12. Apart from their properties as transformations, the above transformations are of interest because of certain applications to plane curves, notably to spirals which it is hoped to bring out in a subsequent note.

Since finishing this note the writer finds that the finite forms of the transformations discussed were given by Laisant in the *Nouvelles Annales de Mathématiques*, 2d series, vol. 7 (1868), p. 318, in the solution of a problem proposed by Haton de la Goupillière, *Nouvelles Annales*, vol. 6 (1867), problem No. 803. The wide divergence between the properties and the points of view of the present note and the solution referred to seem to warrant its presentation to the Society. The above-mentioned volumes of the *Nouvelles Annales* are to be had in the Library of Congress.

Baltimore, 14 April, 1897.

CONTINUOUS GROUPS OF CIRCULAR TRANSFORMATIONS.*

BY PROFESSOR H. B. NEWSON.

(Read before the American Mathematical Society, at the Meeting of April 24, 1897.)

The object of this paper is to present the outlines of a fairly complete theory of the continuous groups of linear fractional transformations of one variable. The method employed is quite different from the methods of Lie. Lie's classic theory is based upon the infinitesimal transformation; I shall make but little use of the infinitesimal transformation, but shall develop the subject from the consideration of the essential parameters of the transformation. The complex plane is chosen because it beautifully illustrates the methods. I have put together some old and some new facts and have sought to build up a general theory.

*Several terms have been proposed to designate the linear fractional transformations of the complex plane. Möbius introduced the term "Kreisverwandtschaft." Mathews' Theory of Numbers, page 107, translates this as "Möbius' Circular Relation." Professor Cole, in *Annals of Mathematics*, vol. 5, page 137, refers to "Orthomorphic Transformation," following Cayley; this seems too general for the special case here considered, since it is applicable to all conformal transformations. Darboux, in his Théorie des Surfaces, vol. 1, page 162, uses "transformation circulaire." It seems to me that "Circular Transformation" is the best yet proposed, for the fundamental property is expressed in the name.
A circular transformation $T$ of the complex plane is represented by

$$z_1 = \frac{az + b}{cz + d}.$$ 

Regarding the straight lines of the plane as circles through the one point at infinity, the fundamental property of this transformation is that it transforms circles into circles. It interchanges among themselves the circles of the plane, but leaves unchanged or invariant the configuration composed of the $\infty$ circles of the plane. (See Forsyth's Theory of Functions, pages 512-524.)

Since any two circular transformations $T$ and $T_1$ each leave invariant this configuration of all circles of the plane, their product, i.e., the transformation which is equivalent to the successive application of the two, must likewise leave the same configuration invariant and hence be a circular transformation.

This conclusion may be verified analytically by eliminating $z_1$ from two circular transformations $T$ and $T_1$ as follows:

$$z_1 = \frac{az + b}{cz + d}, \text{ and } z_2 = \frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}.$$ 

The product of $T$ and $T_1$ is $T_2$ given by

$$z_2 = \frac{(a, a + b, c)z + (a_1 b + b_1, d)}{(c_1 a + d_1, c)z + (c_1 b + d_1, d)}.$$ 

This has the same form as $T$ and therefore is a circular transformation.

But this is just what is known in modern mathematical language as the group property. Hence all circular transformations of the complex plane form a group.

If the coefficients $a, b, c, d$ in (1) be made to vary continuously, all the resulting transformations belong to the above group; and conversely all transformations belonging to the above group are obtained by continuously varying the coefficients in (1). Such a group is called a continuous group. The question of continuity will be more fully discussed later.

The circular transformation $T$ can usually be brought to the normal form*

Where \( m \) and \( n \) are the roots of the quadratic equation
\[
cz^2 + (d - a)z - b = 0,
\]
and
\[
k = \frac{(a + d - \sqrt{(a + d)^2 - 4(ad - be))}}{4(ad - be)}.
\]

\( m, n \) and \( k \) are called the essential parameters of the transformation, \( m \) and \( n \) are the two invariant points and \( k \) is the multiplier of the transformation. Since
\[
k = \frac{z_1 - m}{z_1 - n} \cdot \frac{z - m}{z - n}.
\]

we infer that \( k \) is the anharmonic ratio of the two invariant points and any pair of corresponding points in the transformation. Since \( m, n \) and \( k \) are complex quantities of the form \( a + ib \), it follows that \( T \) involves six real variable parameters.

