THE MATHEMATICAL TRIPOS OF 1906.*

Cambridge University Examination Papers, Easter term, 1906; Containing the Papers for the Mathematical Tripos, Parts I and II. Cambridge, at the University Press. Pp. 739–782. Price 2s. 6d. (For sale through Deighton, Bell and Company, Cambridge, England.)

THE main difference between the average British student of mathematics and his continental brother is summed up in the relative importance of examination and dissertation. While the examination for the German doctorate is a single oral ordeal lasting not over a couple of hours, the tripos for the first degree (baccalaureus artium) occupies several days, and the honor papers make demands on the successful candidates which could seldom be met by a young German doctor. On the other hand, the German student has been engaged on a single problem for one to two years, and his results make a more or less important contribution to the science. The student at Cambridge has concentrated all his skill on short, definite exercises, most of which, however, have a direct bearing on general mathematical development. The principal feature of the examinations is the solution of a large number of strictly original problems.

The names which have become famous among English mathematicians are almost identical with those which have stood high in the examinations, but many of these very men are now active in the movement to change their character by allowing time for the second part, and putting more emphasis on real original work, rather than on "an excessive amount of polishing of mathematical tools." †

The tripos papers of 1906 contain twenty examinations of three hours each. The number of questions varies from four to twenty, and these range in difficulty from a simple corollary in plane geometry to abstruse theorems on the outer boundary of mathematical knowledge. The papers are arranged in three groups. The first group of seven is set for all candidates for the B.A. degree with honors in mathematics. They are taken

* For an account of the development of the mathematical tripos, one should consult the History of the study of mathematics at Cambridge, by W. W. Rouse Ball, Cambridge, 1899, and the inaugural address before the London mathematical society by Dr. Glaisher, in 1888.

† See BULLETIN, volume 12, page 468.

at the end of the third year of residence, but in exceptional cases may be taken at the end of the second year. The subject matter of each examination is announced in advance; it may include a number of divisions. The questions usually consist of two parts, the first being a general theorem, frequently found in the text books, while the second part is a problem, not generally known, and probably having only a distant connection with the first part. The topics of these examinations are: (1) Euclid and conics, elementary modern geometry; (2) algebra and trigonometry; (3) analytic geometry and elementary calculus; (4) statics and dynamics; (5) geometric optics and elementary astronomy; (6) hydrostatics, heat and electricity; (7) problem paper on subjects (1) to (6). It will thus be observed that pure and applied mathematics are given equal weight. In this first part quickness and dexterity are essential to success.

After the first set of seven papers there is an interval of eleven days, at the end of which the list of honor men is published. These then take the second set of seven papers. On the result of the whole fourteen papers the candidates are arranged in order of merit as wranglers, senior optime, and junior optime. This second set of seven examinations may be roughly classified as follows: (8) analytic statics, dynamics of a particle, rigid dynamics, astronomy, hydrostatics, electricity, hydrodynamics; (9) advanced algebra (series, theory of numbers, theory of equations), symbolic operators, advanced analytics of two and of three dimensions; (10) analytic and spherical trigonometry, differential equations, definite integrals, plane curves; (11) analytic statics, dynamics, elasticity, hydrodynamics, electricity and magnetism; (12) theory of functions, elliptic functions, harmonic analysis (Bessel's functions, Fourier's functions, etc.); (13) a mixed paper, both pure and applied; (14) problem paper on (8) to (13). Here again, pure and applied mathematics are given equal weight.

About a third of the wranglers take the second part of the tripos at the end of their fourth year. In the first part a student is required to have a good general knowledge of the whole field of pure and applied mathematics, but in the second part there are four divisions of pure mathematics and four divisions of applied mathematics. The student chooses any two of these eight divisions for further specification. The questions embody the latest work in each division, and the student's rank is determined by the quality of his work in one direction. Sample papers are here added in illustration of the general character of the examination.*

PART I. SIXTH PAPER.

1. Find the conditions of equilibrium satisfied by a body which is wholly or partly immersed in a fluid, and free to turn about a fixed point.

A semi-circular lamina has one of the ends of its diameter smoothly hinged to a fixed point above the surface of a liquid, and floats with its plane vertical and its diameter half immersed. If the inclination of the diameter to the horizon is $\pi/4$, prove that the ratio of the density of the liquid to that of the lamina is $4(3\pi - 4) : 9\pi - 8$.

2. Find the center of pressure of a rectangle, completely immersed in a liquid, with one of its sides horizontal.

A rectangular block whose edges are of lengths $2a$, $2b$, $2c$ is divided by a plane through the center perpendicular to the edges of length $2c$, and the two halves are hinged together along edges parallel to those of length $2a$. The whole is then immersed in a liquid with the line of hinges inclined at an angle θ to the horizon and the dividing plane vertical, the hinges being in the upper face. Prove that the two halves will not separate unless

$$\left\{\left(1 - \frac{\sigma}{\rho}\right)c^2 - \tfrac{2}{3}b^2\right\} \cos\theta > 2bd,$$

where d is the depth of the center of gravity of the block, σ is the density of the block, and ρ that of the liquid.

