perience as a teacher. My criticisms largely arise from the fact that the author seldom indicates the limitations and uncertainties in the mathematical methods adopted, or supplies references whence his readers might obtain the knowledge which the book does not itself supply. Even in the parts of the work dealing with experimental results one is rather struck by the paucity of references to recent work done outside the author's laboratory. The insertion of long lists of authorities in an elementary book may savor of ostentation, but the author goes too far in the opposite direction.

The book seems carefully printed. Of the few errata I have noticed the following are the chief not already referred to :—p. 84, l.19, for "then" read "there"; p. 101, last line, for "negative" read "positive"; p. 108, lines 22 and 23, interchange "upper" and "lower" (?); p. 111, l. 6 from foot, insert "greatest" before "stress"; p. 117, case 4, the force F_x *increases* uniformly (*algebraically*) from one end to the other; p. 126, the conclusion that shearing stress is greatest at the neutral axis would follow from proof given only if breadth ζ constant; p. 135, in lines 9 and 10 of Art. 102, interchange "above" and "below"; p. 158, l. 2 below fig. 101, for 188 read 116; p. 185, l. 3 from foot, for "M" read "M_1"; p. 207, "t" is omitted in denominators of formulæ for f and f' in Art. 142; p. 208, "θ" should be shown in Fig. 136; p. 227, l. 10, for dM/da read dM_p/da.

CHARLES CHREE.

RICHMOND, SURREY,
July 2, 1900.

SCHEFFERS' DIFFERENTIAL GEOMETRY.

Anwendung der Differential- und Integralrechnung auf Geometrie. By DR. GEORG SCHEFFERS, Professor in the Darmstadt Polytechnic School. Erster Band: *Einführung in die Theorie der Curven in der Ebene und im Raum.* Leipzig, Veit and Co., 1901.

THE author of this work has already proved his capacity for writing text-books in a clear and readable manner, in the three volumes of Lie's works which he edited. The present volume is arranged and written in the same attractive and on the whole satisfactory style—for which the author is to be particularly congratulated, inasmuch as the work is intended for students who wish to obtain a bird's-eye view of the vast subject of the applications of the infinitesimal calculus to geometry—a subject which is touched upon, more or less, by nearly every writer of mathematical text-books or papers, and which for that reason is almost inaccessible as a whole to the average student.

There is a remarkable dearth of books on this subject in the English language—practically the only sources of information for the beginner being Salmon's classical (but antiquated) works, and the various larger text-books on the calculus. Important portions of the subject are treated by all the French authors who edit a Cours d'analyse—Houël, Jordan, Picard, Laurent, etc.; but these works fail to be adapted to the wants of the beginner either from being too much condensed—as those of Jordan and Picard—or from being too diffuse, as that of Laurent. In German, the elementary text-books are Hoppe and Joachimsthal—neither of which appeals to the beginner on account of peculiarities of arrangement; while Stahl-Kommerell is much too brief and Bianchi too difficult for one who has only had a course in elementary calculus.

A great mathematical discovery is nearly always the outgrowth of a need, felt by mathematicians in general rather than by a single investigator, for the advance represented by the discovery; so that a work which coördinates and presents in a readable form the principal branches of a many-sided subject paves the way, at least, for a notable advance in the science. In this sense, Professor Scheffers deserves the thanks, both of the students of mathematics, for whom the work is especially intended, and of mathematicians in general.

The volume which has just appeared, and which we shall examine briefly, contains the first grand division of the author's subject: the theory of curves—plane curves, and space curves with their accompanying developables. The second division, which will appear within a year, will contain the general theory of surfaces.

The subjects considered in Part I. of the present volume are largely those given in any complete work on the calculus—the theory of contact of plane curves, curvature, evolute and involute, etc. It is perhaps regrettable that practically nothing is done towards the discussion of asymptotes (as is done, for example, in Houël's Calcul infinitésimal, volume 2), and the treatment of singular points is very meager.