When the invariant points \( m \) and \( n \) coincide, \( T \) can no longer be brought to the above normal form but is then reducible to a second normal form
\[
\frac{1}{z_1 - m'} = \frac{1}{z - m'} + a.
\]
The condition for coincident invariant points is \((a + d)^2 = 4(ad - be)\).
\[
m' = \frac{a - d}{2c} \quad \text{and} \quad a = \frac{2c}{a + d}.
\]

\( a \) is a constant whose properties are to be determined. When the condition for two coincident invariant points is substituted in the equation for \( k \), we get \( k = 1 \). Hence the characteristic anharmonic ratio of a transformation \( T' \) of this kind is unity. Every circular transformation can be brought to the one or the other of these normal forms.

Let us consider two transformations \( T \) and \( T' \) which have no invariant point in common. Their equations are
\[
\frac{z_1 - m}{z_1 - n} = k \frac{z - m}{z - n} \quad \text{and} \quad \frac{z_2 - m'}{z_2 - n'} = k' \frac{z - m'}{z - n'}.
\]
Eliminating \( z_1 \) we have the product \( T' \) in the form
\[
\frac{z_2 - m'}{z_2 - n'} = k' \frac{z - m'}{z - n'}.
\]
where \(k_2, m_2,\) and \(n_2\) are given as follows:

\[
\begin{align*}
\frac{k_2 + 1}{\sqrt{k_2}} &= \frac{(kk'_1 - k - k'_1 + 1)}{\sqrt{kk'_1}} \lambda + \frac{k + k'_1}{\sqrt{kk'_1}}; \\
m_2 &= \frac{k_2 D + A}{C(k_2 + 1)}; \quad n_2 = \frac{k_2 A + D}{C(k_2 + 1)}.
\end{align*}
\]

(5)

\(A, D, C\) and \(\lambda\) stand for the following expressions:

\[
\begin{align*}
A &= kk'_1(n - m_1) - km_1(n - n_1) - k'_1n_1(m - m_1) + m_1(m - n_1), \\
D &= kk'_1(m - m_1) - km_1(n - n_1) - k'_1n_1(m - m_1) + n(m - n_1), \\
C &= kk'_1(n - m_1) - k(n - n_1) - k'_1(m - m_1) + (m - n_1). \\
\lambda &= \frac{n_1 - m}{m_1 - n_1} : \frac{n - m}{m_1 - n_1}.
\end{align*}
\]

(6)

\(i.e., \lambda\) is one of the anharmonic ratios of the four invariant points \((mmn_1n_2)\).

The transformation \(T_2\) is not independent of the order of the components \(T_1\) and \(T\); the value of \(k_2\) is independent of the order of \(T_1\) and \(T\) for \(\lambda\) is unaltered when \(m\) and \(n\) are interchanged with \(m_1\) and \(n_1\) but not so with \(m_2\) and \(n_2\). Hence the two transformations \(T\) and \(T_1\) are non-commutative.

The results obtained may be formulated as follows:

**Theorem 1.** All circular transformations of the complex plane form a six-parameter continuous group. The transformations of the group are non-commutative.

Our task is now to enumerate and discuss all the subgroups of this six-parameter group, to develop their properties and to classify them according to their most characteristic properties.

Lie expounds in "Continuierliche Gruppen," page 113 an axiomatic principle which, for the purposes of this paper, is best stated in the following form: All point transformations of the plane which leave invariant a certain figure or configuration in the plane have the group property. The group may be either a continuous or a discontinuous group. A good example of the latter is the group of 18 linear transformations of the plane cubic into itself. We shall make frequent use of this principle in what follows; but in each case the group obtained will be seen to contain one or more continuously varying parameters and is therefore a continuous group.
According to Lie's principle all transformations leaving invariant a single point \( m \) form a group. Since there are \( \infty^2 \) points in the plane, the \( \infty^6 \) transformations of the six-parameter group \( G_6 \) are distributed into \( \infty^2 \) subgroups, one for each point. Accordingly each such subgroup should contain \( \infty^2 \) transformations and be a four-parameter group \( G_m \) or \( G_n \). This group leaves invariant not only the point \( m \) but also the net of circles through \( m \). The circles of the net are interchanged among themselves, but the net as a whole is unaltered.

