The precise position of these $p$ intervals can be determined when $k$ is an integer either by Van Vleck's method or by the method explained at the beginning of this paper. If, for instance, $k = 3$ we may proceed as follows. We easily find that $R_n$ satisfies the relation:

$$\xi^3 R_n''' + [\xi - (n + 1)(n + 2)] R_n' - (n + 2) R_n = 0.$$ 

At two successive points where $R_n = 0$ $R_n'''$ will therefore have opposite signs unless between the points is question $\xi = (n + 1)(n + 2)$; and we have the theorem:

$J_{n+3}(x)$ vanishes once and only once between two successive positive roots of $J_n(x)$ except between the two roots which include between them the point $x = 2 \sqrt{(n + 1)(n + 2)}$ in which interval $J_{n+3}(x)$ does not vanish at all.

Bessel's equation is clearly only a first example to which the methods of Sturm, which we have discussed, can be profitably applied. Further considerations of this sort, however, with reference especially to Bessel's functions with negative subscripts and to the theory of hypergeometric functions I will reserve for a future occasion. I shall be satisfied if the foregoing discussion helps to emphasize the importance of Sturm's paper.

Harvard University, January 4, 1897.

ON THE TRANSITIVE SUBSTITUTION GROUPS WHOSE ORDERS ARE THE PRODUCTS OF THREE PRIME NUMBERS.

By Dr. G. A. Miller.

[Read at the January meeting of the Society, 1897.]
able to observe a number of important properties of the
given regular groups. We shall first develop the theorem
upon which our method is based.

Let $a_1, a_2, a_3, \ldots, a_n$ and $s_1, s_2, \ldots, s_g$ represent the elements
and the substitutions, respectively, of a transitive group $G$.
The $g$ substitutions may be arranged as follows:

\[
\begin{array}{cccccccc}
1 & s_2 & s_3 & \cdots & s_m & a_1 \\
\alpha & \alpha s_2 & \alpha s_3 & \cdots & \alpha s_m & a_2 \\
\alpha s_2 & \alpha s_3 & \alpha s_4 & \cdots & \alpha s_m & a_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\alpha s_{g-1} & \alpha s_g & \alpha s_{g+1} & \cdots & \alpha s_m & a_n \\
\end{array}
\]

Where the line represented by $a_a$ contains all the substitutions
of $G$ that replace $a_1$ by $a_a$. In particular, the substitutions
of the first line do not contain $a_1$. It should be
observed that the same letters are used to represent the elements
of $G$, and also the lines in this arrangement of its substitutions. The reason for this will appear presently.

By multiplying all the substitutions of the line $a_a$ into any
one of the substitutions which replace $a_a$ by $a_b$, we obtain
each of the substitutions of the line $a_b$. Hence we obtain
merely a new arrangement of the lines by multiplying all
the substitutions of $G$ into any one of its substitutions.*
This new arrangement may be obtained from the given ar-
rangement by transforming the $a$'s representing the lines by
the substitution which has been multiplied. By multiply-
ing each of the substitutions of $G$ by all its substitutions,
we do not only obtain the identical substitution group in
the $a$'s representing the lines, but we obtain identical substitu-
tion in every case, that is, the rearrangement of the $a$'s is
always exactly represented by the substitution which has
been multiplied.

It may be observed that we have not assumed that the
line $a_1$ contains all the other elements of $G$. Our remarks,
therefore, apply to all the cases when the first line is rep-
resented by any letter that does not occur in it, and the fol-
lowing lines by the letters which replace this in any one of
their substitutions. By choosing one of the possible letters
to represent one line, we determine all the others. When
the group represented by $a_1$ is of degree $n - 1$, it is more
convenient to represent its $n$ conjugates by the given letters
and notice that each of the substitutions of $G$ transforms
these $n$ conjugates in exactly the same way as it transforms
its elements. As this notation is not as general as the pre-

ceding, we shall not employ it, and shall therefore not con­
sider it at this place.

