PERIODIC DEVELOPMENTS.

\[ F(z) = \sum_{n=0}^{\infty} a^n z^n, \quad (|a| < 1), \]

is single-valued, provided \(|a|\) is not too large.

The proof is as follows. Evidently

\[ \left| \frac{f(z) - f(z')}{z - z'} \right| = 1 + \sum_{n=1}^{\infty} \frac{z^{n+1} + z^n z' + \cdots + z'^{n+1}}{(a^n + 1) (a^n + 2)} \]

\[ \geq 1 - \sum_{n=1}^{\infty} \frac{|z|^{n+1} + |z|^{n} |z'| + \cdots + |z'|^{n+1}}{(a^n + 1) (a^n + 2)} \]

\[ \geq 1 - \sum_{n=1}^{\infty} \frac{1}{a^n + 1} = 1 - \frac{1}{a + 1} - \left( \frac{1}{a^2 + 1} + \frac{1}{a^3 + 1} + \cdots \right) > 0. \]

Hence \(|f(z) - f(z')| > 0\), q. e. d.

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If we put with Professor S. Newcomb*

(1) \[ E = ev_1 + e^2 v_2 + e^3 v_3 + \cdots \]

(2) \[ \rho - \log a = e\rho_1 + e^2 \rho_2 + e^3 \rho_3 + \cdots \]

where \(E\) stands for the equation of the center and \(\rho = \log r\), then \(v_i\) and \(\rho_i\) will be of the form

(3) \[ iv_i = \frac{1}{2} \sum k_j^{(i)} \sin jz, \]

(4) \[ i\rho_i = \frac{1}{2} \sum h_j^{(i)} \cos jz, \]

\((j = i, \ i - 2, \ i - 4, \ \cdots, \ -i)\).

the coefficients \( k_j^{(i)} \) and \( h_j^{(i)} \) being rational numerical fractions subject to the conditions

\[
k_j^{(i)} = - k_{-j}^{(i)}; \quad h_j^{(i)} = h_{-j}^{(i)}.
\]

We propose to give in this note formulas by which these coefficients can be computed for any value of \( i \) and \( j \).

If we put

\[
E = \sum_{i=1}^{\infty} H_i \sin i \zeta,
\]

\[
\rho = \log a = \frac{1}{2} A_0 + \sum_{i=1}^{\infty} A_i \cos i \zeta,
\]

then the comparison with formulas (1)-(4) gives

\[
H_i = \sum_{m=0}^{\infty} \frac{1}{i + 2m} \exp^{i + 2m},
\]

\[
A_i = \sum_{m=0}^{\infty} \frac{1}{i + 2m} \exp^{i + 2m}.
\]

On the other hand it can be shown* that

\[
H_i = \frac{2\sqrt{1 - e^2}}{i} \sum_{j} \sum_{q} \frac{i^q}{q!} \left( \frac{e}{2} \right)^{j+q} N_{-i-j, q}
\]

where \( j \) and \( q \) assume all integral positive values (zero included) such that

\[
j + q = i, \quad i + 2, \quad i + 4, \ldots
\]

If we develop \( \sqrt{1 - e^2} \) and put

\[
H_i^{(2m)} = \sum_{j} \sum_{q} \frac{i^q}{q!} N_{-i-j, q} \quad (-i + j + q = 2m),
\]

then formula (9) becomes

\[
H_i = \frac{2}{i} \left( \frac{e}{2} \right)^{i} \sum_{m=0}^{\infty} \left( \frac{e}{2} \right)^{2m} \left[ H_i^{(2m)} - 2H_i^{(2m-2)} - \ldots \right.
\]

\[
- \frac{1 \cdot 3 \cdots (2m-3) \cdot 2m}{m!} H_i^{2m} \right].
\]

Comparing this formula with (7) we conclude that

(12) \[ k_i^{(i+2m)} = \left( \frac{i + 2m}{i} \right) \cdot \frac{1}{2i+2m-1} \left[ H_i^{(2m)} - 2H_i^{(2m-2)} \right. \\
- \frac{1}{2!} 2^2 H_i^{(2m-4)} - \cdots - \frac{1 \cdots (2m-3)}{m!} 2^m H_i^0 \right]. \]

By this formula the computation of the coefficients \( k_i^{(i)} \) is reduced to the computation of Cauchy's numbers for which the author has given a general formula.*