If we make \( m_1 = m \) in (3'), (5), and (6), we find \( \lambda = 1 \), \( k_s = kk', \) and \( m_2 = m \). If \( m \) be a fixed point, the equations (3') or (3'') contain only two essential variable parameters \( k \) and \( n \) and, hence, four real variable parameters; thus, it is shown analytically that the group \( G_m \) is four-parameter. The fact that \( k_s = kk' \) is very important as will be seen later. The transformations of the group are non-commutative, for \( n \) is not independent of the order of \( T \) and \( T' \).

**Theorem 2.** All transformations leaving a point \( m \) invariant form a four-parameter group. The law of combination of the essential parameters \( k \) of this group is given by \( k^2 = kk' \). The transformations of the group are non-commutative.

Again by Lie's principle all transformations leaving invariant two distinct points \( m \) and \( n \) form a group. Since there are \( \infty^4 \) such pairs of points, the transformations of the six-parameter group \( G_6 \) are distributed into \( \infty^4 \) subgroups each of which contains \( \infty^2 \) transformations and is a two-parameter group. It is clear that each four-parameter group \( G_m \) contains \( \infty^2 \) of these two-parameter groups \( G_s \) or \( G_{an} \), one corresponding to each point of the plane taken with the fixed point \( m \). Such a two-parameter group leaves invariant not only the points \( m \) and \( n \) but also the pencil of circles through these points.

If we make \( m_1 = m \) and \( n_1 = n \) in (3'), (5) and (6), we get \( \lambda = 1 \), \( k_s = kk' \), \( m_2 = m \) and \( n_2 = n \). Or we may eliminate \( z_1 \) from (3') and get (3'') by multiplication and thus get directly that \( k_s = kk' \). \( T \) in this case has only one essential variable parameter \( k \) and, hence, only two real variable parameters. Thus again the group \( G_{mn} \) is shown to be two-parameter. Since \( k_2 \) is independent of the order of \( k \) and \( k_1 \), the transformations of this group are commutative.

**Theorem 3.** All transformations leaving invariant two points \( m \) and \( n \) form a two-parameter group. The law of combination of the essential parameters of this group is expressed by \( k_s = kk' \). The transformations of the group are commutative.

Thus far we have considered only transformations of the type \( T \) with two invariant points. The normal form of \( T' \)
(4) contains only two essential parameters \( m' \) and \( a \) and hence only four real variable parameters. There are therefore \( \infty^4 \) such transformations in the plane. Taken together do they form a group? This is readily answered in the negative. The product of two transformations \( T' \) and \( T_1' \)

\[(4') \quad \frac{1}{z - m'} = \frac{1}{z - m'} + a, \quad \text{and} \quad \frac{1}{z - m_1'} = \frac{1}{z - m_1'} + a_1,\]

is a transformation of the kind \( T \) with two distinct invariant points, as may easily be verified by eliminating \( z_1 \) from \((4')\).

But if the two transformations \( T \) and \( T_1' \) leave invariant the same point \( m' \), they do form a group. Making \( m_1' = m' \) in \((4')\) and eliminating \( z_1 \) by addition, we get

\[(4'') \quad \frac{1}{z - m'} = \frac{1}{z - m'} + a_2;\]

where \( a_2 = a + a_1 \). The group is evidently two-parameter and its transformations are commutative. Let it be symbolized by \( G_m' \) or \( G_{a_1}' \).

**Theorem 4.** All transformations of the kind \( T' \) leaving a single point invariant form a two-parameter group. The law of combination of the essential parameters is expressed by \( a_2 = a + a_1 \).
The transformations of the group are commutative.

