the general reader. The movement itself, of which all these works are the outgrowths, is undoubtedly of the first importance. In addition to active interest on the part of leading mathematicians and of the strong and influential society appointing and supporting the commission whose report we have considered, other evidences of a widespread and significant participation in the movement are not lacking. Of these only the representation of the Deutsche Mathematiker-Vereinigung on the Commission, and the space devoted to pedagogic matters in its *Jahresbericht*, need be mentioned here. While this movement is directly concerned only with mathematics in Germany, it is nevertheless of international significance, both on account of the fundamental nature of the changes which it proposes and the weight attaching to the names of the societies and men that are engaged in it. In its essentials, it is in close harmony with movements of the day for improvement of the teaching of mathematics in England, in France and in the United States, which agree in demanding:

1. That the presentation take more careful account of the pupils' powers; that it be psychological as well as logical; that consequently the intuitional, the inductive, the concrete, be given prominent place in the teaching of mathematics.
2. That the amount of complex or merely technical work be diminished.
3. That, in addition to its disciplinary value, the importance of mathematics for a clear understanding of nature, and the fundamental character of mathematics in the fabric of modern civilization be brought effectively before the pupils.

J. W. A. YOUNG.

THE UNIVERSITY OF CHICAGO,
January 24, 1906.

VECTOR ANALYSIS.

Vorlesungen über die Vektorenrechnung. Von E. JAHNKE. Leipzig, B. G. Teubner, 1905. xii + 235 pp.

THE entering class at an American college is more or less prepared on elementary geometry and algebra, and perhaps on trigonometry. The problem of collegiate instruction has, therefore, one fairly well-defined premise. All the students who continue their work in mathematics, for no matter what purpose, must study analytic geometry and calculus. From this point on, however, the needs of different students are different, and to a considerable extent the aims and the subjects of instruction must vary. What shall be the place, and what the meaning of further instruction in algebra?

It is possible to proceed in several ways. In the first place the plan usually followed may be adopted and more algebra, similar to that with which the student is already familiar, may be taught. On the other hand the questions of the foundations of algebra and arithmetic may be studied with the object of showing the student the meaning of postulates, the significance of the number system, and the fact that algebra is merely symbolic and devoid of any inherent concrete interpretation. The present fervor for logic may make this course of procedure popular and in the future much time may be devoted to this branch of algebra. The result, however, might turn out to be far from justifying the expenditure in time and exertion. It is difficult to get any real grasp of the meaning of postulates or any considerable enjoyment in the logical development of a subject from them, unless the consequences of other related systems of postulates be also investigated. Thus it would seem better to postpone this second extension of algebra until such time as similar questions in geometry with especial reference to Lobatchevskian and Riemannian geometry shall be taken up. In the third place it may be the aim of the instructor to broaden the student's ideas on the general subject of algebra, to bring out the natural generalization of those ideas, to show how geometry and algebra may be united into a geometric algebra somewhat in contrast to algebraic or analytic geometry.

It seems as if this third way of continuing algebra, no more difficult of execution and considerably more practical than the preceding two, is all too much neglected here in America and in the world at large. The method of advance almost universally followed in respect to algebra is as if on entering college the student were put to work studying the nature of postulates and the formal side of geometry or developing further the synthetic methods of higher geometry instead of proceeding to analytic geometry and through it to the generalizations of modern geometry. If a thorough course (or even two) in analytics — an amount considerably in excess of that required for elementary calculus and its applications — be presented before matters of postulates and higher synthetic methods, why should not some corresponding training in geometric algebra come before the investigation of those other algebraic questions? The reason is perhaps more historical than rational. And although it would doubtless be gross exaggeration to state that geometric algebra is on a par with analytic geometry as regards its general importance, yet in view of its great usefulness in geometry and its present widespread use in physics, not to mention its own intrinsic interest, it certainly deserves a greater relative recognition than is now given to it.

To further this end nothing seems more propitious than the appearance of Jahnke's Lectures on vector analysis. The title is but poorly descriptive of the book, which is far wider in its scope and method. These lectures are really lectures on multiple algebra and form an excellent introduction to the subject — no more than an introduction because the theory of the linear function which is of such fundamental importance is hardly mentioned and the closely allied theory of linear associative algebra is not touched at all. The author's method of treatment is logical and pedagogic from beginning to end. He first considers in detail the algebra of the plane and only later proceeds to corresponding discussions for space. The similarities between the two cases render the study of space much easier after the plane has been treated; and the dissimilarities bring out very clearly the changes that must still be made in order to generalize to space of n dimensions. What is especially remarkable for a German text-book is that not merely does the author work out numerous examples but he actually leaves a larger number as exercises to be worked out by the student. This feature of the book will certainly appeal to any one who may wish to teach this subject, in which collections of problems are hard to find.

