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THE LARGEST LINEAR HOMOGENEOUS GROUP WITH AN INVARIANT PFAFFIAN.

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1. In the December number of the Bulletin (pp. 120–135) I have shown that the second compound of the general 2m-ary linear homogeneous group is a linear group in $C_{2m,2} = m(2m - 1)$ variables which leaves invariant the Pfaffian

$$F = [1, 2, \ldots, 2m].$$

Denoting the variables as follows:

$$(1) \quad Y_{ij} \equiv - Y_{ji} \quad (i, j = 1, \ldots, 2m \; ; \; i \not= j),$$

the second compound was proved to contain exactly $(2m)^2$ linearly independent infinitesimal transformations

$$(2) \quad \sum_{r+s,t} Y_{rt} \frac{\partial f}{\partial Y_{rs}} \partial t. \quad (t, s = 1, \ldots, 2m).$$

The object of the present note is to prove that the largest linear homogeneous group $G$ in the $m(2m - 1)$ variables (1) which leaves invariant the Pfaffian $F$ contains only the $(2m)^2$ linearly independent transformations (2).

2. Let the general infinitesimal transformation of the group $G$ be as follows:

$$(3) \quad \delta Y_{ij} = \sum_{k=1}^{2m} a_{ik}^j Y_{kt} \partial t \quad (i, j = 1, \ldots, 2m \; ; \; i \not= j),$$

where, on account of (1), we may suppose

$$(4) \quad a_{ik}^j = - a_{ik}^j = + a_{ik}.\quad$$

The condition that (3) shall multiply $F$ by a constant $c \partial t$ is as follows:

$$(5) \quad \sum_{i,j,k} \frac{\partial F}{\partial Y_{ik}} a_{ij}^k Y_{kt} = c F.$$ 

Now
\[ \frac{\partial F}{\partial Y_q} = \frac{\partial}{\partial Y_q} \left\{ (-1)^{i+j-1} \prod_{i,j=1}^{2m} i \right\} \]

\[ = (-1)^{i+j-1} \prod_{i,j=1}^{2m} i \]

Comparing the coefficients of the terms in (5) of the type

\[ (-1)^{\sigma} Y_{i_1} Y_{i_2} \cdots Y_{i_{2m-1}} \frac{1}{i_{2m}} \]

where \( i_1, i_2, \ldots, i_{2m} \) is a permutation of \( 1, 2, \ldots, 2m \) and where \( \sigma \) denotes the number of transpositions giving that permutation, we obtain the conditions

\[ (6) \quad a_{i_1}^{i_2} + a_{i_2}^{i_3} + \cdots + a_{i_{2m-1}}^{i_{2m}} = \sigma. \]

Comparing the coefficients of the terms,

\[ Y_{i_1}^{\sigma} Y_{i_2}^{\sigma} \cdots Y_{i_{2m}} \]

we obtain the conditions

\[ (7) \quad a_{i_1}^{i_2} = 0 \quad (i_1, i_2, \ldots, i_{2m} \text{ is any four different integers } \leq 2m). \]

Comparing the coefficients of the terms

\[ Y_{i_1} Y_{i_2} \cdots Y_{i_{2m-1}} \frac{1}{i_{2m}} \]

we find

\[ (8) \quad a_{i_1}^{i_2} - a_{i_2}^{i_1} = 0 \]

\((i_1, i_2, i_3, i_4, \ldots, i_m, \text{ being any four different integers } \leq 2m).\)

We may now obtain a complete set of linearly independent infinitesimal transformations (3), which leave \( F \) invariant. According as every \( a_{i_1}^{i_2} = 0 \), or not every such \( a \) is zero, we obtain two independent types of transformations (3), which together form the desired complete set. We consider the two types in succession:

(a) If any \( a_{i_1}^{i_2} = 0 \), say = 1, where \( r, s, t \) are distinct integers \( \leq 2m \), then by (8) we have

\[ a_{r}^{s} = 1 \quad (r = 1, \ldots, 2m ; \ r \neq s, t). \]