Some sections of this part, however, treat of subjects not usually given in the current text-books, and they deserve special notice. In § 2 the writer explains what is meant by a *movement* in the plane, by showing that it is immaterial whether the operations represented by the familiar equations

$$x' = x \cos \alpha - y \sin \alpha + a,$$
$$y' = x \sin \alpha + y \cos \alpha + b, \qquad (1)$$

be interpreted, 1°, as a change in the position of the coordinate axes with reference to the geometrical figure under consideration, or 2°, as a change of the position of the figure with reference to the (unchanged) axes. This introduces the reader at once, and in the most natural manner, to the conception of a *transformation* of the points of the plane. In § 8 certain *differential invariants* of a plane curve are defined and deduced by means of the definition of a movement contained in § 2. Any function of

$$x, \quad y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{d^2x}, \quad \cdots$$

which remains unchanged, when the curve $y = f(x)$ is subjected to the movements (1), is called a differential invariant of the curve with regard to those movements.* All possible differential invariants (in this sense) are shown—in an entirely elementary manner and without reference to the theory of transformation groups—to be functions of the magnitudes

$$r, \quad \frac{dr}{d\tau}, \quad \frac{d^2r}{d^2\tau}, \cdots, \qquad (2)$$

where r and τ are radius of curvature and angle of contingence, respectively.

The results of this section are applied in § 9 to establish the necessary and sufficient conditions that two plane curves shall be congruent. For since the differential invariants (2) are unchanged by a movement (1), it is clear that two curves can be congruent only when their differential invariants have the *same* values at corresponding points. It is

* The conception "differential invariant" is contained implicitly in all the older works on differential equations. Cayley, in his Theory of Invariants, and Halphen, in his "Thèse sur les invariants différentiels" (1878), established certain classes of differential invariants; but the *general* theory of differential invariants is due to Lie (*Verhand. der Gesell. der Wiss. zu Christiania*, Febr., 1875, also 1882–3; *Archiv for Math.*, 1882–3; *Math. Annalen*, vol. 24).

clear that $dr/d\tau$ is some function of τ; so that if $dr/d\tau = \omega(\tau)$ along the one curve, the necessary and sufficient condition for congruence may be seen to be that $dr/d\tau$ must be the *same* function of r along the other curve.*

All properties of a given curve therefore which are independent of the position of the coördinate axes, find their complete expression in the equation $dr/d\tau = \omega(\tau)$; therefore nothing is more natural than to introduce $dr/d\tau = \omega(\tau)$ as the intrinsic, or natural, equation to a curve.†

In § 11 it is shown that the differential invariants (2) represent, geometrically, the radii of curvature of the evolute, or of the evolute of the evolute, ⋯, of the given curve, as readily appears.

The §§ 14–20 of Part I. are taken up with a brief outline of the theory of ordinary differential equations in the plane: and the introduction of the conceptions *curvilinear coördinates*, *curve-nets*, etc. It is very fortunate for the beginner that the author treats the subject of parametric (curvilinear) coördinates in the plane very fully and clearly—carefully deducing the conditions that two curve families shall form an orthogonal system, an isometric system, etc., in the plane—as most of these results can be extended at once to curve families on a surface. It is just these sections of Part I. which will be most useful and most interesting to the beginner—as those subjects are not systematically treated in the current text-books, at least until the difficult general theory of surfaces is studied. Of especial interest are the paragraphs giving Lie's geometrical interpretation of an integrating factor, the finding of all transformations which leave areas invariant, the finding of all *conform* transformations, and the discussion of Lie's method of integrating *two* ordinary differential equations when a certain relation between their integrating factors is known.

Part II. of this volume scarcely demands a detailed discussion here, as it consists mainly of an excellent exposition of the classical theory of the space-curve, introducing the reader to the conceptions *osculating plane*, *contact*, *curvature*, *osculating sphere*, etc., together with the natural extension to space of the developments in Part I. concerning movements, differential invariants, etc.

In § 3, for instance, the writer explains how the (space) equations analogous to (1) define a movement, and proves incidentally Mozzi's theorem that (in general) any movement in space can be replaced by a screw movement.