The relationships of the groups thus far determined may be symbolized as follows:

\[ G_0 = \infty^2 G_2 = \infty^1 G_2 + \infty^2 G_1', \quad G_4 = \infty^2 G_2 + G_2' \]

or

\[ G_0 = \infty^2 G_2 = \infty^2 (\infty^2 G_2 + G_1'). \]

We now go on to examine more closely the two-parameter group \( G_m \) and shall show that the transformations composing it can be distributed into one-parameter subgroups. The essential parameter \( k \) of the group \( G_m \) may be written \( k = \rho e^{i\theta} \). Here \( \rho \) and \( \theta \) are independent parameters and may vary independently. If we put \( \rho = e^{i\theta} \), where \( c \) is some constant quantity, we have \( k = e^{i\theta} \cdot e^{i\theta} = e^{i(c+c)\theta} \). Since in the group \( G_m \) \( k_2 = k k_1 \), we have \( k_2 = e^{i(c+c)\theta} \cdot e^{i(c+c)\theta} = e^{i(c+c)\theta} \cdot e^{i(c+c)\theta} \), where \( c \) and \( \theta \) are two independent parameters. But if \( c = c_1 \), we have \( k_1 = e^{i(c+c_1)\theta} \cdot e^{i(c+c_1)\theta} = e^{i(c+c_1)\theta} \), where \( \theta_1 = \theta + \theta_1 \).

Here we have the conditions for a one-parameter group; \( k_1 \) is of the same character as \( k \) and \( k_1 \), and there is but one parameter, viz : \( \theta \). The effect of the successive transformations of the group upon a point \( P \) of the plane is to transform it.
into $P_1, P_2, \ldots P_n$ which lie on a curve called by Lie the Bahn-
curve or path curve. By a transformation $T$ every point $P$ is
moved along its path curve. The path curves of this group
are the well-known double spirals of Holzmüller* used so ex-
tensively by Klein and others. Their properties are so well
known that it is unnecessary to develop them here. For a
lucid account in English see a paper by Professor F. N.

Within the group $G_{mn}$ there is a sub-fold group for every
real value of $c$ and a corresponding set of double spiral path
curves.

**Theorem 5.** The transformations of the group $G_{mn}$ are distri-
buted into $\infty$ one-parameter subgroups. The path curves of these
subgroups are double spirals about the invariant points $m$ and $n$.

The chief properties of one of these one-parameter sub-
groups are easily deduced from the expression for the
parameter, $k = e^{(c+i)\vartheta}$. When $\vartheta = 0$, $k = 1$; when the anhar-
monic ratio of the four points $(mnzz^*) = 1$, $z$ coincides with $z^*$.
Every point of the plane is unaltered by such a trans-
formation, which is called the identical transformation of
the group. The two transformations corresponding to
values of $\vartheta$ numerically equal but of opposite signs are called
inverse transformations. Their product is the identical
transformation of the group. When $k = \infty$ or $0$, all points
of the plane are transformed respectively to the invariant
points $m$ or $n$. I have elsewhere called these pseudo-trans-
formations† (*ausgeartete Transformationen, Lie*).

Within the two-parameter group $G_{mn}$ are two one-parame-
ter subgroups of special importance; these are the groups
for which $c = 0$ and $c = \infty$ respectively; i.e., for which
$|k| = 1$ and for which $k$ is real. In the first case, for which
$k = e^{i\vartheta}$, the path curves reduce to coaxial circles having $m$
and $n$ for vanishing points. All transformations of this
one-parameter group are elliptic. In the second case, when
$k$ is real, the path curves reduce to a pencil of circles through
$m$ and $n$. The transformations of this group are all hyper-
bolic. The other one-parameter subgroups of $G_{mn}$ are made
up chiefly of loxodromic transformations.

**Theorem 6.** Every two-parameter group $G_{mn}$ contains one one-
parameter subgroup of elliptic transformations, one one-parameter
subgroup of hyperbolic transformations, and $\infty$ one-parameter
subgroups of loxodromic transformations. For each group of
loxodromic transformations $c$ in the formula $k = e^{(c+i)\vartheta}$ is a constant.