Setting every other \( a = 0 \), we obtain a set of solutions of (6), (7), (8), for which

\[ \left\{ \begin{array}{l}
\delta Y_r = Y_r \delta t \\
\delta Y_i = 0
\end{array} \right. \quad (r = 1, \ldots, 2m ; \ r \neq s, t)
\]

\[ \left\{ \begin{array}{l}
\delta Y_i = 0 \\
\delta Y_j = Y_j \delta t
\end{array} \right. \quad (i, j = 1, \ldots, 2m ; i \neq j). \]
We thus obtain the \(2m(2m - 1)\) infinitesimal transformations (included in the formula (2))

\[
\sum_{r,s,t}^{r=1,\ldots,2m} Y_{rs} \frac{\partial f}{\partial Y_{st}} \delta t \quad \left( s, t = 1, \ldots, 2m \right),
\]

which are therefore linearly independent.

(b) If next
\[a_{rs} = 0 \quad (r, s, t = 1, \ldots, 2m; r \neq s, \neq t),\]
the general transformation (3) becomes

\[
\delta y_i = a_{ij} y_j \delta t \quad (i, j = 1, \ldots, 2m),
\]

where the \(a_{ij}\) are subject to the conditions (6).

Writing for brevity [see (4)],

\[a_{ij} \equiv (ij) = (ji),\]

these conditions (6) become

\[
(i_1i_2) + (i_3i_4) + \cdots + (i_{2m-1}i_{2m}) = c.
\]

We obtain at once the following \(2m\) sets of solutions of these equations, each set being given by one value of \(l\) chosen from 1, 2, \ldots, 2m;

\[
\begin{cases}
(l1) = (l2) = \cdots = (l(l-1)) = (l(l+1)) = \cdots = (l2m) = c \\
(ij) = 0 \\
[i, j = 1, \ldots, 2m; i \neq j].
\end{cases}
\]

These sets of solutions of the equations (6), (7), (8) give rise to the following \(2m\) infinitesimal transformations:

\[
A_u^{j=1,\ldots,2m} Y_{ij} \frac{\partial f}{\partial Y_{ij}} \quad (l = 1, \ldots, 2m).
\]

These transformations are linearly independent if \(m \geq 2\). Indeed, if

\[
\sum_{i=1}^{2m} k_i A_u^{j=1,\ldots,2m} = 0,
\]

upon equating the coefficients of \(\frac{\partial f}{\partial Y_{rs}}\) in the two members, we have

\[k_r + k_s = 0 \quad (r, s = 1, \ldots, 2m; r \neq s).\]

Hence, if \(m \geq 2\), \(k_i = 0 \quad (l = 1, \ldots, 2m)\).
The transformations (9) and (11) make up the \((2m)^3\) linearly independent transformations (2). It follows from the theorem of the next paragraph that there do not exist more than \(2m\) linearly independent transformations of the type (b). We will then have proved the following theorem:

The largest linear homogeneous group in \(C_{2m, 2}\) variables leaving invariant the Pfaffian \([1, 2, \ldots, 2m]\) is identical with the second compound of the general \(m\)-ary linear homogeneous group.

3. Theorem. The \(m(2m - 1)\) quantities

\[
(ij) \equiv (ji) \quad [i, j = 1, 2, \ldots, 2m, i \neq j]
\]

satisfying the \(1\cdot 3\cdot 5 \cdots (2m - 3)(2m - 1)\) equations

\[
\begin{bmatrix}
E_{2m} \\
Q_{2m}
\end{bmatrix}
\begin{cases}
(i_1i_2) + (i_3i_4) + \cdots + (i_{2m-1}i_{2m}) = c_{2m} \\
(1, 2), (3, 4), (5, 6), \ldots, (2l - 1, 2l), \ldots, (2m - 1, 2m)
\end{cases}
\]

can all be expressed in terms of certain \(2m\) of the \((ij)\), for example,

\[
\begin{bmatrix}
Q_{2m}
\end{bmatrix}
\begin{cases}
(1, 2), (3, 4), (5, 6), \ldots, (2l - 1, 2l), \ldots, (2m - 1, 2m)
\end{cases}
\]

but not in terms of fewer than \(2m\) of them if \(m > 1\).

