THE APPLICATION OF MATHEMATICS TO THE THEORY OF ECONOMICS.

Mathematical Investigations in the Theory of Value and Prices. By IRVING FISHER, Ph.D., Yale College. New Haven, 1892. 8vo, pp. 124. (Reprinted from the *Transactions of the Connecticut Academy*, vol. IX, July, 1892.)

WITH the advance of any branch of human knowledge more refined methods of investigation become necessary. As the fundamental ideas assume more definite shape and the principles relating to them find more accurate expression, not only do more powerful methods of analysis become applicable to the problems which present themselves for study, but they become essential to their solution. It is not surprising, then, that students of economics should be making use more and more largely of mathematics in the development of their subject. In fact, the quantitative character of the principal economic notions, labor, land, capital, price, and utility, seems to point to mathematical methods as particularly appropriate to the subject; and it is found that the relations between some of these elements is of a character quite as dependent upon the operations of pure mathematics as are those between acceleration, velocity, and distance in dynamics. If many have regarded with disfavor the application of mathematical methods to economic science, and have distrusted the information thus obtained, they have been victims of a prevalent confusion between pure theory and its application, and have based their judgment upon a misinterpretation or a misapplication of the analytical results. It must be remembered that a purely theoretical result requires many modifications before it can apply to a practical example, and generally can apply only in an approximate manner.

Up to the present time the principal advantages derived from the use of mathematics have been in the way of illustration. Diagrams and formulas have been employed to give greater definiteness to the ideas under consideration, and although not generally essential to the discussions, have materially assisted in abridging or elucidating them. A literature, however, has already begun to exist and promises to rapidly increase in importance, in which mathematical methods applied to economic phenomena play a *rôle* of the highest interest.

The first work in which the mathematical character of the subject and the utility of mathematical methods received adequate recognition was published by Cournot in 1838, under the title "Recherches sur les principe mathématiques de la théorie des richesses." * Jevons furnished a most important contribution in 1871 in his "Theory of Political Economy." † This was followed in 1874 by Walras's "Éléments d'économie politique pure," ‡ and in 1889 by the "Untersuchungen über die Theorie des Preises," § of Auspitz and Lieben. In 1890 appeared an interesting and comprehensive treatise by Marshall, "Principles of Economics," ‖ vol. I., with mathematical foot-notes and appendix, a second edition of which followed in the same year. In none of these works has any mathematics of great difficulty been employed. Nothing more complicated than a problem in maxima and minima of a function of several variables, or the discussion of a simple differential equation of the first order, is to be found. Still, much occurs that is of great interest to the student of mathematics.

A recent addition to the literature of this subject is the paper whose title appears at the head of this article. It further clears up some of the ideas already introduced, supplies some new contributions, briefly discusses the work already done in the same line, and gives a valuable bibliography of previous publications.

One of the most interesting features of the progress already made is the introduction of new units appropriate for the measurement of the quantitative conceptions which are peculiar to economics. The introduction of suitable units is just as necessary here as in the case of physical theory, and without such units important advances on mathematical lines would be altogether impossible. The two most important quantitative conceptions yet introduced are utility and marginal utility. The most careful scientific analysis of these conceptions that has come to the writer's notice is contained in the first few pages of Dr. Fisher's paper. We shall follow closely the lines which he has laid down. We do not begin with complete definitions, but we are led by a series of connected definitions to a full appreciation of the nature of the entities considered and to their adequate quantitative definition.