The continuity of the two-parameter group $G_{mn}$ is based

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† *Kansas University Quarterly*, vol. 5, page 79.
upon the continuity of the complex number system; for there is a transformation of the group corresponding to every value of $k$, which is a complex number. Let the values of $k$ be represented as usual by the points of a complex plane (not to be confused with the plane of our operations). We wish to see how the values of $k$ which give transformations belonging to a one-parameter subgroup are distributed in the plane. We have $k = e^{(c+i\theta)}$ where $c$ is a constant and $\theta$ is variable. The locus of the point $k$ satisfying this equation is a logarithmic spiral about the zero point cutting the axis of real numbers at an angle whose cotangent is $c$. This is a continuous curve from the zero point to the infinity point, and consequently our one-parameter subgroup is a continuous group.

Different values of $c$ give us different spirals each of which corresponds to a one-parameter subgroup of $G_{mn}$. $c$ varies continuously through all real values from $-\infty$ to $+\infty$ so that these spirals lie infinitely close to one another and, as we shall see, cover twice over the entire plane. These spirals all pass through the unit point. For $c = 0$ the corresponding spiral becomes the unit circle; for $c = \infty$ the spiral reduces to the straight line which is the axis of real numbers. The family of spirals for which $c$ is positive fills the entire plane and no two of them intersect except in the unit point. The same is true of the family of spirals for which $c$ is negative. But every spiral of one family intersects an infinite number of times every spiral of the other family. Every point in the plane not on the unit circle or the axis of real numbers lies on two of these spirals; from which we infer that every loxodromic transformation of the group $G_{mn}$ belongs to two distinct one-parameter subgroups. Every hyperbolic transformation in $G_{mn}$ except the involutoric transformation, for which $k = -1$, belongs to three one-parameter subgroups; for two spirals and the axis of reals pass through every point for which $k$ is real. The elliptic transformations in $G_{mn}$ belong only to the elliptic subgroup. The involutoric transformation is common to the hyperbolic and elliptic subgroups. The identical transformation is common to all subgroups; and the two pseudo-transformations are common to all except the elliptic subgroup. Two loxodromic subgroups for which the $c$'s have the same signs have no transformations in common other than the identical and the two pseudo-transformations; while two loxodromic subgroups for which the $c$'s have opposite signs have in common an infinite number of discrete transformations.
Theorem 7. Every one-parameter subgroup in $G_{mn}$ is continuous. Every loxodromic transformation in $G_{mn}$ belongs to two distinct subgroups. Every hyperbolic transformation in $G_{mn}$, except the involutoric transformation, belongs to three distinct subgroups.

This same geometric representation enables us to discuss intuitively the generation of finite transformations by the repetition of an infinitesimal transformation. Every spiral passes through the unit point, and corresponding to the two points on the spiral adjacent to the unit point we have two infinitesimal transformations belonging to a one-parameter group. These are given by $k = e^{(c+1)b}$ and $k = e^{(c+b)}$. The identical transformation divides the one-parameter group into two portions, each of which contains an infinitesimal transformation. Every finite transformation in each portion of a one-parameter loxodromic group can be generated by the repetition of the corresponding infinitesimal transformation. In the elliptic group, for which the spiral reduces to a circle, every transformation can be generated from either elliptic infinitesimal transformation. In the hyperbolic group, for which the spiral reduces to the axis of real numbers, the transformations for which $k$ is negative can not be generated by the repetition of either of the hyperbolic infinitesimal transformations. Every loxodromic transformation in $G_{mn}$ can be generated from either of two distinct infinitesimal transformations, for every loxodromic transformation belongs to two distinct subgroups. Every hyperbolic transformation for which $k$ is positive can be generated from three infinitesimal transformations; while every hyperbolic transformation for which $k$ is negative, except the involutoric transformation, can be generated from two distinct loxodromic transformations.

Theorem 8. Every hyperbolic transformation in $G_{mn}$ for which $k$ is positive can be generated from three distinct infinitesimal transformations; every other transformation in $G_{mn}$ can be generated from two distinct infinitesimal transformations.

