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The Noether theorems. Invariance and conservation laws in the twentieth century, by Yvette Kosmann-Schwarzbach, translated, revised and augmented from the 2006 French edition by Bertram E. Schwarzbach, Sources and Studies in the History of Mathematics and Physical Sciences, Springer, New York, 2011, hardcover, xiv + 205 pp., ISBN 978-0-387-87867-6

In 1915, at the invitation of Felix Klein and David Hilbert, Emmy Noether arrived in Göttingen. Excitement filled the air. Einstein's general theory of relativity was hot off the press, and Hilbert was striving to understand and compete with his rival. Noether's invariant theory expertise was sought to help her mentors resolve certain relativistic conundrums. Her Göttingen sojourn resulted in her remarkable 1918 paper that contains two groundbreaking general theorems connecting symmetries and conservation laws in the calculus of variations. While her earlier research had concentrated in classical invariant theory, and her fame as the founder of modern abstract algebra was soon to follow, Noether's all-too-brief foray into mathematical physics proved no less profound and influential in the development of twentieth century science and mathematics.

Kosmann-Schwarzbach's book, skillfully translated from the 2004 French original, is an in-depth study of the strange history of Noether's Two Theorems: their antecedents, their immediate impact, the long years of neglect and misrepresentation, and their final vindication and comprehension by much of the community at large. The book describes the highlights of Noether's life, her mathematical upbringing and talent, her inability to secure a regular position, her exile from Nazified Germany to Bryn Mawr in the United States, and her tragically shortened career. The book also brings to life that remarkable era when the titans of German science and mathematics founded modern physics. It makes for a fascinating and lively read, as well as being copiously documented and referenced throughout. One also finds summaries of the contents of many key papers, both pre- and post-Noether, including recent contributions, modern formulations and applications, as well as the strikingly large number of less enlightened versions, misquotations, and omissions that sully much of the literature. Besides a new, definitive English translation of Noether's original paper [N], there are photocopies, transcriptions, and English translations of correspondence between Noether and Klein, Klein and Pauli, and Noether and Einstein. All in all, Kosmann-Schwarzbach's book is a valuable and important contribution to the historical and mathematical literature—well worth owning to savor and reference.

The mathematical setting for Noether's Two Theorems is the calculus of variations, dating back to the Bernoullis and Euler, which now underlies much of modern analysis and physics. The (classical) extremals or, more generally, critical points of a variational principle satisfy a system of differential equations known as the Euler–Lagrange equations, which assumes the role played by the vanishing of the gradient in multivariable calculus. Simply stated, Noether's First, and most famous, Theorem states that every one-parameter symmetry group of a variational problem gives

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rise to a conservation law of the associated Euler–Lagrange equations. And, vice versa, every conservation law gives rise to a one-parameter (generalized) symmetry group. Intimations of this correspondence began to appear in the nineteenth and early twentieth century literature: Jacobi noted that translational and rotational symmetries of space give rise to conservation of linear and angular momentum. Ignaz Schütz showed how time translational symmetry induces conservation of energy—a surprisingly recent concept, due in its modern form to Helmholtz, cf. [E]. In 1911, Gustav Herglotz used the Poincaré invariance of special relativity to construct ten conservation laws, followed soon thereafter by Friedrich Engel's nonrelativistic construction using Galilean invariance. But Noether was the first to realize these were all, in fact, particular instances of a completely general correspondence between conservation laws and symmetry properties.

Moreover, to formulate the converse result, Noether had to radically extend Lie's theory of continuous transformation groups. According to Lie, the infinitesimal generators of one-parameter groups of point transformations are vector fields on a manifold, e.g., the space of independent and dependent variables in a variational principle. He and Bäcklund extended such geometrical group actions to contact transformations on the spaces of derivatives (now known, following Ehresmann, as jet bundles)—although Lie was profoundly disappointed by Bäcklund's Theorem [B] that contact transformations were merely prolongations of point transformations or, in the single dependent variable case, first order ones. Noether's breakthrough, almost completely ignored until the 1960's, was to allow the infinitesimal generators, rather than the group transformations, to depend on the derivative (jet) coordinates. The resulting objects are now known as generalized symmetries. (The misnomer "Lie–Bäcklund symmetries" also appears, although a careful reading of the works of Lie and Bäcklund reveals that, unlike Noether, they never ventured beyond geometrical contact transformation groups.) For example, the higher order conservation laws of the Korteweg–deVries equation,[1] whose discovery [MGK] precipitated the modern soliton revolution in nonlinear partial differential equations, arise, via the Noether correspondence, from truly generalized symmetries, namely the associated hierarchy of higher order commuting flows.