In order to obtain a similar expression for the coefficients \( h_j^{(i)} \) we must first derive a development in powers of the eccentricity for the coefficients \( A_i \). To this end we remark that

\[
\frac{d \rho}{de} = \frac{d \log r}{de} = \frac{dr}{r de} = 1 - e^2 \left( \frac{a}{r} \right) - \frac{1 - e^2}{e} \left( \frac{a}{r} \right)^2.
\]

On the other hand we have \( \dfrac{a}{r} = 1 + \sum_{i=1}^{\infty} J_i(ie) \cos i \xi \)

\[
\left( \frac{a}{r} \right)^2 = \frac{1}{\sqrt{1 - e^2}} + \sum_{i=1}^{\infty} G_i^{(i)} \cos i \xi
\]

where \( J_i(ie) \) is a Bessel's function and

(13) \[ G_i^{(j)} = 2 \sum_j \frac{i+j}{j!} \frac{e^j}{2} \left[ 2J_j(ie) - (1 - e^2) G_j^{(j)} \right] \cos i \xi. \]

Hence we may write that

\[
\frac{d \rho}{de} = \frac{1 - \sqrt{1 - e^2}}{e} + \frac{1}{e} \sum_{i=1}^{\infty} \left[ 2J_i(ie) - (1 - e^2) G_i^{(i)} \right] \cos i \xi.
\]

Now, it follows from (6) and (8) that

\[
e \frac{d \rho}{de} = \frac{1}{2} e \frac{d A_i}{de} + \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} e^{i+2m} h_i^{(i+2m)} \cos i \xi
\]

which, compared with the preceding formula, shows that

\[
2J_i(ie) - (1 - e^2) G_i^{(i)} = \sum_{m=0}^{\infty} e^{i+2m} h_i^{(i+2m)} \cos i \xi
\]

and we only need to find the coefficient of \( e^{i+2m} \) in the left hand side to obtain the required expression for \( h_i^{(i+2m)} \).

From (9) and (13) follows that

\[ (1 - e^\theta) G_i^{(2)} = i \sqrt{1 - e^\theta} H_i \]

\[ = 2 \left( \frac{e}{2} \right)^t (1 - e^\theta) \sum_{m=0}^{m=\infty} H_i^{(2m)} \left( \frac{e}{2} \right)^{2m} \]

so that the coefficient of \( e^{i+2m} \) in \( 1 - e^\theta \) \( G_i^{(2)} \) is found to be

\[ \frac{1}{2^{i+2m-1}} \left[ H_i^{(2m)} - 4H_i^{(2m-2)} \right] \]

while the coefficient of the same power of \( e \) in \( 2J_i(\theta e) \) is

\[ (-1)^m \frac{1}{2^{i+2m-1}} \frac{i^{i+2m}}{m! (i + m)!} \cdot \]

Hence, we conclude that

\[ h_i^{(i+2m)} = \frac{1}{2^{i+2m-1}} \left[ 4H_i^{(2m-1)} - H_i^{(2m)} + \frac{(-1)^m e^{i+2m}}{m! (i + m)!} \right] \]

which is the desired expression for the coefficients \( h_i^{(i)} \).

To conclude we will express the coefficients \( h_i^{(i)} \) by means of the \( k_i^{(i)} \). To this end we multiply formula (7) by \( \sqrt{1 - e^\theta} \) and develop the right hand side in powers of \( e \). Thus we obtain

\[ \sqrt{1 - e^\theta} H_i = \sum_{m=0}^{m=\infty} e^{i+2m} \left[ \frac{h_i^{(i+2m)}}{i + 2m} - \left( \frac{1}{2} \right) \frac{h_i^{(i+2m-2)}}{i + 2m - 2} - \cdots \right] \]

\[ - \frac{1.3 \cdots (2m - 3)}{m!} \left( \frac{1}{2} \right)^m \frac{h_i^i}{i} \]

d, therefore,

\[ h_i^{(i+2m)} = \frac{2(-1)^m \left( \frac{i}{2} \right)^{i+2m}}{m! (i + m)!} - \frac{ik_i^{(i+2m)}}{i + 2m} + \frac{1}{2} \frac{ik_i^{(i+2m-2)}}{i + 2m - 2} \]

\[ + \frac{1}{2} \left( \frac{1}{2} \right)^i \frac{ik_i^{(i+2m-4)}}{i + 2m - 4} \cdots + \frac{1.3 \cdots (2m - 3)}{m!} \left( \frac{1}{2} \right)^m \frac{ik_i^i}{i} \]

which formula enables us to compute the values of the \( k_i^{(i)} \) directly from the \( k_i^{(i)} \).

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