3. A cylindrical diving-bell is lowered to a given depth below the surface of a liquid. Find the height to which the liquid rises inside the bell.

A cube of density ρ_0 floats on a liquid of density ρ, and a cylindrical diving-bell, of length l and cross section A, is placed over it and lowered until the cube just touches the top. Show that the height to which the liquid rises inside the bell is

$$[A(\rho - \sigma) - \rho a^2]\,[l(\rho - \sigma) - a(\rho - \rho_0)]\,/\,[\rho(\rho - \sigma)(A - a^2)],$$

where σ is the density of the atmosphere, and a is the length of an edge of the cube.

4. Describe Smeaton's air-pump, and find the density in the receiver after n strokes.

A barometer tube, in the form of a flat-topped cylinder, is placed with its lower end in a vessel containing mercury, and the air is pumped out. Prove that, if x_n is the distance of the mercury from the top of the tube after n strokes,

$$(x_n - h)(ax_n + u) = (x_{n-1} - h)ax_{n-1},$$

where a is the cross section of the tube, u the volume of the barrel of the pump, and h the ultimate value of x_n.

5. Define the terms 'latent heat' and 'specific heat,' and show how the specific heat of a substance may be determined by observing the alteration in temperature of a given mass of water, into which a given mass of the substance at a given temperature is dropped.

A closed vessel of mass 40 grams and specific heat .095 is filled with 4.4 grams of a gas, the whole being at temperature 20° C. The vessel is then enclosed in a chamber through which steam at 100° C. is passed until the temperature becomes uniform, and the increase in mass is found to be .68 gram. Prove that the specific heat of the gas under the given conditions is .173, the latent heat of steam at 100° C. being 536.5.

6. Define *potential*, and find the force and the potential at any point due to a freely electrified spherical conductor.

A curve is drawn on the surface of a freely electrified oval conductor surrounded by air, dividing it into two parts on which the charges are E and E' respectively. The space, enclosed by the portion of the surface of the conductor on which the charge is E' and by the lines of force drawn from every point of the curve to infinity, is filled with a dielectric of S. I. C. K. Prove that the potential of the conductor is now diminished in the ratio $E + E' : E + KE'$.

7. Prove that the energy of a system of charged conductors is $\frac{1}{2}\Sigma EV$.

A condenser is formed of two parallel plates. Prove that, when the potentials of the plates are kept constant, the work done by the system in a small displacement is equal to the increase in the energy of the system.

8. Define the coefficients of potential of a system of conductors, and prove that $p_{rs} = p_{sr}$. Under what circumstances does $p_{rr} = p_{rs}$?

Three conductors A_1, A_2 and A_3 are such that A_3 is practically inside A_2. A_1 is alternately connected with A_2 and A_3 by means of a fine wire, the first contact being with A_3. A_1 has a charge E initially, A_2 and A_3 being uncharged. Prove that the charge on A_1, after it has been connected with A_3 n times, is

$$\frac{E\beta}{\alpha+\beta}\left\{1+\frac{\alpha(\gamma-\beta)}{\beta(\alpha+\gamma)}\left(\frac{\alpha+\beta}{\alpha+\gamma}\right)^{n-1}\right\},$$

where α, β and γ stand for $p_{11}-p_{12}$, $p_{22}-p_{12}$ and $p_{33}-p_{12}$ respectively.

9. State and prove Kirchhoff's laws as to the distribution of electric currents in a system of linear conductors containing given electromotive forces.

A square $ABCD$ is formed of a uniform piece of wire, and the center is joined to the middle points of the sides by straight wires of the same material and cross section. A current is taken in at A and is drawn off at the middle point of BC. Prove that the equivalent resistance of the network between these points is $\frac{29}{48}$ of that of a side of the square.

10. Describe the tangent galvanometer.

A given current sent through the galvanometer deflects the magnet through an angle θ. The plane of the coil is slowly rotated round the vertical axis through the center of the magnet. Prove that, if $\theta > \frac{1}{4}\pi$, the magnet will describe complete revolutions, but if $\theta < \frac{1}{4}\pi$, the magnet will oscillate through an angle $\sin^{-1}(\tan\theta)$ on each side of the meridian.

Part I. Tenth Paper.

1. If $\sum_0^\infty u_n$ is a semi-convergent series of real terms, prove that the infinite product $\prod_0^\infty (1+u_n)$ tends towards a positive limit or towards zero according as $\sum_0^\infty u_n^2$ is convergent or divergent.