The given permutations of the lines in the given arrange­
ment of the substitutions of $G$ are independent of the ele­
ments by which $G$ may be represented. We obtain the
same permutations if we use for the $s$'s the corresponding
substitutions in any simple isomorphic group $G'$, the simple
isomorphism having been established in any one of the
possible ways. The subgroup of $G'$ which corresponds to
the line $a$, can evidently not contain any operations that
generate any self-conjugate subgroup of $G'$ besides identity
and the substitution which corresponds to $t_a$ and is used to
generate a new line in the arrangement of the substitutions
$G'$ must differ from any of the preceding substitutions in
this arrangement. Conversely, if the substitutions of any
group are so arranged that the first line contains a subgroup
which includes no substitutions that generate any self-conju­
gate subgroup besides identity and if the following lines are
formed in the given manner, these lines will be permuted
according to a simply isomorphic substitution group when
each of the substitutions is multiplied by every substitution
of the group.* Hence we have the

**Theorem I.** The number of the transitive substitution groups that
are simply isomorphic to a given group is equal to the number of
its systems of subgroups that satisfy the two conditions: (1) Each
system includes all the subgroups that are brought in correspond­
ence when the group is made simply isomorphic to itself in every
possible way, and (2) A subgroup that belongs to such a system does
not include any operations that generate any self-conjugate subgroup
with the exception of identity. All these transitive groups may be
derived from the given group.

In order that the applications which we shall make of
this theorem may be more easily followed we shall employ
it to determine the transitive groups that are simply iso­
morphic to $(abc)$ all $(de)$. There are only two systems of
subgroups that satisfy the given conditions, viz.: 1 and
the system which includes $(ab)$. To obtain the group that
corresponds to the former, i. e., the simply isomorphic regu­
lar group we may write the substitutions as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$a_1$</th>
<th>$ab$</th>
<th>$a_4$</th>
<th>$de$</th>
<th>$a_8$</th>
<th>$ab.de$</th>
<th>$a_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$abc$</td>
<td>$a_2$</td>
<td>$ac$</td>
<td>$a_5$</td>
<td>$abc.de$</td>
<td>$a_8$</td>
<td>$ac.de$</td>
<td>$a_{11}$</td>
<td></td>
</tr>
<tr>
<td>$acb$</td>
<td>$a_3$</td>
<td>$bc$</td>
<td>$a_6$</td>
<td>$abc.de$</td>
<td>$a_9$</td>
<td>$bc.de$</td>
<td>$a_{12}$</td>
<td></td>
</tr>
</tbody>
</table>

* Dyck, loc. cit.
By multiplying all of these substitutions into each of the two generators \( ab, ac, de \) we obtain the following two arrangements, a substitution (line) being represented by the corresponding letter:

\[
\begin{array}{cccccccc}
  a_4 & a_1 & a_{10} & a_7 & a_{11} & a_8 & a_5 & a_2 \\
  a_6 & a_3 & a_{12} & a_9 & a_{19} & a_7 & a_4 & a_1 \\
  a_5 & a_2 & a_{11} & a_8 & a_{12} & a_9 & a_6 & a_3
\end{array}
\]

The substitutions that correspond to these arrangements are respectively:

\[
\begin{align*}
  a_1 a_4, & \ a_2 a_9, \ a_3 a_3, \ a_4 a_{10}, \ a_5 a_{12}, \ a_6 a_{11} \\
  a_1 a_{11}, & \ a_2 a_{10}, \ a_3 a_{12}, \ a_4 a_3, \ a_5 a_4, \ a_6 a_9
\end{align*}
\]

Hence these are generators of the regular group which is simply isomorphic to \((abc)\) all \((de)\). To obtain the transitive group which corresponds to the latter of the two systems we may write the substitutions of \((abc)\) all \((de)\) as follows:

\[
\begin{align*}
  1 & \ ab & a_1 & de & ab.de & a_4 \\
  abc & ac & a_2 & abc.de & ac.de & a_3 \\
  abc & bc & a_3 & abc.de & bc.de & a_6
\end{align*}
\]

By multiplying all these substitutions into the same two generators as before, we obtain the following two arrangements:

\[
\begin{array}{cccc}
  a_1 & a_4 & a_5 & a_2 \\
  a_3 & a_6 & a_4 & a_1 \\
  a_2 & a_5 & a_6 & a_3
\end{array}
\]

The substitutions of the lines that correspond to these arrangements are:

\[
\begin{align*}
  a_4 a_9, & \ a_5 a_6, \ a_6 a_9, \ a_1 a_5, \ a_2 a_4, \ a_3 a_6
\end{align*}
\]

Hence these are generators of the transitive group of degree six which is simply isomorphic to \((abc)\) all \((de)\).