The two-parameter group $G_{m'}$ likewise contains $\infty^3$ one-parameter subgroups. The law of combination of the parameters in this group is $a_2 = a + a_1$, or in another form $r_2 e^{i\theta} = r e^{i\theta} + r_1 e^{i\theta}$. If now we take $\theta_1 = \theta$, this becomes $r_2 e^{i\theta} = (r + r_1) e^{i\theta}$. We have here the conditions for a one-parameter group; $a_2$ is of the same form as $a$ and $a_1$ and contains only one parameter $r$. It is clear that we have a one-parameter group for every value of $\theta$. The effect of successive applications of transformations of one of these one-parameter groups on a point in the plane is to move it
along a path curve. The path curves of one of these groups consists of the system of circles tangent at \( m \) to each other and to the line through \( m \) which makes with the axis of reals an angle \( \theta \). All transformations of the group \( G_m' \) are parabolic. For details see the above mentioned paper by Professor Cole.

The properties of one of these one-parameter groups are easily determined. Let \( a = re^{i\theta} \); when \( r = 0 \), we have the identical transformation of the group; the two transformations corresponding to two values of \( r \) equal but with opposite signs are inverse transformations. When \( r = \infty \), all points of the plane are transformed to \( m \) and we have a pseudo-transformation. There are two infinitesimal transformations in the group given by the values \( +dr \) and \(-dr \). Each infinitesimal transformation generates its corresponding portion of the group. Two one-parameter subgroups of \( G_m' \) have no transformation in common except the identical and the pseudo-transformation; these are common to all subgroups of \( G_m' \).

**Theorem 9.** All transformations of the two-parameter group \( G_m' \) are parabolic and are distributed into \( \infty^1 \) one-parameter subgroups. The path curves of a one-parameter subgroup are circles through \( m \), touching each other at \( m \).

We have already shown how the four-parameter group \( G_m \) breaks up into \( \infty^2 \) two-parameter groups \( G_{mn} \). We shall now show that the transformations of \( G_m \) may be distributed into \( \infty^1 \) three-parameter subgroups. The law of combination of the parameters \( k \) within the group \( G_m \) is expressed (Theorem 2) by \( kk' = k_r \). Written in another form this is \( e^{(c+i)d} \cdot e^{(c'+i)d} = e^{(c+\theta_1) + (c'+\theta_1)} \). If \( c = c_1 \), we have

\[
e^{(c+i)d} \cdot e^{(c+i)d} = e^{(c+\theta_2)} \text{ where } \theta_2 = \theta + \theta_1.
\]

Hence, we see that if we chose from each two-parameter group \( G_{mn} \) the one-parameter group characterized by a certain value of \( c \), the totality of the transformations comprised in these \( \infty^3 \) one-parameter groups forms a three-parameter group. It is clear at once that there is one such three-parameter group for every value of \( c \). Thus, for example, all the elliptic transformations contained in \( G_m \) form a three-parameter subgroup. The same is true of all hyperbolic transformations.

**Theorem 10.** The \( \infty^3 \) transformations having a common invariant point at \( m \), and for which the value of \( c \) in the formula \( k = e^{(c+i)d} \) is the same, form a three-parameter subgroup of the four-parameter group \( G_m \).
A very important special case of $G_n$ remains to be noted, viz: when the invariant point $n$ is at infinity. All transformations of the group leave invariant the net of circles through the point at infinity. But this net of circles is the net of all straight lines in the plane. Thus the transformations of this group transform straight lines into straight lines; they are therefore projective transformations. These transformations retain the common property of all circular transformations that angular magnitudes are unchanged. The four-parameter group $G_n$ is therefore identical with the projective group of similitude, whose invariant figure is the line at infinity and the two circular points.

This result can also be shown analytically. Let $n = \infty$ in equation (3), whence we have

$$z_1 - m = k(z - m).$$

Equating real and imaginary parts we get

$$x_1 = k'x - k''y - k'm' + k''m' + m',
    y_1 = k''x + k'y - k'm'' - k'''m' + m''.$$

This is a projective transformation the vertices of whose invariant triangle are the point $(m', m'')$ and the two circular points.

The subgroups of $G_n$ give some interesting results. The path-curves of a one-parameter subgroup of loxodromic transformations are logarithmic spirals* around the point $m$, and the constant of the group $c$ in $k = e^{c+i\phi}$ is the cotangent of the angle between the curve and the radius vector. The path-curves of the one-parameter group of elliptic transformations are concentric circles about $m$; and the path-curves of a one-parameter group of hyperbolic transformations are straight lines through $m$.

