UTILITY OF QUATERNIONS IN PHYSICS.

Utility of Quaternions in Physics. By A. McAulay, M.A., Lecturer in Mathematics and Physics in the University of Tasmania. New York, Macmillan & Co., 1893. 8vo, xiv and 108 pp.

This publication is an essay that was sent, in December, 1887, to compete for the Smith's Prizes at the University of Cambridge. It is now published in order to show the advantages of quaternion methods in the treatment of mathematical physics. The work deals with the usual theories of elastic solids, electricity, magnetism, hydrodynamics, and vortex motion. Some advances in these theories have also been made, such as: the expression of stress in terms of strain in the most general case of finite strain in an æolotropic body, including variations of temperature; the most general mechanical results of Maxwell's theory of electrostatics and its equivalent stress; a new equation in vortex motion by means of which the author has endeavored to deduce certain general phenomena that would be exhibited by vortex atoms acting upon one another, and from which he concludes that Hicks's vortex-atom theory is more promising than Sir William Thomson's in respect to a probable explanation of the fundamental facts of gravitation, inertia, etc.

The Introduction is devoted to a discussion of the advantages of quaternions over analytical geometry, and a statement in analytic form of the principal new results of the book. The author says, p. 2: "I believe that physics would advance with both more rapid and surer strides were quaternions introduced to serious study, to the almost total exclusion of Cartesian geometry, except in an insignificant way as a particular case of the former." And again, p. 3: "If only on account of the extreme simplicity of quaternion notation, large advances in the department of physics now indicated are to be expected. Expressions which are far too cumbrous to be of much use in the Cartesian shape become so simple when translated into quaternions that they admit of easy interpretation, and what is perhaps of more importance, of easy manipulation." To emphasize this remark, the author then compares the quaternion equation

$$\rho_1' S\nabla_1 \Psi \mathcal{D} w \Delta = 0$$

with its exact Cartesian equivalent. One of the three equations that are required for this equivalent occupies six lines of print, from which the other two may be derived by attention to symmetry! [Thomson and Tait, p. 463, eq. (7).]

The above equation embodies several features of notation that appear frequently throughout the book. First, the suffixes, that point out corresponding operator and operand, and which are to be omitted after the differentiations are performed. That this notation is convenient is seen by the compact form in which it puts Leibnitz's theorem, viz., $D^n uv = (D_1 + D_2)^n u_1 v_2$. Next is the inversion of the usual order of operator and operand, ρ' being the operand, and ∇ the vector differential operator $i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$. The inverted nabla ($\nabla$) at the end is another instance of this violation of the usual custom, since it operates on all the variables of the term in which it occurs. The desirability of these perversions of custom is, of course, greater in quaternions than in analytics, on account of the non-commutative property of quaternion multiplication; but I call to mind an instance in Johnson's Differential Equations, p. 162, where an fD is used in Mr. McAulay's sense of f_1D_1, the function f being a determinant with D's in the last column.

The last and most important feature of this equation is the new operator, ${}_{\Psi}\Omega$, that Mr. McAulay has introduced into quaternions, and which contributes greatly to the power of that analysis in physics. It is an operator formed from the linear-vector function Ψ by changing its coefficients into differential symbols with respect to those coefficients, after the manner of forming operators from quantics in algebra. The following considerations show that the ordinary symbol ∇ is more appropriate for this operator than the new symbol Ω:

Let Q, which is a function of the nine coefficients of Ψ, be expressed as a function of the three independent vectors Ψi, Ψj, Ψk, where i, j, k are any given rectangular unit vectors. Then the ordinary partial operator, ${}_{\Psi i}\nabla$, is a *linear* function of i, and it may therefore be appropriately and conveniently written in the form ${}_{\Psi}\nabla i$. The double operative symbol found by dropping the i is exactly Mr. McAulay's new operator. Since the total variation of Q is the sum of its partial variations, we have

$$\delta Q = - S\delta\Psi i\,{}_{\Psi}\nabla i\,.\,Q - \text{etc.} = - S\delta\Psi\zeta\,{}_{\Psi}\nabla\zeta\,.\,Q.$$

Mr. McAulay does not mention the conjugate property $Si\,{}_{\Psi}\nabla j = Sj\,{}_{\Psi}\nabla i$, etc., although it is used in his proofs, *e.g.*, in finding 15*a*, p. 32.