Having called attention to the fundamental ideas employed in the following investigation we proceed to state in what sense we shall use several terms which are not always used in exactly the same sense. The term subgroup shall not include either identity or the entire group. Two subgroups of the same group are said to be transform subgroups when one can be transformed into the other by any operation that transforms the group into itself. With respect to regular groups or operation groups this is equivalent to saying that two subgroups are transform when they correspond in any one of the possible simple isomorphisms of
the groups to themselves.* Two or more self-conjugate subgroups may, therefore, be transform. The concepts, transform subgroups and characteristic subgroups, are clearly corresponding extensions of the concepts, conjugate subgroups and self-conjugate subgroups respectively.

It should be observed that the two definitions which Frobenius gives † for a characteristic subgroup (which are equivalent with respect to regular and operation groups), are not equivalent with respect to substitution groups in general. As may be inferred from what has been stated above, we adopt the former of these two definitions for the characteristic subgroup in substitution groups. It may happen that the characteristic and transform subgroups of a given group are identical with its self-conjugate and conjugate subgroups. This is, for instance, the case in the only non-regular transitive group, whose order is the product of two prime factors, as is evident from the fact that this group contains only one self-conjugate subgroup and that all its other subgroups constitute a single system of conjugate subgroups.

Our problem is clearly not as general as that embraced in the given theorem, since we have only to find the transitive groups which are simply isomorphic to given regular groups. It may, therefore, be convenient to replace the theorem by the corollary which we shall directly employ, using the terms as they have been defined above.

COROLLARY. The number of transitive groups that are simply isomorphic to a regular group \((G)\) is equal to the number of such systems of transform subgroups of \(G\) as do not include any self-conjugate subgroup of \(G\). All these groups may be directly obtained from \(G\).

The non-regular transitive groups whose order is \(p^3\).

Since the degree of these groups is \(p^2\) they are included in Sylow’s determination ‡ of all the transitive groups of degree \(p^2\) and order \(p^3\). For the sake of completeness and simplicity we shall re-determine them by means of the given corollary. In the special case when \(p = 2\), it is well known that \((abcd)_s\) is the only group of this kind. In other words, there is only one regular group of order 8 that contains subgroups which do not include substitutions that generate a

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* Frobenius, Sitzungsberichte der Berliner Akademie, 1895, p. 184.
† Ibid, pp. 183 and 184.
self-conjugate subgroup, and all these subgroups belong to the same system of transforms; in fact, they are even conjugate.

When \( p > 3 \) the two non-commutative regular groups contain substitutions of order \( p \) which are not commutative to all the substitutions of the entire group. Since a self-conjugate subgroup of order \( p \) in a group of order \( p^n \) contains no substitution that is not commutative to all the substitutions of the entire group, each of the two non-commutative regular groups must be simply isomorphic to at least one transitive group of degree \( p^2 \). It remains to prove that each of them is simply isomorphic to only one such group, or, in other words, that each of them contains only one system of transform subgroups that does not include any self-conjugate subgroup.

Each of these groups may be supposed to contain the non-cyclical group of order \( p^2 \) as head. The tail of one of them is composed of substitutions of order \( p^2 \), and that of the other is composed of substitutions of order \( p \). Since each of them must be isomorphic to a commutative group with respect to its only self-conjugate subgroup of order \( p \), each of the substitutions of the other subgroups of order \( p \) must be transformed, by all the substitutions of the group, into itself multiplied by all the substitutions of the self-conjugate subgroup. Hence we see that any substitution in either of these groups transforms one of the substitutions of every non-self-conjugate subgroup of order \( p \), to which it is not commutative, in exactly the same way as it transforms any given substitution of such a subgroup.

We can now see directly by means of the following elementary theorem that each of the given non-commutative regular groups of order \( p^3 \) contains only one system of transform subgroups, satisfying the given condition, and is, therefore, simply isomorphic to only one transitive group of degree \( p^2 \).

**Theorem II.** If two groups are placed in simple isomorphism we may add two new corresponding operations \((s_1, s_2)\), one to each, and thus obtain a simple isomorphism between two larger groups, provided:

1. \( s_1 \) and \( s_2 \) transform corresponding generating operations of the two simply isomorphic groups into corresponding operations, and
2. the first power of either \( s_1 \) or \( s_2 \) that occurs in one of the isomorphic groups corresponds to the same power of the other.