Within $G_n$ all loxodromic transformations with constant $c$ form a three-parameter subgroup of logarithmic spiral motions with constant angle $\phi$. All elliptic transformations in $G_n$ form the three-parameter group of all rotations in the plane. All parabolic transformations in $G_n$ form the two-parameter group of all translations in the plane. Together all elliptic and all parabolic transformations in $G_n$ form the three-parameter group of all Euclidian motions in the plane. All hyperbolic transformations in $G_n$ form the three-parameter group of all affine transformations (i.e., dilations) of the plane.

Theorem 11. All circular transformations leaving the point at infinity invariant are projective transformations, and the four-parameter group $G_x$ is identical with the four-parameter projective group in the plane whose invariant figure is the line at infinity and the two circular points.

It seems to be a favorite method with Klein to express whenever possible projective groups in terms of the complex variable both in the plane and on the Neumann sphere; see for example Nicht-Euclidische Geometrie, vol. 2, page 184 ff and Höhere Geometrie, vol. 2, page 229 ff, and many other places.

We come now to the consideration of another group type of great importance. According to Lie's principle all transformations leaving a circle invariant form a group. Consider first a group of hyperbolic transformations leaving invariant m and n and every circle of the pencil through m and n. Choose one of these circles C and another point $n_1$ on C. The group of hyperbolic transformations with invariant points at m and $n_1$ also leaves C invariant. Thus all hyperbolic transformations having one invariant point at m and the other also on C leave C invariant and form a two-parameter group. In this two-parameter group is included the one-parameter parabolic group whose invariant point is m and whose invariant line is the tangent to C at m. In like manner there is a two-parameter group for every point on C. All the transformations contained in these $\infty^1$ two-parameter groups form a three-parameter group leaving C but no point on C invariant. $\infty^2$ of these transformations are parabolic; these are distributed into $\infty^1$ one-parameter groups, but taken together do not form a two-parameter group.

There are also $\infty^5$ elliptic transformations which leave C invariant. Let m be any point within C and n its inverse point with respect to the circle C. The one-parameter group of elliptic transformations having its invariant points at m and n has C among its pencil of invariant circles. In like manner all one-parameter groups of elliptic transformations whose invariant points are a pair of inverse points with respect to C leave C invariant. There are $\infty^5$ such pairs of points, and hence there are $\infty^5$ elliptic transformations in the group leaving C invariant.

Theorem 12. There are $\infty^5$ circular transformations which leave invariant any given circle; these form a three-parameter group. This group is composed of all hyperbolic transformations whose invariant points are on the circle, of all elliptic transformations whose invariant points are a pair of inverse points with re-
spect to the circle, and of all parabolic transformations whose invariant point is on the circle and whose invariant line is a tangent to the circle at the invariant point.

The transformations of this group $G_c$ are distributed into subgroups as follows: The elliptic transformations are distributed into $\infty^2$ one-parameter subgroups, but not into two-parameter subgroups. The hyperbolic transformations are distributed into $\infty^1$ two-parameter subgroups; each of these two-parameter groups breaks up into $\infty^1$ one-parameter subgroups, one of which is parabolic.

Since a straight line is considered as a circle through the point at infinity, it follows at once that there is a three-parameter group of transformations leaving a straight line invariant. This group $G_L$ is in all respects similar to the group $G_c$.

A very important special case of the group $G_L$ is when the line $L$ is the axis of real quantities.* $G_L$ then becomes the group of real projective transformations of the points on a real line. The properties of the real projective group are at once known from the properties of $G_c$.

**Theorem 13.** The three-parameter group of real projective transformations of the points on a line is a special subgroup of the six-parameter group of circular transformations of the points of the complex plane.