When Ψ is self-conjugate, the vectors Ψi, Ψj, Ψk are not independent. This means that Q may be expressed in various ways as a function of these vectors without altering its actual value, so that ${}_{\Psi}\nabla$ is indeterminate unless we make some supposition as to the form of the function. This we may do by assuming it as symmetric in $Si\Psi j$ and $Sj\Psi i$, etc. We then have the operator for the case of Ψ self-conjugate that Mr. McAulay defines, and which he says has "a slightly different meaning" from the general symbol. When Ψ is not self-conjugate, then $\frac{1}{2}({}_{\Psi}\nabla + {}_{\Psi'}\nabla)i = {}_{\bar{\Psi}}\nabla i$, and $\frac{1}{2}({}_{\Psi}\nabla - {}_{\Psi'}\nabla)i = {}_{V.\epsilon}\nabla i = \frac{1}{2}V_{\epsilon}\nabla i$, where $\overline{\Psi} = \frac{1}{2}(\Psi + \Psi')$, and $V.\epsilon = \frac{1}{2}(\Psi - \Psi')$.

I may remark here, in connection with linear-vector functions, that Mr. McAulay's suggestion of the name *Hamiltonian* for them does not seem so appropriate as the algebraic term *nonion* or the geometric term *linear strain*. Hamilton's name is connected with the whole subject, and not with one part more than with another. It is better to distinguish what others have introduced into the subject than what Hamilton has done, and it would be appropriate, in carrying out this idea, to call the operator ${}_{\Psi}\nabla$ a "McAulayan."

The symbolic vectors ζ, ζ that have been used to express the value of δQ above, and which Prof. Tait pronounces purely and entirely Hamiltonian, are also freely used throughout the book. One of these vectors stands for ∇, the other for the variable ρ upon which it operates. Thus, $\sigma = -\zeta S\zeta\sigma$, and $Q(\zeta, \zeta) = Q(i, i) + Q(j, j) + Q(k, k)$, where Q is a linear function in each of its variables. In general, if $\phi\omega = -\Sigma\beta S\alpha\omega$, then $Q(\zeta, \phi\zeta) = \Sigma Q(\alpha, \beta)$. These and other formulas involving this symbolic vector furnish valuable suggestions as to important transformations in subsequent work.

Equations (8), (9) on page 19 are generalizations of what Tait and Hicks have proved, and deserve attention on account of their analogies. The extension of Mr. McAulay consists in making Q a sedenion (linear-quaternion function) instead of a quaternion—an extension that is readily made on account of the common distributive properties of the two numbers. Instead of the proof by infinitesimals, one may regard Q as at first a linear function of its position, when the theorems are obvious for finite rectangular figures. The extension to Q any continuous function and for any figures follows at once by the usual method of limits, *i.e.*, that method which regards the actual distribution of Q as the limit of a distribution which is linear throughout small rectangular elements of volume, the form of the linear function changing as one passes from each element of volume to an adjacent element.

These theorems are analogous to the well-known theorem for integration over a line AB, viz., $\int_A^B dQ = Q_B - Q_A$. Calling dQ a perfect line differential, and $Q(Vd\Sigma\nabla)$, $Q(\nabla)ds$, perfect surface and volume differentials of Q [∇ operating on Q], then all three theorems are included in the statement that a perfect differential taken over a given figure equals its corresponding integral (into an element of boundary) taken over the boundary of the figure.

The importance of these theorems may be inferred from the fact that they include the theorems of Green and Stokes as particular cases. Mr. McAulay has put them in compact forms that are easily remembered and applied. The advantage of ∇ and the full quaternion methods over any half-hearted attempts at quaternions in the way of vector analysis and *Curl* operators, is well illustrated in the case of these theorems.

In the subject of potentials, it is shown that Poisson's equation holds for a quaternion distribution, *i.e.*, that $\nabla^2 q = 4\pi Q$, where $q = \iiint uQds$, u being the reciprocal of the distance between a given point and any point of the volume.

Passing to the subject of elastic solids we find ample illustration of the superiority of quaternions. If every point of a body receive a displacement η, then $\chi\omega = \omega - S\omega\nabla.\eta$ is the radius vector of an ellipsoid into which the radius vector ω of a unit sphere would strain by the displacement. This unit sphere is, of course, the magnified image of an indefinitely small sphere at the point considered. The author takes $\chi\omega = q\psi\omega q^{-1}$, where ψ represents the pure strain and $q(\ )q^{-1}$ the subsequent rotation. The value of this rotation when the strain is small is incorrectly given (p. 26) as $V\theta(\ )$; it should be $V(1+\theta)(\ )$.

