Since \( S_i T = c^{-1}b^{-1}bc \) is the transform of \( c \) by \( bc \), it is of period three.

The final relation (10) becomes

\[
(b^{-1}c^{-1}b\cdot b^{-1}cbe)^2 = (c^{-1}bcb^{-1}\cdot b^{-1}cbe)^2 = (c^{-1}bcb^2cbe)^2
\]

\[
e^{-1}b(e^2)\cdot b^{-1}c = I.
\]

Since \( S_i \) is commutative with \( S_j \), the condition \( S_j^3 = I \) follows from \((b^{-1}c^{-1}b^2c^{-1})^3 = I \) or \((e^2b^2b)^3 = I \).

\textbf{The University of Chicago,}

\textit{December 11, 1902.}

---

\textbf{NOTE ON A PROPERTY OF THE CONIC SECTIONS.}

\textbf{BY PROFESSOR H. F. BLICHFELDT.}

\textit{(Read before the San Francisco Section of the American Mathematical Society, December 20, 1902.)}

It is easily proved that if \( P, Q, R \) are any three points on the conic \( Ax^2 + By^2 = 1 \), and \( O \) the center of the conic, then the areas of the triangles \( OPQ, OPR, OQR \) will satisfy an equation independent of the position of the points \( P, Q, R \). If \( a, b, c \) are the areas in question, this equation is

\[
(1) \quad a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 16ABa^2b^2c^2 = 0.
\]

Now we can prove that such an invariant relation is possible for no plane curves except the central conics; i.e., if we seek a plane curve \( C \) and a point \( O \) in its plane such that, if \( P, Q, R \) are any three points on \( C \), the triangles \( OQR, ORP, OPQ \) are connected by a relation independent of the coordinates of the points \( P, Q, R \), we find \( C \) to be a central conic section and \( O \) its center.

To prove this theorem, let \( O \) be the origin of coordinates, and let the coordinates of \( P, Q, R \) be respectively \( x_1, y_1; x_2, y_2; x_3, y_3 \). Then twice the areas of the three triangles are

\[
2a = \pm (y_2x_3 - y_3x_2), \quad 2b = \pm (y_3x_1 - y_1x_3), \quad 2c = \pm (y_1x_2 - y_2x_1),
\]
which expressions are functions of the three independent variables \(x_1, x_2, x_3\); \(y\) being considered a given function of \(x\) for points on the curve.

As \(a, b, c\) must satisfy a relation independent of \(x_1, x_2, x_3\), the Jacobian \(\frac{\partial(a, b, c)}{\partial(x_1, x_2, x_3)}\) must vanish. If \(y'_1\) represents \(\frac{dy_1}{dx_1}\), etc., we find

\[
y'_2[y_2[x_1y_1 - x_2y_1 + x_1y_2(y'_2 - y'_1)] + x_3(x_2y_1y'_1 - x_3y_3y'_2)]
\]

\[
+ x_2[(x_1y_2 - x_2y_1)y'_1y'_2 + y_1y_3(y'_2 - y'_1)] + y_3(x_2y_1y'_1 - x_3y_3y'_2) = 0,
\]

say

\[
y'_2(y_2k + x_2f) + x_3m + y_3l = 0,
\]

\(k, l, m\) being functions of \(x_1, x_2\) only, and therefore independent of \(x_3\).

Two cases (\(a\)) and (\(\beta\)) may now present themselves as follows:

(\(a\)) The functions \(k, l, m\) are not all identically zero. In this case the equation (2) gives, when integrated,

\[
y''_2k + 2y'_3x_3l + x_3^2m = f(x_1, x_2).
\]

If we give to \(x_1\) and \(x_2\) arbitrary constant values, the equation (3) represents a conic section with its center at \(O\).

(\(\beta\)) The functions \(k, l, m\) are all zero. We must then have \(x_2y'_1 - y_2 = 0\). Giving to \(y'_1\) a definite constant value, we obtain the equation of a straight line—a special case of (3).

The theorem stated above is therefore proved.

It may be noticed that \(f(x_1, x_2)\) in (3) may be multiple valued. The equation will then represent a series of similar conics similarly placed. If these are finite in number, say \(n\), we find that, if \(P, Q, R\) be located anywhere on this system of curves, the areas \(a, b, c\) of the three triangles considered will satisfy an equation of degree

\[
6n + 18n(n - 1) + 6n(n - 1)(n - 2)
\]

at most, whose left-hand member is composed of factors of form similar to (1), as the reader may prove without much difficulty.

Stanford University,
November, 1902.