First we use the term utility in a general way, to refer to that property of any commodity by virtue of which an individual is led to desire its possession. The utility of a certain amount of a commodity tends to make an individual willing to forego a certain amount of some other commodity. We lay down the preliminary definition: *For a given in-*

* Paris, Hachette. 8vo, pp. 212.
† London, Macmillan & Co. 8vo, pp. 267.
‡ Lausanne, Rouge. 8vo, pp. 547.
§ Leipzig, Duncker & Humblot. Pp. 550.
‖ London, Macmillan & Co. 8vo, pp. 770.

dividual, at a given time, the utility of a units of A exceeds the utility of b units of B if the individual has a preference for a to the exclusion of b rather than for b to the exclusion of a. This definition is analogous to that which is the basis of inequality in the case of any other quantities, e.g., of forces. If two forces opposite in direction act upon the same particle, that is the greater in the direction of which motion is generated. It follows from this definition, that when the given individual has no preference for one of the commodities to the exclusion of the other, their utilities are equal.

Let us see now how this definition of utility applies to the case of an individual who enters the market to purchase supplies of certain commodities *for a given period of time.* We will suppose that at the moment when we observe him he is exchanging some of a commodity A for another B. We may suppose prices to be such that he gives one gallon of A and receives two bushels of B, then a second gallon for two more bushels, and so on, until finally he has given a gallons and received b bushels. At what point does he stop? Although the *exchange values* of a gallons of A and b bushels of B are equal, their utilities to the individual considered are not. He prefers b to the exclusion of a for his act proves his preference. We write therefore,

$$\text{ut. of } b > \text{ut. of } a.$$

Why then did he cease to buy B? He sold exactly a gallons for b bushels. By stopping here he has shown that the utility of a small increment of B, say another bushel, is less to him than the utility of the corresponding number of gallons of A. But by not stopping before, he has shown that the utility of a small decrement, say a single bushel, is greater than the utility of the gallons buying it. Now, by the principle of continuity, if the small increment or decrement be diminished indefinitely in size, the two inequalities just given approach a common limiting equation,

$$\text{ut. of } db = \text{ut. of } da,$$

db and da being here exchangeable increments.

Since the last increment db is exchanged for da at the same rate as a was exchanged for b, that is, at the market rate, we have

$$a/b = da/db,$$

each ratio being the price of a in terms of b, or as the equation may also be written,

$$db/b = da/a.$$

Dividing the first equation by this, member by member, we obtain

$$b\frac{\text{ut. of } db}{db} = a\frac{\text{ut. of } da}{da}.$$

No confusion will arise if we write this last equation in the form

$$b\frac{dU}{db} = a\frac{dU}{da}.$$

The differential coefficients here introduced are known as *marginal utilities. The marginal utility of a commodity is to be defined as the utility of an infinitesimal increment of that commodity, on the margin of a given rate of consumption, divided by the amount of that increment;* or, more exactly, it is the *limit of the ratio* thus found. It will vary with the rate of consumption, that is, with the amount of the commodity available during a given period of time to satisfy the necessities or desires of the individual considered. *For infinitesimal increments on the margin of the same amount of the commodity it may be assumed that the utility is directly proportional to the size of the increment.* It is plain, then, that the ratio which determines marginal utility does not vary with the absolute value of the infinitesimal increment. The equation deduced above may be expressed thus: *For a given purchaser at the time of purchase the quantity of the commodity purchased multiplied by its marginal utility is equal to the like product for the commodity sold.*

The marginal utility of the two commodities will appear from the preceding theorem to be in the same ratio as their prices. This is true for any given instant, but not in general. If we assume that one of the two commodities exchanged is money, the amount paid being m, we have $a\,dU/da = m\,dU/dm$, where dU/dm is the marginal utility of money. In the case of a given individual this may vary in value in the same way as the marginal utility of any other commodity, and with different individuals in the same circumstances it may have altogether different values.

We wish now to compare two marginal utilities with one another. These marginal utilities may relate to different commodities, or they may relate to the same commodity on the margin of different rates of consumption. In either case we select a new commodity, which must be supposed, of course, to be uninfluenced as to marginal utility by the variations which may be supposed in connection with the given commodity or commodities. We take two *equal* infinitesimal increments of the given commodity or commodities: they will have utilities proportional to the marginal utilities in question. Having assumed now any convenient rate of consumption for the new commodity, compare infinitesimal increments of it which have respectively the *same utilities* as the increments previously considered. Infinitesimal increments of the same commodity on the margin of the same amount have utilities proportional to their magnitudes. The ratio of these increments is, then, the same as the ratio of the infinitesimal utilities of the original commodity or commodities, and consequently the same as that of the marginal utilities.

