THE ABSTRACT GROUP $G$ SIMPLY ISOMORPHIC WITH THE ALTERNATING GROUP ON SIX LETTERS.

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1. A slight correction of a theorem due to De Séguier* leads to the result that $G$ is generated by three operators $a$, $b$, $c$, subject only to the relations

1. $a^2 = I, b^4 = I, c^3 = I, (ae)^3 = I,$
2. $(ab^{-1}ab)^3 = I, (ab^{-2}ab^2)^2 = I,$
3. $(cb^{-1}ab)^2 = I, (cb^{-2}ab^2)^2 = I.$

But these generators are not independent, since

4. $a = cb^{-1}bc.$

A simple verification of (4) results from the correspondence

$a \sim (12)(34), \quad b \sim (12)(3456), \quad c \sim (123)$

between the generators of the simply isomorphic groups.

It is shown in this section that $G$ is generated by the two operators $b$ and $c$, subject to the complete set of generational relations

5. $b^4 = I, c^3 = I, (b^{-1}cbe^{-1})^2 = I, (b^2c)^4 = I.$

These relations follow from (1), (2), (3); for, by the above correspondence, $b^{-1}cbe^{-1} \sim (14)(23), \quad b^2c \sim (1235)(46)$.

If $a$ be defined by (4), relations (1), (2), (3) follow from (5).

$a^2 = cb^{-1}cbe^{-1}b^{-1}cbe = c(b^{-1}cbe^{-1})^2c^{-1} = I,$
$(ae)^3 = cb^{-1}cbe^{-1} = I.$

*Journal de Math., 1902, p. 262. For $y = 2, \ldots, n - 3$ in his formula (6), should stand $y = 1, \ldots, n - 4.$
As an auxiliary result, we note that

\[(6) \quad c(bcb^{-1}cb) = (bcb^{-1}cb)ce.\]

The condition (6) may be given the successive forms

\[e \cdot ccb^{-1} \cdot c^{-1}b^{-1} = I,\]
\[e \cdot ccb^{-1} \cdot c^{-1}b^{-1} = (bcb^{-1}b^{-1})^2 = I \quad \text{[by (5)]}.\]

Since (6) may be written \(e \cdot cbab^{-1} = ba\), we have

\[(7) \quad e \cdot ba = ba \cdot c.\]

In view of (6) and (5), we get

\[(8) \quad (cb^{-1}cb)^3 = e^{-1} \cdot b^{-1} \cdot c^{-1}bcb^{-1} \cdot e^{-1}b^{-1}c = I.\]

To verify (3), we note that, by (8),

\[ab^{-1}ab = ab^{-1} \cdot ab^{-1} \cdot c^{-1}b \cdot c \cdot ab^{-1} = (c^{-1}b)^4 = I \quad \text{[by (5)]}.\]

To verify (3), we transform by \(b^{-1}c^{-1}\) and get

\[c^{-1}b^{-1}c^{-1} \cdot b^{-1}c^{-1} \cdot cb^{-1} \cdot cb^{-1} \cdot c^{-1}b^{-1}c^{-1}b^{-1} = (c^{-1}b^{-1}c^{-1})^2 = I \quad \text{[by (5)]}.\]

To verify (3), we note that

\[ab^{-1}ab = ab^{-1} \cdot c^{-1}b \cdot c^{-1}b^{-1}c = (ab^{-1}b^{-1}c)^{-1} = c^{-1}b^{-1}c^{-1}b^{-1}c^{-1}b^{-1}c^{-1}.\]

Cubing the inverse and transforming by \(c\), we get (8).

To verify (3), we note that

\[ab^{-2}ab^2 = ac^{-1} \cdot ab^{-2}a = ab^{-1} \cdot b^{-2}a^{-1}b^{-1}c^{-1} = (ab^{-1}b^{-1}c^{-1}b^{-1}c^{-1})^{-1} = c^{-1}b^{-1}c^{-1}b^{-1}c^{-1}b^{-1}c^{-1}.\]
Transforming its square by $cb^2$, we get
\[
\begin{align*}
&bcb^{-1}c^{-1} \cdot b^{-1}c^{-1}b^{-1}c^{-1} \cdot bcb^{-1}c^{-1} \\
&= cbc^{-1}b^{-1} \cdot b^{-1}c^{-1}b^{-1} \cdot cbc^{-1}b^{-1} \cdot b^{-1}c^{-1}b^{-1} \\
&= cbc^{-1}b^{-1}c^{-1}b^{-1}c^{-1}b^{-1}c^{-1}b^{-1}.
\end{align*}
\]

Transforming by $cb$ and taking the inverse, we get $(b^2c)^4 = I$.