Somewhat later, the reviewer [O3] revisited Noether's result and proved that, as long as the system of Euler–Lagrange equations is "normal", meaning that it can, in some coordinate system, be placed in Cauchy–Kovalevskaya form, then the Noether correspondence matches nontrivial symmetries to nontrivial conservation laws. "Abnormal" systems include the underdetermined systems covered by Noether's Second Theorem—less commonly known and appreciated, but of equal profundity and importance to modern physics and elsewhere. It deals with the case when the variational principle admits an infinite-dimensional symmetry group whose generators depend on one or more arbitrary functions, e.g., the gauge symmetry groups arising in relativity and physical field theories. The conservation laws associated with one-parameter subgroups of such an infinite-dimensional symmetry group are necessarily trivial, meaning that they provide no information on the behavior of solutions. In this way, Noether was able to explain the triviality of the energy conservation law in general relativity that perplexed Einstein, Hilbert, and

[1] A model for water waves, first written down, in fact, twenty years earlier by Boussinesq, cf. [D].

Klein. Moreover, Noether's Second Theorem states that a variational problem admits such an infinite-dimensional symmetry group if and only if its Euler–Lagrange equations are underdetermined, in the sense that a nontrivial combination of their derivatives vanishes identically. In general relativity, this relation is a form of the Bianchi identity of (pseudo-)Riemannian geometry. These results motivate an interesting, still open problem [O3]: normal systems match nontrivial symmetries with nontrivial conservation laws; underdetermined systems match nontrivial symmetries with trivial conservation laws; it is still not known whether there are any overdetermined systems that match trivial symmetries with nontrivial conservation laws through the Noether correspondence.

Despite her close association with Klein and Hilbert, despite their subsequent promotion of her talents and achievements, and despite the fundamental and groundbreaking impact of her Two Theorems, Noether's original paper remained mostly unread, unappreciated, and even unquoted for much of the twentieth century, her accomplishments either ignored or obscured by claims of later lesser lights. We will never know how much of this neglect was purposeful (because she was a woman, because she was Jewish, because she was not a member of the academic establishment, etc.) and how much was because other researchers were unable or unwilling to come to grips with her deep mathematics and profound insight. While Courant and Hilbert's influential text [CH] includes a version of her First Theorem with attribution, her contributions are either downplayed or completely omitted from the contemporaneous works of Weyl, Pauli, Wigner, Carathéodory, Cartan, etc., as Kosmann-Schwarzbach's book fully documents. Even those Noether partisans that tout her enduring fame as a pioneering woman mathematician and algebraist fail, almost uniformly, to properly appreciate or understand the true extent of her analytical accomplishments. An oft-cited 1951 paper by the (University of Minnesota) physicist Edward Lee Hill [H], billed as a comprehensive survey of the analysis and applications of conservation laws in physical field theories, included only a simplified form of Noether's First Theorem, and no mention whatsoever of the Second Theorem. Hill's paper inspired a host of mediocre follow-up papers claiming to generalize Noether's First Theorem while in reality only re-establishing or proving special cases of her original result. (I found and documented many of these whilst composing the historical notes in my own book [O3].) While the true extent of Noether's contributions has become much better understood and appreciated over the last 25 years, one, frustratingly, still finds similarly unenlightened works appearing in print to this day; e.g., [Ne].

The strange neglect of Noether's work also served to retard the development of a number of areas of mathematics, physics, and engineering. For instance, researchers in elasticity, including Eshelby [Es] and Rice [R], who were well aware of the use of material symmetry properties for formulating frame-indifferent constitutive relations, nevertheless ended up constructing the associated conservation laws (path-independent integrals) by hand; only later did the simpler and more powerful Noether mechanism become known [KS] and, subsequently, generalized symmetries were applied to construct new conservation laws [O1, O2]. In a similar vein, the famous analytical identities of Pokhozhaev [P], generalized and extended by Pucci and Serrin [PS], turned out to be direct consequences of the basic Noether identity, again much later producing significant generalizations [V].