In the first chapter the beginnings of Grassmann's point analysis, the addition and subtraction of points affected with numerical coefficients, is taken up. This leads at once to the free vector as the difference of two points and to the addition and subtraction of vectors. The third chapter discusses localized vectors and the multiplication of points. Next comes the multiplication of free vectors followed by a large and various set of applications to problems in mechanics and physics. A sixth chapter on regressive multiplication completes the treatment of the plane. From this brief indication of the contents of the first eighty-eight pages of the Lectures, it can be seen that the reader has been brought step by step to the different and manifold ideas and methods underlying geometric algebras of the non-linear-associative type. The study of the complex variable will furnish a correspondingly simple case of this latter type.

Proceeding to space, the author follows exactly the same program, weaving together side by side the point analysis and the vector analysis. Here, however, there are two classes of directed quantities (vectors and bivectors or lines and planes) and the product of three vectors or four points is a scalar; whereas in the plane there was only a single kind of vector and the product of two vectors was a scalar. Among the applications may be mentioned the theory of moments of inertia, of the screw with Chasles's theorems, and of the null-system. Further there is a treatment of differentiation, about thirty pages in the whole, less than ten of which are on divergence, curl, and Gauss's theorem. Stokes's theorem seems not to be mentioned. This insignificant amount of attention paid to space differentiation and integration, which makes the book much less useful than many another to the physicist, leaves greater room for the purely algebraic and geometric side and is, therefore, particularly to be commended from the standpoint of one wishing to learn or to teach multiple algebra as a continuation of elementary algebra and of a little analytic geometry.

There is one point on which a little adverse criticism or at any rate a word of caution seems necessary. On page 110 the author says: "In most books on vector analysis it is a question of vector and scalar products, not of outer and inner products. While the inner and scalar products are identical, there is an essential difference between the outer and vector products. If $[\mathbf{ab}]$ be an outer product, $[\mathbf{ab}]$ is a vector of higher dimensions than $\mathbf{a}$ or $\mathbf{b}$ individually. On the other hand $V\mathbf{ab}$ or $\mathbf{a} \times \mathbf{b}$ is a simple vector $\mathbf{c}$. In this kind of multiplication the conception of dimensions is lost." In a paper read at the congress at Heidelberg, I have analysed this argument in detail and come to the conclusion that if it be not wrong it is, at least from the point of view of the physicist, so inadequate as to be worthless.* There is no need of repeating the analysis here. The trouble arises through the use of the term dimensions in two different senses. There is just one fundamental proposition underlying the idea of physical dimensions, namely, that the physical dimensions of a product are the product of the dimensions of the individual factors. This is independent of the

* "On products in additive fields," Verhandlungen des III. internationalen Mathematiker-Kongresses in Heidelberg, 1904, § 5, pp. 209–211.

analysis by which such product is represented. It is even independent of the kind of product. Thus if **a** and **b** are vectors representing length, the scalar product **a b**, the inner product [**a**|**b**], the vector **a** × **b**, the bivector [**ab**], the dyad **ab**, the quaternion **ab**, all have the dimensions $[L]^2$. It will not do for the physicist to lose sight of the fact for a moment. The followers of Grassmann, however, use the term dimension in a more geometric sense. If **a**, **b**, **c** be vectors representing lengths, **ab** is a bivector or vector of the second class (zweiter Stufe) and **abc** is a trivector, a scalar, a quantity of the third class. Again, by the law of regressive multiplication, (**abc**)**a** becomes a vector of the first class. As a matter of fact the physical dimensions of (**abc**)**a** are $[L]^4$. As a geometer one may take great delight in the theory of geometric dimensions in multiple algebra, but as a physicist he must discard it as confusing. This is precisely what is done by the physicists who use Grassmann's notation.

The whole matter may, however, be let pass with good grace inasmuch as the book is otherwise singularly free from blemishes and well fulfils the requirements for an elementary text on geometric or multiple algebras. Jahnke's lectures supplemented first with the elementary geometric theory of imaginary numbers and of quaternions and second with Gibbs's theory of dyadics, where the ideas of products are considerably more extended than anywhere else, would form an excellent introductory course on higher algebra and would furnish, in addition, a very respectable knowledge of some parts of projective geometry, mechanics, and physics.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
January, 1906.

CELESTIAL MECHANICS.

An Introduction to Celestial Mechanics. By F. R. MOULTON, Ph.D., Instructor in Astronomy in the University of Chicago. New York, The Macmillan Company, 1902. xv + 384 pp.

THE need of a work written in the English language which would properly introduce astronomical students into the subject of celestial mechanics has been keenly felt for some time. In many of our universities the teaching of astronomy is confined to practical and to what is commonly called theoretical