2. In a paper entitled “The abstract group simply isomorphic with the group of linear fractional transformations in a Galois field,” communicated November 2, 1902 to the London Mathematical Society, the writer shows that the group $G$ is generated by three operators subject to the relations

\[(9) \quad T^2 = I, \quad S_1^2 = I, \quad S_j^3 = I, \quad S_jS_j = S_jS_j, \]

\[(10) \quad (S_jT)^3 = I, \quad (S_jT)^4 = I, \quad (S_jS_jTS_j^{-1}S_jT)^2 = I. \]

From these we obtain relations (1), (2), (3), if we set
\[a = T, \quad b = S_jT, \quad c = S_j.\]

This is evident for relations (1). Also,
\[
(ab^{-1}ab)^3 = (S_j^{-1}TS_jT)^3 = S_j^{-1} \cdot TS_jT \cdot S_jTS_jT \cdot S_j^{-1}TS_jT
\]
\[= S_j^{-1} \cdot S_jTS_jT \cdot TS_j^{-1}T \cdot S_jTS_jT \cdot S_j^{-1}TS_jT = (S_jT)^4 = I.
\]

Also (2) and (3) follow from (9) and $(S_jT)^4 = I$, while (3) follows from (9) and the first and third relations (10). We thus obtain a new proof that (9) and (10) define $G$.

We may readily derive directly from (9) and (10) a complete set of relations between the two generators $b = S_jT$ and $c = S_1$ of $G$. We note that, from (10),
\[
T = S_1TS_1T = S_1 \cdot TS_j^{-1} \cdot S_jTS_1 = cb^{-1}cbe.
\]

We therefore have
\[S_i = c, \quad T = cb^{-1}cbe, \quad S_j = be^{-1}b^{-1}c^{-1}be^{-1}.
\]

Then $(S_jT)^4 = I$ follows from $(5_1)$, $S_i^3 = I$ from $(5_2)$, $T^2 = I$ from $(5_3)$, $S_jS_j = S_jS_1$ from $(5_4)$. Thus
\[
S_iS_j = cbe^{-1}b^{-1} \cdot c^{-1}be^{-1} = beb^{-1}c^{-1} \cdot c^{-1}be^{-1} = be \cdot b^{-1}c^{-1}be^{-1} = be \cdot cb^{-1}c^{-1}b = S_jS_1.
\]
Since $S_1T = c^{-1}b^{-1}cbe$ is the transform of $c$ by $bc$, it is of period three.

The final relation (10) becomes

$$(bc^{-1}b^{-1}c \cdot b^{-1}cbe)^2 = (c^{-1}bcb^{-1} \cdot b^{-1}cbe)^2 = (c^{-1}bc^2cbe)^2$$

$$= c^{-1}b(e^2b^2b^{-1}c = I.$$ 

Since $S_j$ is commutative with $S_j'$, the condition $S_j^3 = I$ follows from $(b^{-1}c^{-1}b^2c^{-1})^3 = I$ or $(c^2b^2b)^3 = I$. 

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NOTE ON A PROPERTY OF THE CONIC SECTIONS.  

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It is easily proved that if $P$, $Q$, $R$ are any three points on the conic $Ax^2 + By^2 = 1$, and $O$ the center of the conic, then the areas of the triangles $OPQ$, $OPR$, $OQR$ will satisfy an equation independent of the position of the points $P$, $Q$, $R$. If $a$, $b$, $c$ are the areas in question, this equation is

$$a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 16ABa^2b^2c^2 = 0.$$  

Now we can prove that such an invariant relation is possible for no plane curves except the central conies; i.e., if we seek a plane curve $C$ and a point $O$ in its plane such that, if $P$, $Q$, $R$ are any three points on $C$, the triangles $OQR$, $OPR$, $OPQ$ are connected by a relation independent of the coordinates of the points $P$, $Q$, $R$, we find $C$ to be a central conic section and $O$ its center. 

To prove this theorem, let $O$ be the origin of coordinates, and let the coordinates of $P$, $Q$, $R$ be respectively $x_1$, $y_1$; $x_2$, $y_2$; $x_3$, $y_3$. Then twice the areas of the three triangles are

$$2a = \pm (y_2x_3 - y_3x_2), \quad 2b = \pm (y_3x_1 - y_1x_3),$$

$$2c = \pm (y_1x_2 - y_2x_1).$$