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THE ELEMENTARY TREATMENT OF CONICS BY MEANS OF THE REGULUS.

BY PROFESSOR CHARLOTTE ANGAS SCOTT.

(Read before the American Mathematical Society, February 25, 1905.)

INASMUCH as projective plane geometry depends on properties of space (since the theorem on triangles in perspective in a plane cannot be established by plane geometry without the use of metric concepts), there is no logical objection to the employment of space considerations in proving theorems of plane geometry. The regulus supplies extremely simple proofs of the properties of a conic, whether this be considered as a locus or an envelope; these proofs have the advantage of connecting the points of a conic and the tangents of a conic from the first, instead of leaving the connection to be proved later by an elaborate chain of reasoning. The method makes unnecessary also the treatment of the tangent as the limiting position of a chord when the determining points become indistinguishable, which is open to some objections in pure geometry; not on account of the assumption as to continuity, for this has already been admitted in the proof of the fundamental theorem of projective geometry, but with regard to elegance and directness of proof. This note contains the application of the method to the proofs of the theorems of Chasles, Brianchon, and Pascal, and of polar and involution properties.

The regulus is the system of lines (rays) that meet three non-incident lines, the directors. By means of the fundamental theorem of projective geometry it is proved that the regulus is crossed by a second regulus; any three rays of either serve as directors of the other; through any point on a ray of either there passes a ray of the other.

Three directors, u, v, w, and three rays a, b, c (that is, two incident triads of lines) give rise to an interesting figure on which the proofs depend. The pairs of lines au, bv, cw determine points A, B, C and planes α, β, γ; the three planes meet in a point O, the three points lie in a plane ω. The point $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$ and the plane $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$, that is, O and ω, are pole and polar. The pairs of points (bw, cv), (cu, aw), (av, bu), that is, $\overset{b\ \ c}{\underset{v\ \ w}{\times}}$, $\overset{c\ \ a}{\underset{w\ \ u}{\times}}$, $\overset{a\ \ b}{\underset{u\ \ v}{\times}}$, lie on lines through O, namely, the lines $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$; the pairs of planes $\overset{b\ \ c}{\underset{v\ \ w}{\times}}$, $\overset{c\ \ a}{\underset{w\ \ u}{\times}}$, $\overset{a\ \ b}{\underset{u\ \ v}{\times}}$ meet in lines in ω, namely, BC, CA, AB. The points are named as shown in Fig. 1, the planes are named to correspond, *e. g.*, the point av is A', the plane av is α'; the planes α', β', γ' determine the point O', etc.

The triangles $A'B'C'$, $B''C''A''$ are perspective from O; $A''B''C''$, BCA from O'; ABC, $B'C'A'$ from O''. It is easily shown that O, O', O'' are collinear, ω, ω', ω'' are coaxial; and that if the rays and directors are associated in the reversed order $\begin{pmatrix} a & c & b \\ u & v & w \end{pmatrix}$, the three new centres of perspective lie on the axis of $\omega\ \omega'\ \omega''$, and the axis of the new planes is $O\ O'\ O''$. These facts, however, are not needed for the present purpose.

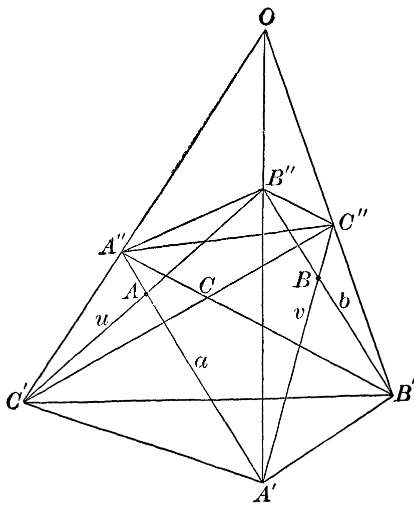

FIG. 1.