There is still another type of three-parameter group consisting entirely of elliptic transformations which is closely related to the group $G_c$. This is the group of transformations of the complex plane which corresponds to the three-parameter group of rotations of a sphere about its centre. Every rotation of a sphere when projected stereographically upon the equatorial plane produces an elliptic transformation in that plane. Klein discusses the group of rotations of the sphere on pages 32–36 of his Ikosaeder, and on page 35 gives an analytic proof of the group property. The relation of this group in the plane to the group $G_c$ is shown as follows:

The invariant points of a one-parameter elliptic subgroup of $G_c$, being inverse points with respect to $C$, form a pair of corresponding points in a hyperbolic involution on the line joining the two points with the centre $O$ of the circle. The double points of the involution are the two points where the line cuts the circle. Every line through the centre of the circle $C$ is the bearer of such an involution; and all the

*Poincaré has investigated many of the properties of these groups $G_c$ and $G_L$ in his papers in Acta Mathematica, vols. 1 and 3, "Theorie des groupes fuchsiens" and "Theorie des groupes kleiniens."
one-parameter groups of elliptic transformations whose invariant points are a pair of corresponding points in one of these involutions belong to the three-parameter group $G_0$.

Let us consider a similar system of elliptic involutions on all lines of a pencil through $O$, such that the product of the distances from the centre of a pair of corresponding points is the same in all the involutions. Thus $OP \cdot OP' = -k^2$, constant for all the involutions. When a sphere is projected stereographically upon the equatorial plane, every pair of opposite points on the sphere project into a pair of corresponding points in one of these involutions. Thus all one-parameter groups of elliptic transformations whose invariant points are a pair of corresponding points in one of these involutions form a three-parameter group.

This three-parameter group leaves no figure of the plane invariant; but if it were allowable to use the language of projective geometry in speaking of the complex plane, we should say that this group leaves invariant an imaginary circle with centre at $O$ and radius equal to $ki$. We shall, therefore, designate this group as $G_0$.

This completes the discussion of the subgroups of the general circular group. It remains to be shown that there are no other types of subgroups besides those discussed above. I shall attempt no formal proof, but shall only bring forward some general considerations bearing upon the question.

A circular transformation transforms points into points and circles into circles. We have considered all possible groups which leave invariant one or two points; a transformation leaving invariant more than two points is identical. We have also considered all possible groups of transformations leaving a circle invariant. If there be a continuous group characterized by the invariance of some curve other than a circle, such a curve must be the path curve of a one-parameter group. The only other path curve besides the circle is the double spiral of Holzmüller. This has two singular points and is invariant only under those transformations whose invariant points are these two singular points; hence, there is only one one-parameter group leaving invariant such a double spiral. These considerations indicate that there are no other subgroups of the general circular group.

Lie's theory of continuous groups based upon the infinitesimal transformation is better adapted than the method of this paper for determining the complete list of types of subgroups of a given group. It may be likened to a net which
gathered in its meshes all types of subgroups and lets none escape. My list of groups should be verified or corrected by the application of Lie's methods.

I append here a list of the group types discussed in the foregoing pages with a brief characterization of each.

1. The six-parameter group $G_6$ of all circular transformations.
2. The four-parameter group $G_4$ leaving a single point invariant.
3. The two-parameter group $G_2$ of type $T$ leaving a pair of points invariant.
4. The two-parameter group $G_2'$ of type $T'$ leaving a single point invariant.
5. The one-parameter group $G_1c$ of type $T$ and constant $c$ in $k = e^{(c+i)d}$ leaving two points invariant. (a) The one-parameter group of elliptic transformations for which $c = 0$. (b) The one-parameter group of hyperbolic transformations for which $c = \infty$ and $\theta = 0$.
6. The one-parameter parabolic group $G_1'$ with constant $\theta$ in $a = re^{i\theta}$ leaving a single point invariant.
7. The three-parameter group $G_3$ of type $T$ and constant $c$ in $k = e^{(c+i)d}$ leaving a single point invariant.
8. The three-parameter group $G_3$ of elliptic, hyperbolic and parabolic transformations leaving a circle invariant.
9. The two-parameter group $G_2c$ of hyperbolic transformations leaving invariant a circle and a point on it.
10. The three-parameter group of elliptic transformations $G_3c$.

The real projective transformations of the plane and of space may be treated in the same spirit and by the same methods here employed for the circular transformations. The writer hopes to be able in the near future to publish the full results of his investigations in these fields.

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PLÜCKER'S COLLECTED PAPERS.


The Kgl. Gesellschaft der Wissenschaften zu Göttingen, of which Plücker was a corresponding member, recently under-