Turning to the mathematics, with the underlying jet and transformation group machinery in hand, the identity that underlies both of Noether's results essentially reduces to integration by parts! To illustrate, let me outline the basic construction, omitting some calculational details, which can all be found in [O3, Chapter 5]. A variety of more sophisticated geometric formulations using, for example, differential forms have been proposed, but the underlying idea is inevitably based on Noether's original argument, as I now explain.

Given a system of differential equations, by a *conservation law* we mean a divergence expression

$$\operatorname{Div} P = 0 \tag{1}$$

that vanishes on all (classical) solutions. In dynamical ordinary differential equations, when the only independent variable is the time t, a conservation law $D_t P = 0$ produces a constant of the motion or first integral: $P =$ const. on solutions. For dynamical partial differential equations, the conservation law takes the form

$$D_t T + \operatorname{Div} X = 0,$$

the indicated divergence now being with respect to the spatial variables $x = (x^1, \ldots, x^p)$. In this case, T is a *conserved density* and its integral $\int T\,dx$ is constant for solutions with suitable boundary behavior, e.g., no flux or sufficiently rapid decay at large distances. On the other hand, in two-dimensional equilibrium mechanics, there is no time coordinate, and a conservation law (1) provides a path- or surface-independent integral, of use in fracture mechanics in that behavior near a singularity, e.g., a crack tip or a dislocation, can be deduced by far field measurements obtained by moving the path of integration away from the singularity.

The starting point for Noether is an nth order *variational principle*

$$I[u] = \int_\Omega L(x, u^{(n)})\,dx, \tag{2}$$

in which $\Omega \subset \mathbb{R}^p$ is a "nice" domain, while $u\colon \Omega \to \mathbb{R}^q$ is a sufficiently smooth function. The notation $u^{(n)}$ is shorthand for the derivatives $\partial^J u^\alpha / \partial x^J$, $\alpha = 1, \ldots, q$, of orders $0 \le \#J \le n$. The basic problem of the calculus of variations is to find a function $u(x)$ that minimizes the functional $I[u]$ subject to prescribed boundary conditions on $\partial\Omega$. Classical smooth minima satisfy the Euler–Lagrange equations $E[L] = 0$, obtained by taking the first variation of the functional.

Now suppose that the variational principle admits a continuous symmetry group G. (Discrete symmetries, while obviously important in their own right, fall outside the purview of Noether's Two Theorems.) To begin with, we assume that G is a (connected) Lie group acting on the space of independent and dependent variables (x, u). The group acts on functions, $u(x) \mapsto g \cdot u(x)$, by pointwise transforming their graphs. We call G a *symmetry group* of the variational principle (2) if the group transformations do not change the value of the integral over all subdomains Ω. As in all matters Lie, the first step in the analysis is to write down the infinitesimal form of the postulated invariance. To this end, a vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \tag{3}$$

is called an *infinitesimal generator* of G if its flow $g_t = \exp(t\mathbf{v})$ forms a one-parameter subgroup. We substitute the group transformations g_t into the variational problem, and, by invariance, set the derivative of the resulting expression equal to zero. Since this must hold on all domains Ω, we arrive at the *infinitesimal invariance criterion*

$$\text{pr } \mathbf{v}(L) + L \text{ Div } \xi = 0. \tag{4}$$

Here pr $\mathbf{v}$ denotes the *prolonged action* of $\mathbf{v}$ on the derivatives of u (or, equivalently, to the jet space), whose explicit formula can be found in [O3, Theorem 2.36].

