12. The formulas of Euler (and Rodrigues), which give rationally in terms of three parameters the nine coefficients of an orthogonal transformation in space, are available for expressing the vertices of any polar triangle of the conic

$$x_1^2 + x_2^2 + x_3^2 = 0.$$

This gives a means of discussing curves of the second and third orders, at least, which contain infinitely many inscribed polar triangles of a conic. Professor White's preliminary communication exhibited the method as applied to conics. The further question was raised, what sort of curve is the locus of points whose coördinates are the eulerian parameters of polar triangles inscribed in a single conic, or of orthogonal transformations which rotate the axes through the surface of an orthogonal quadric cone.

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SOME GROUPS IN LOGIC.

BY PROFESSOR E. W. DAVIS.

(Read before the Chicago Section of the American Mathematical Society, January 2, 1903.)

DE MORGAN has pointed out* that his eight forms of proposition identical with the $A, E, I, O$, and their contranominals of the older logic, can be derived from any one by the three operations of reversing the subject, reversing the predicate, denying the copula. If, in fact, we denote the operations in question by $s, p,$ and $f$ respectively, we have

$$Ap = E, \quad Af = O, \quad Afp = I;$$

while $sp$ changes any proposition to its contranominal $X \leq Y$ to $Y \leq X$; or, what is the same thing to $X \leq Y$. Here $\leq$ is the sign of implication and the bar written over a letter or sym-

* Formal Logic, p. 63 et seq.
bol reverses or denies the meaning of the same. All the relations can be simply expressed by a diagram

\[
\begin{array}{c}
X \preceq Y \\
\downarrow \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ }\end{array}
\]

This scheme differs from De Morgan’s in that he assumes always the existence of both a subject and its contrary, whereas our assumption, following Mr. C. S. Peirce, is that \( X \preceq Y \) means ‘not \( X \) or else \( Y \),’ while \( X \preceq Y \) means ‘\( X \) but not \( Y \).’

There is also a group of order 32 which operating upon any one of 24 forms of valid syllogism will produce the rest, together with 8 non-valid syllogisms. To see this, start with Barbara

\[(X \preceq Y)(Y \preceq Z) \preceq (X \preceq Z),\]

or say

\[A_1 A_2 \preceq A_3.\]

Let now \( a_1 \) mean the operation of performing \( sf \) upon \( A_2 \) and \( A_3 \), with similar significations for \( a_2 \) and \( a_3 \). Then, \( B \) standing for Barbara, \( Ba_1 \) and \( Ba_2 \), are valid, while \( Ba_3 \) is not.† Moreover

\[G_* = \{1, a_1, a_2, a_3\}\]

is a 4-group. Were \( B \) instead of Barbara, any valid syllogism it would still be true that of \( Ba_1, Ba_2, \) and \( Ba_3 \), two would be valid while one was invalid. If \( B \) were invalid the operations \( a \) might lead to none that were valid, could at most lead to one that was.

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† Conversion of the propositions with negative copula reveals the fact that \( B, Ba_1, Ba_2, Ba_3 \), are in the four figures of the older logic.
Suppose \( x \) to be the operation of interchanging \( X \) and \( \bar{X} \), with like significations for \( y \) and \( z \). These 3 operations generate a group \( G_{32} \), such that when performed upon any syllogism \( B \) the resulting 8 syllogisms are valid or invalid according as \( B \) is.

The group is commutative with \( G_4 \) and the product of the two is the \( G_{32} \) referred to above.

Let \( s, p, f \) be the operations of performing \( s, p, f \) upon \( A \). It is evident that

\[
\begin{align*}
x &= s_1s_3, & y &= p_1s_2, & z &= p_2p_3, & a_1 &= s_2p_3f_2s_3p_3f_3.
\end{align*}
\]

These 9 operations generate a group \( G_{512} \), the product of \( G_{32} \) by the group

\[
G_{16} = \{x_1s_1, p_1, p_3, f_1\}.
\]

The new syllogisms got by operating with this upon the 32 syllogisms derived from \( Barbara \) above are all invalid.

The operation called conversion interchanges the subject and predicate of a proposition, at the same time reversing each. The meaning is, of course, left unchanged. Immaterial to the argument likewise is it whether the order of the propositions be \( A_1A_2 \) or \( A_2A_1 \). Let the operation of converting \( A_i \) be denoted by \( c_i \) while the interchange of \( A_1 \) and \( A_2 \) is denoted by \( t \). Then \( \{c_1, c_2, c_3, t\} \) is of order 16. We call it \( H_{16} \). Finally

\[
G_{512} \times H_{16} = G_{8192},
\]

whereof

\[
G_{32} \times H_{16} = H_{512}
\]

contains the operations which performed upon \( Barbara \) lead to all the valid syllogisms of \( BG_{8192} \) taking account of conversion and the order of the premises, viz., to 384.

The same sort of group building can of course be applied to more complicated sets of statements, and is a very simple consequence of the binary character of deductive logic.

University of Nebraska, January 11, 1903.