Since each of the non-commutative groups of order \( p^3 \) is

---

generated by any two of its substitutions that are not commutative, we may obtain generating substitutions of each of the two possible non-regular transitive groups of order $p^3$ by multiplying all the substitutions of each regular non-commutative group of order $p^3$ into two of its substitutions that are non-commutative and observing how the lines, with respect to some non-self-conjugate subgroup of order $p$, are permuted by this operation. If we arrange the work so that the first line contains one of the multiplied substitutions we evidently obtain a substitution whose degree is less than $p^3$ as one of the two generators.

The non-regular transitive groups whose order is $pq^2$.

Every group of order $pq^2$ contains either a self-conjugate subgroup of order $q^2$ or it contains such a subgroup of order $p$. When the same group contains a self-conjugate subgroup of each of these orders it is commutative and can therefore not be represented as a non-regular transitive group. We shall first determine the non-regular transitive groups which contain a self-conjugate subgroup of order $q^2$.

According to Sylow's theorem each of these groups contains only one system of conjugate subgroups of order $p$. From this and the given corollary we have that every regular group of order $pq^2$ that contains no self-conjugate subgroup of order $p$ is simply isomorphic to one and only one transitive group of degree $q^2$. As none of the required groups can be of a prime degree it remains only to find those which are of degree $pq$. In other words, it remains to determine the number of systems of transform groups of order $q$ in the regular groups of order $pq^2$ that contain no self-conjugate subgroup of order $p$.

As the subgroups of order $q$ must be contained in the self-conjugate subgroup of order $q^2$, there can be only one such subgroup when the given self-conjugate subgroup is cyclical. That is, there is no non-regular group of degree $pq$ and order $pq^2$ that contains a self-conjugate subgroup of order $q^2$. This is also directly evident. When the given self-conjugate subgroup of order $q^2$ is non-cyclical it contains $q + 1$ subgroups of order $q$. We need to consider only the two cases when the remaining substitutions of the groups permute either $q - 1$ or all of these subgroups. By means of Theorem II we readily find that there is in each of these cases only one system of transform subgroups of order $q$ that does not include any self-conjugate subgroup.

Hence each of these regular groups is simply isomorphic to one and only one non-regular transitive group of degree $pq$.

We have now determined the numbers of the non-regular transitive groups that are simply isomorphic to the different regular groups of order $pq^2$ that contain a self-conjugate subgroup of order $q^2$. By employing the lists of the regular groups we may state results as follows:

Of degree $q^2$, $p$ being odd, there are $\frac{1}{2} (p + 5)$ groups when $q - 1$ is divisible by $p$ and there is only one such group when $q + 1$ is divisible by $p$. When $p = 2$ there are 3 such groups.

Of degree $pq$, $p$ being odd, there are $\frac{1}{2} (p + 1)$ groups when $q - 1$ is divisible by $p$ and there is only one group when $q + 1$ is divisible by $p$. When $p = 2$ there is only one such group. When neither $q = 1$ nor $q + 1$ is divisible by $p$ there are no such groups.

It remains to find the non-regular transitive groups that are simply isomorphic to the non-commutative regular groups of order $pq^2$ which contain a self-conjugate subgroup of order $p$. Since $p$ has primitive roots, there must be one and only one such group of degree $p$, when $p - 1$ is divisible by $q^2$. When this condition is not satisfied there is no group of this degree. It now remains only to find the possible groups of degree $pq$.

One of the three regular groups which satisfy the given conditions contains no substitution of order $q^2$. This contains only two systems of transform subgroups of order $q$. One of these systems consists of a characteristic subgroup. Hence this regular group is simply isomorphic to only one transitive group of degree $pq$. The other two of the given regular groups contain substitutions of order $q^2$.

The one which transforms the substitutions of the self-conjugate subgroup of order $p$ according to the cyclical group of order $q - 1$ contains only commutative substitutions of order $q$. The other, which transforms the substitutions of the self-conjugate subgroup of order $p$ according to the cyclical group of order $q^2$ contains one conjugate system of groups of order $q$. Hence this is simply isomorphic to a transitive group of degree $pq$ while the preceding is not simply isomorphic to such a group. The last is also simply isomorphic to one group of degree $p$ as has been observed above.