In order now to establish a unit of utility we must select some standard commodity, some standard rate of consumption, and some standard amount of the commodity which is to possess the unit utility. For example, to the man who consumes 200 loaves of bread a year an additional loaf possesses a utility which might be adopted as a unit. This is all, however, under the assumption, as is also all our preceding discussion, that the utility of bread, and of every other commodity, is independent of that of every other commodity. Dr. Fisher introduces the term *util*, which he defines in the following manner: *A util is the marginal utility of any arbitrarily chosen commodity on the margin of some arbitrarily chosen quantity of that commodity for a given individual at a given time.*

To get the *total utility* of a given amount of a given commodity we summate the utilities of its several parts. For example, the utility of 100 loaves of bread is the sum of the utilities of each loaf.

$$\begin{aligned}\text{ut. of } x &= \text{ut. } (dx_1) + \text{ut. } (dx_2) + \ldots\ldots + \text{ut. } (dx_n)\\ &= \int_0^x \text{ut. } (dx)\\ &= \int_0^x \frac{dU}{dx}\, dx.\end{aligned}$$

The total utility of a given quantity of a commodity at a given time and for a given individual is the integral of the marginal utility times the differential of that commodity. To illustrate this, let us suppose a curve constructed, the abscissa of each point of which indicates the amount of the commodity consumed by an individual during some stated period of time, and the ordinate the corresponding marginal utility of the commodity. Then the total utility of the amount of the commodity included between any two given values of the abscissa will be the area included between the corresponding ordinates; for each infinitesimal increment of commodity will have for its utility the area vertically over it. This curve, which we will call the marginal utility curve, or Jevons's utility curve, affords us a graphical representation of these most important quantitative economic conceptions.

There are two other important quantities which have simple graphical representations in connection with Jevons's utility curve. These are *utility value* and *consumer's rent.* The utility value of a commodity is the product of the amount of the commodity by its marginal utility,

$$x \frac{dU}{dx}.$$

The name is suggested by money-value, which is quantity times price. It is represented by the rectangle contained by the abscissa and ordinate of the corresponding point of the curve.

The consumer's rent is total utility minus utility-value

$$\int_0^x \frac{dU}{dx}\, dx - x \frac{dU}{dx}.$$

It is the total utility diminished by that utility which the commodity would have if it were all rated at the same degree of utility as the last or the least useful increment. In the figure it is represented by the triangular area limited by the curve and situated above the rectangle denoting utility-value.

The last economic quantity which we shall at present introduce is *price.* In an exchange between a commodity and money we have, as we have already seen, the equation $a\frac{dU}{da} = m\frac{dU}{dm}$, a denoting the amount of commodity for which an amount of money m is paid. The price of the commodity is then

$$p_a = \frac{m}{a} = \frac{\frac{dU}{da}}{\frac{dU}{dm}},$$

or the ratio of the marginal utility of the commodity to that of money.

From an obvious extension of the last equation we may draw the following conclusion: *In an ideal market at the instant that any individual discontinues trading, the mar-*

ginal utilities to him of all the commodities are in proportion to their prices.