The stress function is ϕ, i.e., ϕi is the stress upon unit of surface normal to i. It is a tension or pressure according as the angle between i and ϕi is acute or obtuse. The stress upon the surface $d\Sigma$ is therefore $\phi d\Sigma$. By the theorems of differentiation already referred to, the sum of all such forces acting upon a closed surface of the body is the volume differential $\phi\nabla ds$ taken throughout the enclosure; hence $\phi\nabla$ is the *force* per unit volume on the element ds.

More generally $V\rho\phi d\Sigma$ is the vector moment of $\phi d\Sigma$ about the origin, and the sum of these moments for all points of the closed surface equals the volume differential, $V\rho\phi\nabla ds$, taken throughout the enclosure. Considering the partial variations of ρ and ϕ separately, this is a moment per unit volume of $V\rho_1\phi\nabla_1 + V\rho\phi_1\nabla_1$. The last term is that due to the force $\phi\nabla$ already found; hence $V\rho_1\phi\nabla_1$, ($= V\zeta\phi\zeta = 2\epsilon$, where $V.\epsilon$ is the rotary part of ϕ), is the *couple* per unit volume on the element ds.

These preliminaries being established with respect to a body subject to a strain χ, due to a displacement η, and having at every point a stress ϕ, the author now gives to every point, $\rho' = \rho + \eta$, a small additional displacement, $\delta\eta$. It is assumed that the potential energy of the body is the sum of the potential energies of its elements, $= \iiint w ds_0$, where w is the potential energy per unit of unstrained volume. Computing δw from this and from the work done by stresses on the surface and throughout the enclosure, $\delta \omega$ comes out a function of $\delta\eta$ and $\overline{\phi}$ (the pure part of ϕ) only In other words, the potential energy is independent of the stress couples. It is similarly found, by expressing $\delta\eta$ in terms of the strain, to be independent of the rotary part of the strain, $q(\ )q^{-1}$, viz., $\delta w = -mS\delta\psi \,.\, \psi^{-1}\zeta\varpi\zeta$, where $m = ds/ds_0$, the volume modulus of the strain, and ϖ is the pure stress due to the pure strain ψ. It is also shown that the rotation $q(\ )q^{-1}$ merely rotates the stress along with it, or that $\overline{\phi}\omega = q\varpi(q^{-1}\omega q)q^{-1}$.

Thus, finally, w is a function of the pure strain ψ only, so that $\delta w = -S\delta\psi\zeta\,{}_{\psi}\nabla\zeta \,.\, w$. Comparing this with the preceding value of δw, remembering that $\delta\psi$ is arbitrary, we have

$$m(\varpi\psi^{-1} + \psi^{-1}\varpi) = 2{}_{\psi}\nabla w.$$

This equation is explicitly solved by Mr. McAulay, giving ϖ in terms of ψ and w, which are supposed known. A simpler solution is also obtained by taking w as a function of $\psi^2 = \Psi$, transforming the value of δw so found back in terms of $\delta\eta$, and comparing with the value of δw already found in the same terms. This gives, directly,

$$m\overline{\phi}\omega = 2\chi\,{}_{\Psi}\nabla w\chi'\omega = 2\rho_1{}'S\rho_2{}'\omega S\nabla_1\,{}_{\Psi}\nabla w\nabla_2.$$

One of the three Cartesian equivalents of this equation is given in the Introduction, where it occupies ten lines of print.

The author then finds the general equations of internal equilibrium. The particular case for no bodily forces is given in Thomson & Tait, p. 463, eq. (7). [See the first equation in this review.]

In considering variations of temperature, Mr. McAulay states that the above results are still true, w being then a function of the temperature as well as a function of ψ. In treating the thermodynamics of the subject, an entropy function, f, is found by application of the second law, which is of the same form as for gases, viz., $\delta f = \delta H/t$.

Next comes the case of small strains, when the various general processes are much simplified. The results for small strains are, of course, well known in their Cartesian form, but the author adds, "it cannot be bias that makes these quaternion proofs appear so much more natural and therefore more simple and beautiful than the ordinary ones."

