S^1_\theta\times\mathbb{R}_p\times\mathbb{R}_z,\ker(dz-p\,d\theta)),$$

for which the $g_t(\theta)$ differ by pre-compositions with isotopies of S^1, can be realized by an ambient contact isotopy by Corollary 4.2.

After a suitable such isotopy, supported in an arbitrarily small neighborhood of K_2, we may assume that $g_1(\theta) \geq 0$ has the property that it vanishes precisely inside a finite number of intervals $[a_i, a_i + \epsilon]$ where $\epsilon > 0$ is arbitrarily small. (Roughly, reparameterize $g(\theta)$ in small neighborhoods of $g^{-1}(0)$ in order to shrink the domains where the function vanishes.) For $\epsilon > 0$ sufficiently small we are guaranteed the existence of round Darboux balls centered at $(\theta, p, z) = (a_i, 0, 0)$ of radius $r = 2\epsilon$ that are entirely contained inside U. Obviously these Darboux balls cover the non-transverse part of the knot. Arguing as in the proof of Lemma 4.4, in particular using the contactomorphism between $(\mathbb{R}^2_{(x,y)} \times \mathbb{R}_z, dz - y\,dx)$ and $(\mathbb{R}^2_{(x,y)} \times \mathbb{R}_z, dz - (1/2)(y\,dx - x\,dy))$, we get Darboux balls where the incoming transverse arc is a normal to the southern hemisphere of the ball while the outgoing transverse arc is normal to the northern hemisphere. As in Step II of the same proof, one can then deform these arcs so that they are both passing through the south pole and north pole, respectively, where they are contained inside the Reeb chord $\{(x, y) = 0\}$.

The part of the knot outside of these Darboux balls is positively transverse. One can connect these arcs by positively transverse arcs inside the Darboux balls to form a closed transverse knot $K_{tr} \subset U$. The sought neighborhood is finally given by the union consisting of a suitable standard neighborhood of K_{tr} together with the previously constructed Darboux balls. □

Proof of Theorem A. In view of Lemmas 4.3, 4.4, and 4.6, it suffices to produce a contact isotopy of a knot

$$K \subset (\{(0,0)\} \times S^1) \cup B^3_\epsilon(\{((0,0), \theta_0)\}) \cup \ldots \cup B^3_\epsilon(\{((0,0), \theta_N)\})$$

that squashes it onto the transverse knot $\{(0,0)\} \times S^1$, where we can assume that $K \cap B^3_\epsilon(\{((0,0), \theta_i)\}$ is an unknotted arc, and where K coincides with the transverse knot $\{(0,0)\} \times S^1$ outside of these balls.

The contact isotopy can be taken to fix the arcs

$$K \setminus \left(B^3_\epsilon(\{((0,0), \theta_0)\}) \cup \ldots \cup B^3_\epsilon(\{((0,0), \theta_N)\})\right),$$

while, inside each Darboux ball, it acts on K by the rescaling

$$(x, y, z) \mapsto (e^{-t}x, e^{-t}y, e^{-2t}z).$$

(Here we consider a Darboux ball centered at the origin.) Corollary 4.2 is used in order to ensure that this isotopy is induced by an ambient contact isotopy. □

5. Smooth C^0-limits of Legendrians are Legendrian (proof of Theorem D)

The statement that the image is a Legendrian is an immediate consequence of Lemma 5.1 together with Corollary C.

Lemma 5.1. *Under the assumptions of the theorem, the Legendrian Λ is squashed onto K by the sequence $\varphi_i \colon M \to M$ of contactomorphisms. (See Definition* 1.5.*)*

Proof. For $i_0 \gg 0$, we may assume that the contactomorphisms $\varphi_i \circ \varphi_{i_0}^{-1}$ with $i \geq i_0$ all are arbitrarily close in C^0-distance to the identity.

Part (1) of the definition: We need to show that for $i_0 \gg 0$, there exists a tubular neighborhood of K that contains the image of $\varphi_i(\Lambda)$ for all $i \geq i_0$, in which the latter is smoothly isotopic to K. This follows from Lemma 5.2.

Part (2) of the definition follows immediately from the C^0-convergence. □

Lemma 5.2. *Consider a smooth knot $K \subset M$ and a fixed tubular neighborhood $\mathcal{N} \supset K$. Let $\phi\colon M \to M$ be a smooth map which is sufficiently C^0-close to a homeomorphism ψ that satisfies $\psi(\Lambda) = K$. Then we may assume that $\phi(\Lambda) \subset \mathcal{N}$ is smoothly isotopic to K inside of $\mathcal{N}$.*

Proof. It suffices to show that $\pi_1(\mathcal{N} \setminus \phi(\Lambda)) = \mathbb{Z}^2$ since the existence of a smooth isotopy inside $\mathcal{N}$ from $\phi(\Lambda)$ to K is then a consequence of the classical fact that the Hopf link is detected by the fundamental group of its complement; see [Neu61]. To that end, note that $\mathcal{N}$ is a solid torus, i.e., the complement of an unknot in S^3.

