Real curve	$u' = .22314$	
X	$u = .\ 5596$	
$\cosh 2u$	1.6945	
$-\cosh 2u'$	−1.1012	
$-\cosh (1.2528 + u)$	−3.1441	
$+\cosh (1.5316)$	2.4209	
	− .1299	
Y		
$\sinh 2u$	1.3679	
$-\sinh 2u$	− .4612	
$-\sinh (1.2528 + u)$	−2.9808	
$\sinh (1.53157)$	2.2047	
	.1306	
Imaginary curve	$u' = .5$	
X'	$u\ -.4$	
	or	
$\sinh (3.6931 + u + u')$	$+\sin h\ 1.5931$	2.3591
$-\sinh (1.2523 + u')$	$-\sin h\ 1.7528$	2.7983
		− .4392
Y'	or	
$\cosh (\ .6931 + u + u')$	$\cos h\ 1.5931$	2.5622
$-\cosh (1.2528 + u)$	$-\cos h\ 1.7528$	−2.9711
		− .4089

LIE'S GEOMETRY OF CONTACT TRANSFORMATIONS.*

SOPHUS LIE.—*Geometrie der Berührungstransformationen.* Dargestellt von SOPHUS LIE und GEORG SCHEFFERS. Erster Band. Mit Figuren im Text. Leipzig, Druck und Verlag von B. G. Teubner. 1896. 8vo. Pp. xi+693.

Lie was asked in conversation once what constitute the necessary and sufficient characteristics of a mathematician. He replied forthwith: "Phantasie, Energie, Selbstvertrauen,

*Lie uses in the German *Berührungstransformation;* his French pupils translate it *transformation de contact*, Forsyth translates it *tangential transformation*, and Klein *contact transformation*. The latter English translation is retained here as a more precise designation, since by such a transformation not only is the property of tangency, but also in general the order of contact preserved.

Selbstkritik." The illustrious Norwegian thereby unconsciously drew an impressionistic portrait of his own genius, as the striking originality and wonderful scope of his mathematical discoveries testify. Lie's phenomenal creative power and indomitable energy are evidenced not only by the numerous memoirs which have appeared from his pen in the journals and proceedings of learned societies of Norway, France and Germany during the last twenty-five years, but even more emphatically by the fact that the above-named volume is the last of six* tomes aggregating over four thousand royal octavo pages that have come from his hand in the interval between 1888 and 1896, the first five of which appeared in the years 1888–'93.

The last one may with all propriety be called Lie's Geometry or the Geometry of Lie's contact transformations. Transformations applied to certain given differential equations had appeared in the science of mathematics as early as Euler, Lagrange and Legendre, but none of these mathematicians studied the transformations in themselves or established general propositions concerning the particular transformations used, much less concerning general categories of such transformations. Ampère studied more general transformations than those of Euler and Lagrange which he applied to partial differential equations, but the idea of a contact transformation appears neither in his writings nor in those of his successors up to the time of Lie's activity. None of Lie's predecessors gave a definition of a contact transformation, much less a systematic theory of contact transformations. To him is due both the credit of inventing the idea of a contact transformation and that of establishing the theory of such transformations in an independent existence. He has also given a precise development of the theory of infinitesimal contact transformations

*SOPHUS LIE: Theorie der Transformationsgruppen. Bearbeitet von Sophus Lie unter Mitwirkung von Prof. Dr. Engel. i. Abschnitt, x+632, 1888; ii. Abschnitt, vi+554, 1890; iii. Abschnitt, xxvii+830, 1893. Leipzig, Teubner.

SOPHUS LIE: Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen. Bearbeitet und herausgegeben von Dr. Scheffers, xiv+568, 1891. Leipzig, Teubner.

SOPHUS LIE: Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen. Bearbeitet und herausgegeben von Dr. Scheffers, xii+810, 1893. Leipzig, Teubner.