Step 2 applies integration by parts to the left-hand side of the preceding equation, resulting in *Noether's identity*

$$\text{pr } \mathbf{v}(L) + L \text{ Div } \xi = Q \cdot E(L) - \text{Div } A, \tag{5}$$

in which $Q = (Q^1, \ldots, Q^q)$, with components

$$Q^\alpha = \varphi^\alpha - \sum_{i=1}^{p} \xi^i \frac{\partial u^\alpha}{\partial x^i}, \qquad \alpha = 1, \ldots, q,$$

is known as the *characteristic* of the infinitesimal generator $\mathbf{v}$, while $A = (A^1, \ldots A^p)$ is a well-defined p-tuple of functions that depends on L, ξ^i, φ^α, and their derivatives. (The geometrical interpretation of the characteristic is that the solutions to the first order system of partial differential equations $Q = 0$ are precisely the group-invariant functions.) Now comes the punchline: if the infinitesimal invariance criterion (4) holds, then Noether's identity (5) implies

$$\text{Div}\, A = Q \cdot E(L). \tag{6}$$

But this is a conservation law since the right-hand side vanishes whenever u solves the Euler–Lagrange equations!

There are two key extensions to the original Noether formulation. The first, noted in the subsequent paper by Bessel-Hagen [BH] (which was written upon the suggestion of Noether herself), is to relax the infinitesimal invariance condition by allowing a divergence term on the right-hand side of (4):

$$\text{pr } \mathbf{v}(L) + L \text{ Div } \xi = \text{Div } B, \tag{7}$$

which amounts to requiring that the variational problem (2) be invariant under the group action modulo boundary contributions. The resulting conservation law then takes the form

$$\text{Div}(A - B) = Q \cdot E(L).$$

However, while allowing divergence symmetries permits a cleaner formulation, it does not lead to any new conservation laws that cannot already be deduced through Noether's original correspondence. The second, more profound extension is to allow the infinitesimal generator coefficients ξ^i, φ^α, to also depend on derivatives of the field variables; the resulting infinitesimal generator $\mathbf{v}$ forms a *generalized* or *higher order symmetry*. By including generalized symmetries, Noether's First Theorem also admits a converse: every conservation law comes from a generalized symmetry generator. The first step in the proof is to show, again by integration by parts, that the conservation law is equivalent, modulo trivial laws, to one in *characteristic form* (6). One then runs the preceding argument in reverse to produce the corresponding (generalized) symmetry generator.

Turning to infinite-dimensional symmetry groups, one theoretical challenge is that the underlying theory of Lie pseudo-groups remains, to this day, in poorly developed shape [Op]. Remarkably, unlike finite-dimensional Lie groups, we still do not have an abstract object that adequately represents an infinite-dimensional

Lie pseudo-group, and so they are inextricably tied to their action on a space. Despite this theoretical complication, applications to the construction of conservation laws and identities through Noether's Two Theorems remains reasonably straightforward.

For linear systems of Euler–Lagrange equations, the infinite-dimensional symmetry group obtained by linear superposition of solutions produces the so-called reciprocity relations as conservation laws, of importance in continuum mechanics and elsewhere [O3]. As for Noether's Second Theorem, suppose there is a nontrivial differential relation

$$\mathcal{D}_1 E_1(L) + \cdots + \mathcal{D}_q E_q(L) = 0 \tag{8}$$

among the Euler–Lagrange equations, where the $\mathcal{D}_\alpha$'s are certain differential operators. Multiplying (8) by an arbitrary function $h(x)$ and integrating by parts produces a conservation law in characteristic form (6), with $Q_\alpha = \mathcal{D}_\alpha^*[h]$, where * denotes the formal adjoint of the differential operator. The conservation law is, in fact, trivial since it vanishes identically on solutions to the Euler–Lagrange equations. However, applying the preceding Noether argument produces a nontrivial infinite-dimensional variational symmetry group that depends on the arbitrary function $h(x)$ and its derivatives. To establish a converse, the only subtle point is to show that, even if the original infinite-dimensional symmetry group depends nonlinearly on an arbitrary function $h(x)$ and its derivatives, one can always produce one that depends linearly on $h(x)$, from which the corresponding differential relation (8) readily follows. Again, full mathematical details can be found in [O3].

In conclusion, Kosmann-Schwarzbach's masterful historical and mathematical study is a most welcome addition to the literature, furnishing new insight into the sociology and curious history of twentieth century science and mathematics, supplying a deeper appreciation of Noether's profound genius, and providing an invaluable resource for clearing up misconceptions and misreadings of Noether's wonderful theorems.

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