Consider nested closed tubular neighborhoods

$$K \subset \mathcal{N}_1 \subsetneq \mathcal{N}_2 \subsetneq \mathcal{N}$$

that hence satisfy the property that the inclusion $\partial\mathcal{N}_2 \subset \mathcal{N} \setminus \mathcal{N}_1$ is a homotopy equivalence between a torus and a fattened torus.

We consider the tubular neighborhood $\mathcal{N}_\Lambda \coloneqq \psi^{-1}(\mathcal{N}_2)$ of Λ. For ϕ sufficiently C^0-close to ψ, we may assume that $\phi(\partial\mathcal{N}_\Lambda) \subset \mathcal{N} \setminus \mathcal{N}_1$ is satisfied. Since the map is a C^0-approximation of ψ, it is clearly homotopic to ψ. Hence, it follows that

$$(\phi|_{\partial\mathcal{N}_\Lambda})_* = (\psi|_{\partial\mathcal{N}_\Lambda})_*\colon \pi_1(\partial\mathcal{N}_\Lambda) \to \pi_1(\mathcal{N} \setminus \mathcal{N}_1)$$

is an isomorphism of fundamental groups. In other words, the inclusion $\phi(\partial\mathcal{N}_\Lambda) \subset \mathcal{N} \setminus \mathcal{N}_1$ also induces an isomorphism of fundamental groups

$$(\iota_{\phi(\partial\mathcal{N}_\Lambda)})_*\colon \pi_1(\phi(\partial\mathcal{N}_\Lambda)) \to \pi_1(\mathcal{N} \setminus \mathcal{N}_1).$$

First we claim that the rank of $\pi_1(\mathcal{N} \setminus \phi(\Lambda))$ is at least equal to two. This follows since the previously established isomorphism

$$(\iota_{\phi(\partial\mathcal{N}_\Lambda)})_*\colon \pi_1(\phi(\partial\mathcal{N}_\Lambda)) \cong \mathbb{Z}^2 \to \pi_1(\mathcal{N} \setminus \mathcal{N}_1)$$

of groups factors through $\pi_1(\mathcal{N} \setminus \phi(\mathcal{N}_\Lambda))$. (Recall that $\phi(\partial\mathcal{N}_\Lambda) \subset \mathcal{N} \setminus \mathcal{N}_1$ and that there is a homeomorphism $\mathcal{N} \setminus \phi(\mathcal{N}_\Lambda) \cong \mathcal{N} \setminus \phi(\Lambda)$ since $\mathcal{N}_\Lambda \supset \Lambda$ is a tubular neighborhood.)

Second, we claim that the inclusion

$$\iota\colon \mathcal{N} \setminus \mathcal{N}_1 \hookrightarrow \mathcal{N} \setminus \phi(\Lambda)$$

induces a surjection

$$\iota_*\colon \pi_1(\mathcal{N} \setminus \mathcal{N}_1) \cong \mathbb{Z}^2 \to \pi_1(\mathcal{N} \setminus \phi(\Lambda))$$

of fundamental groups. Namely, since the inclusion $\mathcal{N} \setminus \phi(\mathcal{N}_\Lambda) \subset \mathcal{N} \setminus \phi(\Lambda)$ is a deformation retract, considering the composition of inclusions

$$\mathcal{N} \setminus \phi(\mathcal{N}_\Lambda) \subset \mathcal{N} \setminus \mathcal{N}_1 \subset \mathcal{N} \setminus \phi(\Lambda)$$

we see that the isomorphism

$$\pi_1(\mathcal{N} \setminus \phi(\mathcal{N}_\Lambda)) \cong \pi_1(\mathcal{N} \setminus \phi(\Lambda))$$

factors through ι_*.

Finally, the fact that the surjective group homomorphism $\iota_* \colon \mathbb{Z}^2 \to \pi_1(\mathcal{N} \setminus \phi(\Lambda))$ in addition is injective now follows by purely algebraic considerations, using the previously established fact that the rank of $\pi_1(\mathcal{N} \setminus \phi(\Lambda))$ is at least equal to two. (The rank of $\mathbb{Z}^2$ is equal to two and that any quotient of $\mathbb{Z}^2$ by a non-trivial subgroup has rank strictly less than two). □

Now that we know K is Legendrian, it remains to show that K is the contactomorphic image of Λ. We establish this by showing that $\phi_i(\Lambda)$ is Legendrian isotopic to K for $i \gg 0$. We may assume that $\phi_i(\Lambda) \subset J^1K$ is contained inside a standard contact neighborhood of K, and that $\phi_i(\Lambda)$ is *smoothly* isotopic to $j^10 = K$ inside the same neighborhood. We prove the following.