It is at once seen that O, ω are harmonic to the pairs of lines au, bv, cw, that is, to the two triads of rays a, b, c and directors u, v, w, hence to the complete system of transversals to a, b, c (directors) and transversals to u, v, w (rays). This shows that the rays and directors intersect in pairs at points on ω, lie in pairs (the same pairs) on planes through O; the two of a pair d, x thus associated are harmonic with regard to O and ω. Any chord through O joins a point on a ray d to a point on the associated director x, and is therefore divided harmonically by O, ω. Moreover, if O be any point on a chord that meets a ray d and director x, then the point dx lies on the polar plane of O.

Any line p, not a director, that meets one ray meets one other, but no more. For let p pass through A (*i. e.*, au); the plane pu meets v, w, at V, W; the line VW is therefore a ray, and is met by p. The only possible exception arises when VW is itself the ray a; then the line p meets the regulus at the point A only (or, the line p meets only one ray of the regulus); p is a tangent line, and the plane α (*i. e.*, au), in which lie all tangent lines through the point A (*i. e.*, au), is a tangent plane. The facts proved above as to the relation of O, ω may be stated in the form: The points of contact of tangent planes from the point $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$ lie on the plane $\begin{pmatrix} a & b & c \\ u & v & w \end{pmatrix}$.

In order that a tangent plane may contain a line q, it must contain any two points S, S' on the line; hence the point of contact must lie on both σ and σ', that is on a line q'; q, q' are conjugate lines. Hence if there is one tangent plane through q, there is precisely one other, unless q' is itself a ray or director, which happens only when q is a ray or director, and then q' coincides with q.

The ranges of points determined on two directors u, v by the rays $a, b, c, d, \cdots$ are sections of the axial pencil $x.abcd \cdots$ (where x is any other director) by the transversals u, v; hence they are projective. The regulus is therefore the system of lines that join corresponding points of projective ranges on two non-incident lines. Again, the axial pencils $u.abcd \cdots$, $v.abcd \cdots$ have a common section by the transversal x, hence they are projective. The regulus is therefore the system of lines determined by corresponding planes of two projective axial pencils whose axes are non-incident.

Let the regulus be cut by a plane ω; each ray is thus associated with a particular director; the two, d, x, meet in a point D on ω, and lie in a plane δ (a tangent plane) through O. The projective axial pencils $u.abcd \cdots$, $v.abcd \cdots$ are cut in projective flat pencils, centers A and B; hence the plane section of a regulus (which is a point system of the second order, since a line through A cuts one other ray) is the locus of the intersection of corresponding rays of two projective flat pencils. Consider also the line system composed of the tangent lines in the plane ω; these are the intersections of the tangent planes au, bv, cw, dx, ey, etc. by ω, they are therefore the projections of rays $a, b, c, d, e, \cdots$ (or of directors $u, v, w, x, y, \cdots$) from O on ω; call them $a', b', c', \cdots$. Since the ranges $u.abcd\cdots$, $v.abcd\cdots$ are projective, their projections from O on ω are projective. Hence the lines $a', b', c', d', \cdots$ in the plane ω connect the corresponding points of projective ranges on the two tangents a', b'; that is, the line system (which, by what has been said about tangent planes, is seen to be of the second order) is composed of the lines that join corresponding points of projective ranges. Moreover, as to the relation of the two systems: the flat pencil $A.ABCD\cdots$ in the plane $\omega =$ the axial pencil $u.abcd\cdots =$ range $v.abcd\cdots =$ range $v'.a'b'c'd'\cdots$ in the plane ω. Thus the section (aggregate of points and lines) has the property that the pencil subtended at any point A by the points C, D, E, F is projective with the range determined on any tangent b' by the tangents c', d', e', f', which is Chasles's theorem. The section will now be called a conic.

Since the pencil determined by four points is projectively

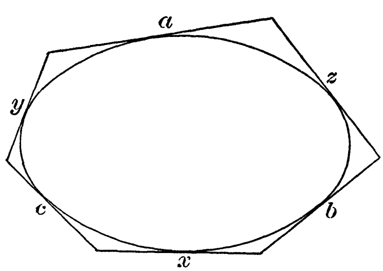

FIG. 2.