We may now write down the so-called *equations of exchange.* Considering the special case in which the market contains n consumers and m commodities, the amounts of the commodities $K_a, K_b, \ldots K_m$, and the incomes of the consumers $K_1, K_2, \ldots K_n$, all being given, we find the following equations for the price and marginal utility of each commodity, and for the amount purchased by each consumer.

$$\left\{\begin{array}{l} a_1 + a_2 + a_3 + \ldots + a_n = K_a, \\ b_1 + b_2 + b_3 + \ldots + b_n = K_b, \\ \cdots\cdots\cdots\cdots\cdots \\ m_1 + m_2 + m_3 + \ldots + m_n = K_m. \end{array}\right.$$

$$\left\{\begin{array}{l} a_1 p_a + b_1 p_b + \ldots + m_1 p_m = K_1, \\ a_2 p_a + b_2 p_b + \ldots + m_2 p_m = K_2, \\ \cdots\cdots\cdots\cdots\cdots \\ a_n p_a + b_n p_b + \ldots + m_n p_m = K_n. \end{array}\right.$$

$$\left\{\begin{array}{l} \dfrac{\frac{dU}{da_1}}{p_a} = \dfrac{\frac{dU}{db_1}}{p_b} = \ldots = \dfrac{\frac{dU}{dm_1}}{p_m}, \\ \dfrac{\frac{dU}{da_2}}{p_a} = \dfrac{\frac{dU}{db_2}}{p_b} = \ldots = \dfrac{\frac{dU}{dm_2}}{p_m}, \\ \cdots\cdots\cdots\cdots\cdots \\ \dfrac{\frac{dU}{da_n}}{p_a} = \dfrac{\frac{dU}{db_n}}{p_b} = \ldots = \dfrac{\frac{dU}{dm_n}}{p_m}. \end{array}\right.$$

$$\left\{\begin{array}{l} \dfrac{dU}{da_1} = f_{a_1}(a_1), \quad \dfrac{dU}{db_1} = f_{b_1}(b_1), \quad \ldots \quad \dfrac{dU}{d_{m_1}} = f_{m_1}(m_1), \\ \dfrac{dU}{da_2} = f_{a_2}(a_2), \quad \dfrac{dU}{db_2} = f_{b_2}(b_2), \quad \ldots \quad \dfrac{dU}{dm_2} = f_{m_2}(m_2), \\ \cdots\cdots\cdots\cdots\cdots \\ \dfrac{dU}{da_n} = f_{an}(a_n), \quad \dfrac{dU}{db_n} = f_{bn}(b_n), \quad \ldots \quad \dfrac{dU}{dm_n} = f_{mn}(m_n). \end{array}\right.$$

These equations are equal in number to $2mn + m$, the number of unknown quantities, and will suffice to determine them if we know, as we must assume, the nature of the functions f. Similar equations can be constructed representing the general case in which producers as well as consumers are considered.

The preceding ideas are developed with much skill in Dr. Fisher's paper. Its most conspicuous feature, however, consists in the systematic representation of different questions in the equilibrium of supply and demand through the agency of an elaborate mechanism in the construction of which the greatest ingenuity is displayed. The equilibrium is brought about by means of a liquid in which float a number of cisterns representing the individual consumers and producers. These are made to fulfil the requisite conditions and relations through a series of connecting levers. This dynamical solution of economic problems is both novel and instructive. When more general considerations are introduced, however, under which the utilities of the various commodities are supposed to influence one another, this mechanical representation becomes impossible.

In the general case of mutually dependent commodities, there is nothing to correspond to the notion of total utility, although that of marginal utility is preserved. The idea of marginal utility is thus seen to be of a more fundamental character than that of total utility. In the case where two commodities alone are considered, a convenient method of investigation is by means of two families of curves orthogonal to each other. These are called *indifference curves* and *lines of force*, or curves having at every point the direction of maximum increase of utility. The position of equilibrium for any individual is found to be the point at which his *income line* is cut orthogonally by a line of force.

Dr. Fisher does not refer to one of the most important applications of the mathematical methods, viz., the theory of monopoly revenue and taxation. This theory presents several points of peculiar interest, both theoretical and practical. We must refer the reader for an account of it to Marshall's treatise, already mentioned.

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NEW YORK, *May* 20, 1893.