Vol. 1 of the Theory of Transformation-Groups was reviewed by Dr. Chapman in the BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY, vol. 2, No. 4, January, 1893; the Lectures on Continuous Groups, by Mr. Brooks in the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, vol. 1, No. 10, July, 1895.

and groups of contact transformations, these constituting but a part of his more general and beautiful theory of continuous groups. With great injustice to Lie these transformations have been referred to as the contact transformations of Jacobi and Lie. As well, in fact with better propriety could we speak of the method of fluxions of Wallis and Newton or the infinitesimal calculus of Fermat and Leibnitz. All honor to Jacobi for his additions to our knowledge of the means of transforming the so-called canonical systems of partial differential equations in dynamics. It is true that Jacobi used a particular contact transformation. So did Apollonius of Perga two hundred years before the present era. But we have no evidence that either Jacobi or Apollonius knew it, let alone the conception of the most general transformation of this kind; the idea of a contact transformation is Lie's.

To be completed in two volumes the work is intended to give an extended account of Lie's first geometrical investigations* in the years 1869–72. The text of the first volume before us is preceded by a preface of five pages and a table of contents, and followed by an analytical alphabetical index of subjects and also one of the references to more than a hundred mathematicians of the past and present; the marginal references indicating concisely the contents of the paragraphs constitute another marked feature in the way of indices.

The preface is more than an indication of the contents and scope of the sequel. It presents in precise form Lie's ideas relative to the notions that have been of most power in the development of mathematics; his oft-repeated regrets that the science has of late years been broken into two distinct branches—analysis and geometry, to such an extent that one phase almost asserts its independence of the other, and a plea for a kind of reciprocity between analysis and geometry by which they will advance side by side and enrich each other step by step as in the earlier days. Mathematics is broader than geometry, and deeper than analysis, and while Lie is preëminently a geometer, and professionally a professor of geometry,† yet no living mathematician pleads more earnestly for the union of the two great forces in mathematics, and he has taken particular pains to develop

* The best known of these are: "Om en Classe geometriske Transformationer," *Ges. der Wiss. zu Christiania*, October, 1870. Ueber Complexe, insbesondere Linien- und Kugelcomplexe mit Anwendung auf die Theorie partieller Differentialgleichungen," *Math. Ann.*, vol. 5, 1872.

† Professor der Geometrie an der Universität Leipzig.

a rigorous analytical theory * of contact transformations alongside the more obvious geometrical theory.

It is characteristic of Lie's lectures that every advance synthetically is duplicated analytically and *vice versa*. Not the least important result of the dual development of his theory of continuous groups is its wide application to many apparently heterogeneous subjects and hence its great usefulness as a unifying principle in the science. This demand that analysis become the handmaid of geometry and geometry the servant of analysis, was the theme of Lie's inaugural lecture at Leipzig in 1886. How far this theme has been the burden and inspiration of his own mathematical investigations is best shown by the following sentence from the preface, " Charakteristich für meine Richtung dürfte es besonders sein, das ich nach dem Vorbilde von Monge die geometrischen Begriffe, namentlich die von Poncelet und Plücker eingeführten, für die Analysis verwertet und andererseits Lagrange's, Abel's and Galois' Ideen über die Behandlung der algebraischen Gleichungen auf die Geometrie und besonders auf die Theorie der Differentialgleichungen ausgedehnt habe."

The first volume falls into the three grand divisions: I. Contact Transformations in the Plane, chapters 1–5, pp. 1–176; II. Geometry of Lineal Elements in Space, chapters 6–10, pp. 177–480; III. Introduction to the Geometry of Surface Elements. Partial Differential Equations of the First Order, chapters 11–14, pp. 480–687.

I. In the first division of the work is introduced the notion *contact transformation in the plane*, and methods are developed for finding all transformations of this kind. The most important properties of contact transformations are exhibited and their usefulness in geometry and ordinary differential equations illustrated by simple but instructive examples. The infinitesimal contact transformations are defined and all forms of them determined. Applications of these infinitesimal contact transformations to optics and mechanics are given, and finally, their importance in geometry is further shown by a beautiful application in the solution of the difficult problem—to find the form of the elements of arc of all surfaces for which the differential equation of the third order of the geodesic circles† written in

* See vol. 2 of the Theory of Transformation-Groups.

†These so-called geodesic circles are the curves having constant geodesic curvature and were first studied by Minding. They must not be confused with those geodesic circles studied by Gauss under the definition that their points have a constant geodesic distance from fixed points of the