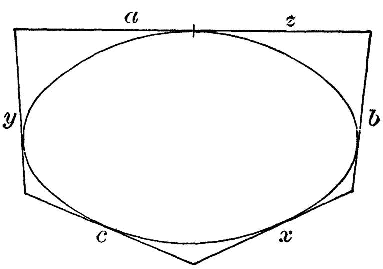

FIG. 3.

the same wherever on the section its vertex may lie, and similarly for the range determined by four tangents, it is unnecessary to specify the vertex or base; it is sufficient to speak of the four points or four lines. Since also the range determined by four rays on any director of the regulus depends only on the rays, it is sufficient to speak of the four rays. What has been proved above can be stated in the form: Four points of a conic are projective with the four rays, or four directors, that pass through them; four tangents to a conic are projective with the four rays, or four directors, that pass through their points of contact.

If four rays a, b, c, d are projective with four directors u, v, w, x, their intersections are coplanar. For let y be the director through the point D in which the plane au, bv, cw (that is, ABC) meets d; then the rays a, b, c, d are projective with the points A, B, C, D, and these are projective with the directors u, v, w, y. Hence $u\,v\,w\,x = u\,v\,w\,y$, and y is therefore the same as x.

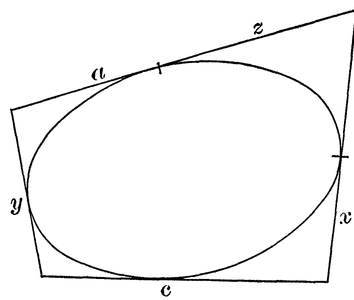

FIG. 4.

The polar properties follow at once from the harmonic relation borne to the rays and directors by O, ω. A point T in the plane ω has a polar plane τ, which by harmonic symmetry of the whole figure to $O\omega$ must pass through O. The section of τ by ω is the polar line of T with regard to the section considered, and it has already been shown that any chord through T, and therefore a chord of the section, is harmonically divided by $T\tau$. Hence follows the usual quadrilateral construction for pole and polar.

To prove Brianchon's theorem, take the six tangents as pro-

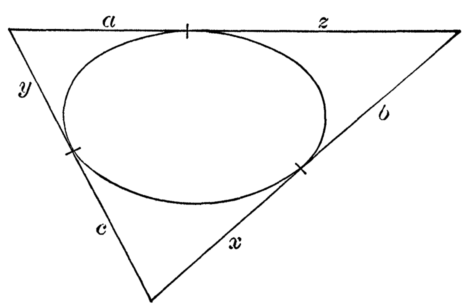

FIG. 5.

jections alternately of rays and directors, as shown in Fig. 2. In space, the lines $\underset{y\ z}{\overset{b\ c}{\times}}$, $\underset{z\ x}{\overset{c\ a}{\times}}$, $\underset{x\ y}{\overset{a\ b}{\times}}$ meet at a point S; projecting the figure from O on to ω, we obtain three concurrent lines joining opposite vertices of the circumscribing hexagon.

To prove the special cases of this theorem for the circumscribing pentagon, quadrilateral, triangle, regard the necessary number of tangents (one, two or three) as projections of both a ray and a director, as shown in Figs. 3, 4, 5; the proof then applies without change, the point az being the point of contact of the tangent.

To prove Pascal's theorem, take alternately the rays and directors that pass through the six points, Fig. 6. The common lines of the planes $\underset{y\ z}{\overset{b\ c}{\times}}$, $\underset{z\ x}{\overset{c\ a}{\times}}$, $\underset{x\ y}{\overset{a\ b}{\times}}$ lie in a plane σ; hence in the section by the plane ω the common points of the lines $\underset{y\ z}{\overset{b\ c}{\times}}$, $\underset{z\ x}{\overset{c\ a}{\times}}$, $\underset{x\ y}{\overset{a\ b}{\times}}$ lie in a line, the line $\sigma\omega$. The special cases of this theorem, which arise when sides of the inscribed hexagon (one, two, or three) are replaced by tangents, require no change in the proof. Moreover, the same proof applies when the section by the plane ω consists of a ray and a director, that is, when the theorem to be proved becomes Pappus's theorem, that if the vertices of a hexagon are taken to lie alternately on two intersecting lines, the three intersections of opposite sides are collinear.