Proposition 5.3. *The Legendrian knot $\varphi_i(\Lambda) \subset J^1K \subset M$ for $i \gg 0$ has the same classical invariants as the Legendrian $K = j^10$ (rotation number, Thurston–Bennequin invariant relative to a big Reeb push-off K' of K, smooth isotopy class) when considered inside the standard contact neighborhood J^1K of the Legendrian knot $K = \phi_\infty(\Lambda)$.*

Remark 5.4. In the case when the contact manifold (M, ξ) satisfies $H_1(M) = H_2(M) = 0$ and the absolute Thurston–Bennequin invariant thus is well-defined, the same ideas as the proof of Proposition 5.3 can be used to show something stronger: if the Legendrian Λ_0 can be squashed onto the Legendrian Λ_1, and Λ_1 can be squashed onto Λ_0, then Λ_0 and Λ_1 are contactomorphic.

Proof. If we take $i_0 \gg 0$ sufficiently large, then $\varphi_i(\Lambda) \hookrightarrow U_K \subset (M, \xi)$ for all $i \geq i_0$, where $U_K \hookrightarrow J^1K$ is contactomorphic to standard contact neighborhood of K in which the latter is identified with j^10. Furthermore, we may assume that $\varphi_i \circ \varphi_{i_0}^{-1}$ is ϵ-close to the identity on some neighborhood $B_r(K) \subset J^1K$, while $\varphi_{i_0}(\Lambda) \subset B_{r/2}(K)$, for some fixed $r > 0$ and $\epsilon > 0$ arbitrarily small.

First we show that for any knot $K' \subset B_r(K) \setminus B_{r/2}(K)$ in the same homology class as K, the relative Thurston–Bennequin numbers $\mathtt{tb}_{K'}(\varphi_{i_0}(\Lambda))$ and $\mathtt{tb}_{K'}(K)$ as computed inside U_K are the same. Since Lemma 3.3 implies $\mathtt{tb}_{K'}(\varphi_{i_0}(\Lambda)) \leq \mathtt{tb}_{K'}(K)$, it suffices to prove $\mathtt{tb}_{K'}(\varphi_{i_0}(\Lambda)) \geq \mathtt{tb}_{K'}(K)$.

Consider the sequence $(\varphi_i \circ \varphi_{i_0}^{-1})^{-1}$ of inverses of the previously defined contactomorphisms, which C^0-converges to $\varphi_{i_0} \circ \varphi_\infty^{-1}$. Lemma 5.1 implies this sequence of contactomorphisms squashes the Legendrian K onto $\varphi_{i_0}(\Lambda)$. As before, we again assume the contactomorphisms to be ϵ-close to the identity on $B_r(K)$ for $i \geq i_0$.

For $i \gg 0$ the squashing property implies that

$$(\varphi_i \circ \varphi_{i_0}^{-1})^{-1}(K) \subset U_\Lambda$$

where $U_\Lambda \hookrightarrow J^1\Lambda$ is a standard contact neighborhood of $\varphi_{i_0}(\Lambda)$ in which the latter is identified with j^10. Again Lemma 3.3 implies

$$\mathtt{tb}_{K'}(\varphi_{i_0}(\Lambda)) \geq \mathtt{tb}_{K'}((\varphi_i \circ \varphi_{i_0}^{-1})^{-1}(K)) = \mathtt{tb}_{\varphi_i \circ \varphi_{i_0}(K')}(K).$$

Since $\varphi_i \circ \varphi_{i_0}|_{K'}$ is ϵ-close to the identity, we may assume that $\varphi_i \circ \varphi_{i_0}(K')$ is homologous to K' inside $B_r(K) \setminus B_{r/2}(K)$. This immediately implies that $\mathtt{tb}_{\varphi_i \circ \varphi_{i_0}(K')}(K) = \mathtt{tb}_{K'}(K)$ is satisfied.

This finishes the proof of the equality of relative Thurston–Bennequin numbers

$$\mathtt{tb}_{K'}(\varphi_{i_0}(\Lambda)) = \mathtt{tb}_{K'}(K).$$

in $U_K \hookrightarrow J^1K$.

Since the smooth isotopy types are clearly the same, it remains to establish an equality between rotation numbers. Recall the Thurston–Bennequin inequality

$$\mathtt{tb}(F(\varphi_{i_0}(\Lambda))) + |\mathtt{rot}(F(\varphi_{i_0}(\Lambda)))| \leq -1,$$

where $F\colon J^1S^1 \hookrightarrow (\mathbb{R}^3, \xi_{st})$ is a contact embedding that takes j^10 to the standard unknot Λ_{st} [Ben83]. Then

$$\mathtt{tb}(F(\varphi_{i_0}(\Lambda))) = \mathtt{tb}(F(j^10)) = \mathtt{tb}(\Lambda_{st}) = -1,$$

which implies the vanishing of the rotation number. □

Proposition 5.3 combined with the classification result of Legendrian knots inside J^1S^1 in [DG07] due to Ding–Geiges produces the sought-after Legendrian isotopy from $\varphi_i(\Lambda)$ to K for $i \gg 0$ confined inside the standard contact neighborhood $J^1K \subset M$. Note that, if we compute the relative Thurston-Bennequin invariant $\mathtt{tb}_{K'}$ inside the jet-space with respect to K' given as a Reeb-flow push-off of the zero section, then this agrees with the definition of the Thurston-Bennequin invariant used by Ding–Geiges (e.g., the zero-section has vanishing Thurston–Bennequin invariant).

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