---

* This criterion does not hold, for obvious reasons, for circles and minimal straight lines.

† That this is equivalent to Whewell's form $s = f(\tau)$ is readily verified.

Then in § 12 all differential invariants of a space curve—with regard to the movements—are determined; and they are shown to be functions of the radius of curvature r, the radius of torsion ρ, and the derivatives of these functions with respect to the arc length s. Thence are obtained the intrinsic (or natural) equations of space curves as the two independent equations which contain the differential invariants of lowest orders of the curve. In the next section the author gives Lie's method of reducing to the integration of a Riccati equation the problem of finding the finite equations of a space curve from its given intrinsic equations.

These sections, as well as § 18, 19 in which the curve is to be determined from the spherical indicatrix of its tangents, principal normals, or binormals, will offer considerable, but by no means insuperable, difficulties to the beginner.

In order to complete the development of the theory of the space curve—in particular, the theory of the evolutes or involutes of a given space curve—it is necessary to examine the curve in connection with its accompanying developable surfaces. The more important of these developables are, respectively, the envelopes of the osculating planes and of the normal planes of the curve. The author gives, therefore, in the first section of Part III. a brief discussion of the ruled surface, and, in particular, of the developable, deducing afterwards the theory of the evolutes and involutes of the space curve in his usual clear manner.

All theorems which are proved with the arc length s chosen as the variable parameter of which x, y, and z are functions, fail of course for the special curves for which s is everywhere zero. Hence the minimal curves (or minimal straight lines) demand a special discussion, and the development of the theory of these curves according to Legendre, Enneper, Weierstrass, and Lie closes the volume.

The type work of the book is excellent; very few misprints have been observed. It is furnished with an appendix containing several tables of useful formulæ, and with a good index. The author has enriched the general theory of curves by several original developments—notably in the discussion of the trajectories of a curve family * in the plane, and in the integration of the intrinsic equations † of a space curve. The historical references, given whenever a new definition is introduced, will be of great value to the

* G. Scheffers, *Ber. der math.-phys. Klasse der kgl. Sächs. Gesell. der Wiss. zu Leipzig*, October 24, 1898.

† Same publication, January 8, 1900.

reader who wishes to penetrate deeper into this branch of the science. The figures in the text are drawn with great care, and the illustrative examples worked out fully and clearly. The reader of English text-books will miss the long lists of problems usually given at the ends of the chapters in such works ; but we predict for the book a very useful career in the lecture room, especially in the hands of an energetic teacher who can supply himself with an abundance of illustrative problems.

J. M. PAGE.

UNIVERSITY OF VIRGINIA,
November 11, 1900.

NOTES.

A NEW edition of the Annual Register of the Society will be issued in January. Forms for furnishing necessary information have been sent to each member, and a prompt response will be of great assistance to the Secretary.

AT the annual general meeting of the London mathematical society held November 8, 1900, the following officers were elected : Dr. E. W. HOBSON, president ; Lord KELVIN, Professor W. BURNSIDE, and Major P. A. MACMAHON, vice-presidents ; Dr. J. LARMOR, treasurer ; Mr. R. TUCKER, and Professor A. E. H. LOVE, honorary secretaries ; Mr. J. E. CAMPBELL, Lieut.-Col. A. C. CUNNINGHAM, Professor E. B. ELLIOTT, Dr. J. W. L. GLAISHER, Professor M. J. M. HILL, Messrs. A. B. KEMPE, H. M. MACDONALD, A. E. WESTERN and E. T. WHITTAKER, additional members of the council. The subject of LORD KELVIN'S address as retiring president of the society was "The transmission of force through a solid."

AT the anniversary meeting of the Royal Society of London, on November 30, Sir WILLIAM HUGGINS was elected president. Among the new members of the council is Professor E. B. ELLIOTT. A Royal medal was presented to Major P. A. MACMAHON for his contribution to mathematical science.

THE National academy of sciences held its autumn meeting at Brown University, Providence, R. I., on November