Hence we have that there are two transitive groups of degree $pq$ and order $pq^2$ that contain a self-conjugate subgroup of order $p$ and there is one such group of degree $p$, when $p - 1$ is divisible by $q^2$. When $p - 1$ is divisible by $q$
but not by $q^2$ only one of these three groups occurs. This is of degree $pq$. When $p - 1$ is not divisible by $q$ there is no such group. This completes the determination of the non-regular transitive groups of degree $pq^2$. The generating substitutions can be found by the method which has been explained above.

The non-regular transitive groups whose order is $pqr$.

We may assume $p > q > r$. Since every group of order $pqr$ contains only one subgroup of order $p$ there can be no transitive group of degree $qr$ and order $pqr$. All of the given groups that do not contain an operation of order $qr$ include also a self-conjugate subgroup of order $q$.† From Sylow's theorem it follows that these groups contain only one subgroup of each of the orders $p$ and $q$. Hence we have that each of the non-commutative groups of order $pqr$ that do not contain an operation of order $qr$ is simply isomorphic to one and only one transitive group of a lower degree and that the degree of each of these groups is $pq$. It remains to consider those groups of order $pqr$ which contain operations of order $qr$.

Since $p$ has primitive roots, there is one and only one group of degree $p$ and order $pqr$, provided $p - 1$ is divisible by $qr$. When this condition is not satisfied, there is no such group. We see directly that this group contains more than one subgroup of each of the orders $q$ and $r$. It is therefore simply isomorphic to a transitive group of degree $pr$, and also to one of degree $pq$.

If we establish a simple isomorphism between $r$ non-cyclical transitive groups of order $pq$, and also between $q$ non-cyclical transitive groups of order $pr$, we obtain two heads, to each of which we may add a substitution which merely interchanges the systems of intransitivity, and thus obtain the two remaining non-commutative groups of order $pqr$ that contain substitutions of order $qr$. From this construction it is evident that the former contains a self-conjugate subgroup of order $r$, and is simply isomorphic to only one transitive group of degree $pr$ while the latter contains a self-conjugate subgroup of order $q$ and is simply isomorphic to only one transitive group of degree $pq$. As the non-regular transitive groups occur under the same conditions as the simply isomorphic regular groups, the determination of the non-regular transitive groups of order $pqr$ is complete.

† Hölder, Mathematische Annalen, vol. 43, p. 370.
Summary.

We have now found all the possible non-regular transitive groups whose orders are the products of three prime factors by employing, in most cases, the regular groups of these orders. The results are as follows:

<table>
<thead>
<tr>
<th>Order</th>
<th>Degree</th>
<th>Number</th>
<th>Conditions</th>
</tr>
</thead>
</table>
| $p^3$ | $p^3$  | 2      | when $p > 2$.
|       |        | 1      | when $p = 2$. |
| $pq^2$ | $p$   | 1      | when $p - 1$ is divisible by $q^2$. |
| $q^3$ | $p + 5$ | when $p > 2$ and $q - 1$ is divisible by $p$. |
|       | 2      | when $p > 2$ and $q + 1$ is divisible by $p$. |
|       | 3      | when $p = 2$. |
| $pq$  | $p + 1$ | when $p > 2$ and $q - 1$ is divisible by $p$. |
|       | 2      | when $p - 1$ is divisible by $q^2$. |
|       | 1      | when $p - 1$ is divisible by $q$ but not by $q^2$. |
|       | 1      | when $p > 2$ and $q + 1$ is divisible by $p$. |
|       | 1      | when $p = 2$. |
| $pqr$ | $p$    | 1      | when $p - 1$ is divisible by $qr$. |
| $pr$  | 2      | when $p - 1$ is divisible by $qr$. |
|       | 1      | when $p - 1$ is divisible by $q$ but not by $r$. |
| $pq$  | $r + 2$ | when $p - 1$ is divisible by $qr$ and $q - 1$ is divisible by $r$. |
|       | $r + 1$ | when $p - 1$ is divisible by $r$ but not by $q$ and $q - 1$ is divisible by $r$. |
|       | 2      | when $p - 1$ is divisible by $qr$ but $q - 1$ is not divisible by $r$. |
|       | 1      | when $p - 1$ is divisible by $r$ but not by $q$ and $q - 1$ is not divisible by $r$. |
|       | 1      | when $p - 1$ is not divisible by $r$ but $q - 1$ is divisible by $r$. |

Paris, December 7, 1896.