The last part of the theorem follows from the linear independence of the \(2m\) infinitesimal transformation of type (b) above.

The first part of the theorem will be proved by induction. For \(m = 2\), it is evident; for the equations \([E]\) are as follows:

\[
\]

Supposing the first part of the theorem to be true for a given value of \(2m\), we can prove it true for the next value \(2(m + 1)\). Indeed, applying this hypothesis to certain equations of the set \([E_{2m+2}]\), viz.:

\[
(i_1i_2) + (i_3i_4) + \cdots + (i_{2m-1}i_{2m}) = c_{2m+2} - (2m + 1, 2m + 2),
\]

where \(i_1, i_2, \ldots, i_{2m}\) is a permutation of \(1, 2, \ldots, 2m\), it follows that the quantities

\[
(ij) \quad [i, j = 1, 2, \ldots, 2m; i \neq j]
\]

can be expressed in terms of the quantities \(Q_{2m}\) and that \(c_{2m+2}\) is expressible in terms of the quantities \(Q_{2m}\) together with \((2m + 1, 2m + 2)\).

Consider next the equations of the set \([E_{2m+2}]\)

\[
(i_1i_2) + (i_3i_4) + \cdots + (i_{2m-3}i_{2m-1}) + (j, 2m + 1) + (2m, 2m + 2) = c_{2m+2},
\]
where \(i_1, i_2, \ldots, i_{2m-2}, j\) form a permutation of \(1, 2, \ldots, 2m - 1\).

It follows that every

\[(j \ 2m + 1) \quad [j = 1, 2, \ldots, 2m - 1]\]

is expressible in terms of the \(Q_{2m+1}\) and \((2m \ 2m + 2)\) and hence in terms of the \(Q_{2m + 2}\).

From the equation

\[
(1 \ 2) + (3 \ 4) + \cdots + (2m - 5 \ 2m - 4) + (2m - 3 \ 2m) + (2m - 2 \ 2m + 1) + (2m - 1 \ 2m + 2) = c_{2m + 2}
\]

we have \((2m - 1 \ 2m + 2)\) expressed in terms of the quantities \(Q_{2m + 2}\) (by using our earlier results). Hence, from the equation

\[
(1 \ 2) + (3 \ 4) + \cdots + (2m - 3 \ 2m - 2) + (2m \ 2m + 1) + (2m - 1 \ 2m + 2) = c_{2m + 2}
\]

we obtain \((2m \ 2m + 1)\) expressed in terms of the \(Q_{2m + 2}\). We have, therefore, every

\[(j \ 2m + 1) \quad [j = 1, 2, \ldots, 2m + 2]\]

expressed in terms of the \(Q_{2m + 2}\).

Finally, from the equations

\[(i_1 \ i_2) + \cdots + (i_{2m-1} \ i_{2m}) + (j \ 2m + 2) + (2m - 1 \ 2m + 1) = c_{2m + 2},\]

where \(j, i_1, i_2, \ldots, i_{2m}\) form a permutation of \(1, 2, \ldots, 2m - 2, 2m\), we are able to express

\[(j \ 2m + 2) \quad [j = 1, 2, \ldots, 2m - 2, 2m]\]

in terms of the \(Q_{2m + 2}\).

Combining our results, we find that every

\[(i \ j) \quad [i, j = 1, 2, \ldots, 2m + 2; i \neq j]\]

is expressible in terms of the quantities \(Q_{2m + 2}\).

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