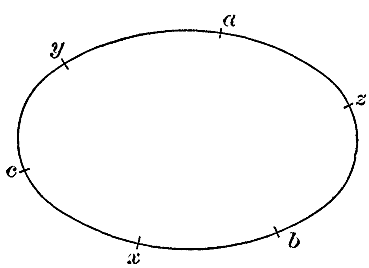

FIG. 6.

It has been mentioned that if a chord meets d, x, the point dx lies on the polar plane of any point on the chord. Hence if chords of a conic, A_1A_2, B_1B_2, C_1C_2, $\cdots$, are concurrent in S, then $A_1B_1C_1D_1 = A_2B_2C_2D_2$; for let a, b, c, d be the rays through A_1, B_1, C_1, D_1, and u, v, w, x the directors through A_2, B_2, C_2, D_2, and let the points au, bv, cw, dx (which are known to lie on σ) be U, V, W, X. Then $A_1B_1C_1D_1 = abcd = UVWX$ (in the section σ) $= uvwx = A_2B_2C_2D_2$. This proof is unaffected by coincidence; D_1D_2 may be the same as A_2A_1. Thus $A_1B_1C_1A_2 = A_2B_2C_2A_1$. Conversely, if in the plane ω $A_1B_1C_1A_2 = A_2B_2C_2A_1$, the chords A_1A_2, B_1B_2, C_1C_2 are concurrent. For since $A_1B_1C_1A_2 = abcd$, and $A_2B_2C_2A_1 = uvwx$, we have $abcd = uvwx$, and therefore by a result obtained earlier, the points au, bv, cw, dx lie on a plane σ. Hence the planes au, bv, cw, dx meet in a point S. But the planes au, dx meet in a line in the plane ω, hence the point S lies in ω. Since then the planes

au, bv, cw, dx meet at a point S in ω, the section of these planes by ω gives chords of the conic concurrent in S. Thus we have the theorem : If points of a conic are projectively paired, they lie on concurrent chords, which is the foundation of the theory of involution on a conic.

For the sake of brevity, as little detail as possible has been given in this note, the design being simply to draw attention to this mode of proving fundamental properties of the conic.

BRYN MAWR COLLEGE,
June, 1905.

ARZELÀ'S CONDITION FOR THE CONTINUITY OF A FUNCTION DEFINED BY A SERIES OF CONTINUOUS FUNCTIONS.

BY PROFESSOR E. J. TOWNSEND.

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§ 1. A FUNCTION defined by a series whose terms are continuous functions may or may not be itself continuous. It may in fact be discontinuous at a set of points everywhere dense within the interval of definition. It is important to establish a criterion by which the continuity of such a function may be determined. Conditions which are sufficient, although not necessary, are to be found in any extensive work on calculus. Arzelà was the first to formulate a set of conditions which are both necessary and sufficient.* In his first discussion of the subject, however, he was not sufficiently rigorous.† A later and more rigorous development was given, differing from the first in some particulars.‡ Still more recently he has revised his first set of proofs and maintains that they are now sufficiently rigorous to be valid.§ It is the purpose of this paper to present in substance the final results of Arzelà's investigations.

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*Intorno alla continuità della somma di infinite funzioni continue, Bologna, 1884.

†See Schoenflies, Punktmannigfaltigkeiten, p. 225, footnote. Also Arzelà, Sulla serie di funzioni di variabili reali, Bologna, 1902, p. 6.

‡Sulla serie di funzioni, part 1, Bologna, 1899, p. 10 et seq.

§See Sulla serie di funzioni di variabili reali, Bologna, 1902.