tors in any group in which more than half the operators are of order 2.

A \((2^a, 2^\beta)\) isomorphism between \(G_1\) and the direct product of the dihedral rotation group of order \(2^{\beta+1}\) into an operator of order 2 can be established in such a manner as to obtain a group in which the number of operators of order 2 is either \(3 + 2^a + 2^{\beta+1} + 2^{a+\beta}\), or \(3 + 2^{a+1} + 2^{\beta+1} + 2^{a+\beta-1}\), \(\beta > 0\). In fact, it is possible to form other such isomorphisms, but these two seem especially useful in this connection. Moreover, by establishing a \((2^a, 2^\beta)\) isomorphism between \(G_1\) and a group of order \(2^{\beta+2}\), which is constructed in the same way as \(G_1\), we arrive at groups which contain any of the following three numbers of operators of order 2: \(3 + 2^a + 2^\beta + 2^{a+\beta}\), \(3 + 2^{a+1} + 2^{\beta+1} + 2^{a+\beta-2}\), \(3 + 2^{a+1} + 2^\beta + 2^{a+\beta-1}\).

From the above results it follows directly that there are groups of order \(2^m\) which contain any prescribed number of operators of order 2 which satisfies the conditions that it is \(\equiv 3 \mod 4\) and less than 124. By other considerations this limit can readily be extended, but my methods seem too special to be given here. It would be interesting to find a number \(\equiv 3 \mod 4\) which could not equal the number of operators of order 2 in any group of order \(2^m\), or to prove the non-existence of such a number.

ON THE ARITHMETIC NATURE OF THE COEFFICIENTS IN GROUPS OF FINITE MONOMIAL LINEAR SUBSTITUTIONS.

BY DR. W. A. MANNING.

(Read before the American Mathematical Society, September 7, 1905.)

Professor Maschke* has proved (with a certain restriction) that the coefficients of finite linear substitution groups can, by proper transformations, be made rational functions of roots of unity. Professor Burnside † has also recently written on this subject. In this note it is proved that linear groups all of

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whose elements are finite monomial substitutions, that is, of the form

\[ x_i^j = a_{ij} x_j \quad (i, j = 1, 2, \ldots, n), \]

\((n\) is the number of variables) can be written so that all the coefficients are roots of unity.

**Theorem I.** Any non-vanishing element of the principal diagonal of a monomial substitution of finite period is a root of unity.

Consider the \(k\)th power of a substitution \(S\). If \(a\) lies in the principal diagonal of \(S\), \(a^k\) occupies the same place in the principal diagonal of \(S^k\). Hence \(a\) is a root of unity.

**Theorem II.** The product of two non-vanishing elements of the same or any two substitutions of a group \((G)\) of finite monomial linear substitutions, which are symmetrically placed with respect to the principal diagonal, is a root of unity.

The product of these two elements stands in the principal diagonal of the product of the two substitutions.

**Theorem III.** If \(G\) has no coefficient \(a_{ik}\) zero for all its substitutions, it can be transformed into another monomial group such that the non-vanishing element in the first column of every substitution of \(G\) becomes a root of unity.

Since no element \(a_{ik}\) is zero for every substitution of \(G\), we can choose \(n - 1\) substitutions

\[ A^{(2)} = (a^{(2)}_{ik}), \quad A^{(3)} = (a^{(3)}_{ik}), \ldots, A^{(n)} = (a^{(n)}_{ik}), \]

in which none of the coefficients

\[ a^{(2)}_{12}, \ a^{(3)}_{13}, \ldots, a^{(n)}_{1n} \]

vanish. Transform \(G\) by the canonical substitution

\[ x_i' = x_i/p_i \quad (i = 1, 2, \ldots, n). \]

The transformed group \((G')\) is monomial. If \(p_i\) is an arbitrary root of unity, and if \(p_2 = a^{(2)}_{12}, p_3 = a^{(3)}_{13}, \ldots, p_n = a^{(n)}_{1n}\), we have in place of \(A^{(2)}, A^{(3)}, \ldots, A^{(n)}\) substitutions with \(p_1\) as the only non-vanishing element in the first row of each. Now apply Theorem II to all the substitutions of \(G'\), and the present theorem follows.

**Theorem IV.** If \(G'\) has no coefficient everywhere zero the non-vanishing elements of every substitution are roots of unity.
Consider the first column of a product $AB$. The elements of this column are obtained by multiplying the rows of $A$ into the first column of $B$. Let $A$ and $B$ run through all the substitutions of $G'$. Every coefficient is seen to be the quotient of two roots of unity, that is, a root of unity.

Suppose that a certain coefficient $a_{ik}$ vanishes in every substitution of $G$. We may assume that the variables of $G$ have been so permuted that the $n - r$ last elements in the first row of all the substitutions of $G$ are zero, and that no other row has more than $n - r$ elements which vanish for every substitution of $G$.

**Theorem V.** Every substitution of $G$ is in the form

\[
\begin{bmatrix}
N_1 & 0 & 0 & \cdots \\
0 & 0 & \cdots & N_2 \\
0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots
\end{bmatrix},
\]

where $N_1$ and $N_2$ are monomial matrices, without further transformation.

There now are $r - 1$ substitutions $A^{(2)}, A^{(3)}, \ldots, A^{(r)}$ in which the coefficients $a_1^{(2)}, a_1^{(3)}, \ldots, a_1^{(r)}$ do not vanish. From the products

\[A^{(2)}B, A^{(3)}B, \ldots, A^{(r)}B,
\]

\[a_{i2}^{(2)}b_{2i} = 0, \quad a_{i3}^{(3)}b_{3i} = 0, \quad \ldots, \quad a_{ir}^{(r)}b_{ri} = 0 \quad (i = r + 1, \ldots, n),
\]

where $B$ is any substitution of $G$. Hence the last $n - r$ coefficients of the first $r$ rows of all the substitutions of $G$ are zero. Since these substitutions are monomial the first $r$ elements in the last $n - r$ rows are also everywhere zero.

The group in the variables $x_1, x_2, \ldots, x_r$ has by hypothesis no coefficients that are everywhere zero, so that for it Theorem IV holds.

We continue in this way with the group in the last $n - r$ variables, and finally have the

**Theorem VI.** The coefficients of a group of monomial linear substitutions of finite period may, by means of transformations which leave them monomial, be made roots of unity.

Stanford University,
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