The general problem of isotropic bodies, the transformation to orthogonal co-ordinates, which is easier in quaternions than in Cartesian geometry, Saint-Venant's torsion problem, and a general treatment of wires, complete the chapter on elastic solids.

As an illustration of the general character and scope of the work this synopsis is perhaps sufficient, but I cannot resist adding a brief outline of the author's treatment of the general problem of electrostatics in the section on electricity and magnetism.

He considers that Maxwell starts with space uniformly filled with a substance called electricity, whose resultant displacement, **D**, at any point is linearly connected with the electromotive force, **E**, at that point, i.e., $\mathbf{D} = K\mathbf{E}/4\pi$, where the nonion K depends upon the state of the medium at the point considered. It is further assumed that this displacement and force bear to each other the ordinary energy relation between work, force, and displacement; i.e., if w be the potential energy per unit volume, then $\delta w = -S\mathbf{E}\delta\mathbf{D}$. This, by the principle of conservation of energy, may be integrated on the supposition that the displacement, and therefore the force, has increased uniformly from 0 to its given value. This gives $w = -S\mathbf{DE}/2$; from this, and the preceding value of δw, he finds $\delta w = -S\mathbf{D}\delta\mathbf{E}$, and hence that K is self-conjugate.

As a further consequence of the conservation of energy, the line integral of **E** round a closed curve is zero, i.e., $\mathbf{E} = -\nabla v$.

The charge in any space is the surface integral of the displacement outwards. Thus, if σ, D, are surface and volume densities, $\sigma = SUd\Sigma_a\mathbf{D}_a + SUd\Sigma_b\mathbf{D}_b$, a, b denoting the two faces of the element; also $D = -S\nabla\mathbf{D}$.

Two values of $W = \iiint wds$ are then found, one directly by substituting the value of w given above, and the second by substituting in the first $\mathbf{E} = -\nabla v$ and changing the perfect volume differential into a surface differential. From twice the second, minus the first, a third value of W is obtained which is the one used by the author in subsequent work.

The author next, following the example of Helmholtz in the particular case of K a scalar, gives to every point of space a small displacement $\delta\eta$, vanishing at infinity, and finds the consequent increment δW in W. He then, in order to find the equivalent stress $\phi\omega$, puts δW in the form $-\iiint S\delta\eta_1\phi\nabla_1 ds$. The value of ϕ thus found is

$$\phi\omega = -V\mathbf{D}\omega\mathbf{E}/2 - {}_\psi\nabla\omega \,.\, w.$$

The latter part of this stress is unknown, depending upon the unknown relation between ψ, the pure strain of the medium, and w. Its value is found by Mr. McAulay under one or two assumptions as to the relations of these quantities.

When ω is in the plane of **D**, **E**, the first part represents the vector ω turned through 180° round the bisector of **D** and **E** as axis, and lengthened in the ratio of $T\mathbf{DE} : 2$. When ω is perpendicular to **D**, **E**, the first part represents ω altered in the ratio $S\mathbf{DE} : 2$.

In the ordinary case of air, $K = 1$, and $\phi\omega = -\mathbf{E}\omega\mathbf{E}/8\pi$ = vector ω turned 180° round **E** as axis, and lengthened in the ratio $T\mathbf{E}^2 : 8\pi$. This is Maxwell's result (vol. I, chap. V), as may be seen by taking ω alternately parallel and perpendicular to **E**. This result is, of course, immediately obtained by following Maxwell's method in that chapter.

I can hardly make substantial addition to what Prof. Tait has said in praise of the book (*Nature*, Dec. 28, 1893). It is of undoubted scientific value, and the work of a man of genuine power and originality. I find fewer obscurities in the work than in many of the so-called elementary text-books upon the same subjects. It can hardly be doubted that this book will go far towards accomplishing the author's purpose of arousing serious interest in quaternion analysis.

With regard to the fancied difficulties of the subject, I can speak from experience with large classes of average students that it is as easily acquired as analytical geometry. When the fundamental ideas of quaternions have drifted by their weight into the elementary schools, so that students may attack the subject with minds properly prepared for it, I believe that quaternion methods will be found clearer and simpler to the average student than analytical methods. It should not be forgotten that all elementary text-books are built upon the lines of analytical geometry, and that the great need of quaternions at present is a carefully prepared series of text-books that lead up to and finally employ the highest quaternion analysis in all departments of mathematics and physics, to the exclusion of analytical geometry, except as a special case of the former. A. S. HATHAWAY.

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