If $u_0 = u_1 = u_2 = 0$ and when $n > 1$

$$u_{2n-1} = \frac{-1}{\sqrt{n}}, \qquad u_{2n} = \frac{1}{\sqrt{n}} + \frac{1}{n} + \frac{1}{n\sqrt{n}},$$

show that $\sum_0^\infty u_n$ and $\sum_0^\infty u_n^2$ are both divergent, but that $\prod_0^\infty (1 + u_n)$ is convergent.

2. Show that a rational integral congruence of degree n cannot have more than n roots incongruent to a prime modulus.

Prove that, if n is an odd prime,

$$x^{n-1} - 1 - [x^2 + 1(n-1)]\,[x^2 + 2(n-2)] \cdots [x^2 + \tfrac{1}{2}(n-1)\tfrac{1}{2}(n+1)] \equiv 0 \pmod{n}$$

is an identical congruence.

What arithmetic theorems result from this?

3. Investigate the solution of the quartic $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$, and obtain the auxiliary cubic in the form $4a^3\theta^3 - Ia\theta + J = 0$.

If ϕ is the cross-ratio of the four roots, show that the correspondence between θ and ϕ may be written in the forms

$$(\phi+1)^2 I = 12a^2\theta^2(\phi^2-\phi+1), \quad (\phi+1)^2 J = 4a^3\theta^3(2\phi - 1)(\phi - 2).$$

4. Show that the hessian of a quantic is a covariant of that quantic.

If a ternary cubic is the product of three linear factors, prove that its hessian is the product of the cubic with a constant.

5. If u is a homogeneous function of $x_0, x_1, x_2, \cdots, x_m$ of degree n prove that

$$\left(x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} + \cdots + x_m \frac{\partial}{\partial x_m} \right) u = nu.$$

Let

$$\Omega \equiv x_0 \frac{\partial}{\partial x_1} + 2x_1 \frac{\partial}{\partial x_2} + 3x_2 \frac{\partial}{\partial x_3} + \cdots + mx_{m-1} \frac{\partial}{\partial x_m},$$

and

$$O \equiv mx_1 \frac{\partial}{\partial x_0} + (m-1)x_2 \frac{\partial}{\partial x_1} + (m-2)x_3 \frac{\partial}{\partial x_2} + \cdots + x_m \frac{\partial}{\partial x_{m-1}},$$

and let G be a homogeneous algebraic function of $x_0, x_1, \cdots, x_m$ of degree n such that the sum of the suffixes in each term is w.

Prove that $(\Omega O - O\Omega) G = (mn - 2w) G$.

6. If x, y, z are connected by a single equation, express

$$\frac{\partial z}{\partial x},\quad \frac{\partial z}{\partial y},\quad \frac{\partial^2 z}{\partial x^2},\quad \frac{\partial^2 z}{\partial x \partial y},\quad \text{and}\quad \frac{\partial^2 z}{\partial y^2}$$

in terms of differential coefficients of x with respect to y and z.

If

$$\frac{\partial(Z, z)}{\partial(x, y)} = 0, \quad \text{where} \quad \frac{\partial^2 z}{\partial x \partial y} - Z \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = 0,$$

show that the same relations hold good on permuting the letters, that is to say

$$\frac{\partial(X, x)}{\partial(y, z)} = 0, \quad \text{where} \quad \frac{\partial^2 x}{\partial y \partial z} - X \frac{\partial x}{\partial y} \frac{\partial x}{\partial z} = 0, \quad \text{etc.}$$

7. Find the equation of a pair of tangents to $x^2/a^2+y^2/b^2=1$, the chord of contact being $\lambda x + \mu y + \nu = 0$.

Find the equation of a circle touching these two tangents and meeting the ellipse in the chord $x \cos \phi + y \sin \phi = p$, concurrent with the chords of contact; and if ϕ is the constant a, show that the locus of the intersection of the above-mentioned tangents is the conic

$$x^2/(a^2 \cos^2 \alpha) - y^2/(b^2 \sin^2 a) = (a^2 - b^2)/(a^2 \cos^2 a + b^2 \sin^2 a),$$

that is to say, an hyperbola confocal with a given ellipse.

8. Find the point equation of the circular points at infinity in any system of homogeneous coördinates, and deduce the corresponding tangential equation.

A parabola circumscribes a triangle, show that its axis touches a curve of the third class (a three-cusped hypocycloid) given by

$$\lambda^2 \frac{\partial \Omega}{\partial \lambda} + \mu^2 \frac{\partial \Omega}{\partial \mu} + \nu^2 \frac{\partial \Omega}{\partial \nu} = 0,$$

and the tangent at the vertex always touches the curve of the sixth class defined by the rationalized form of

$$\lambda^{\frac{1}{2}} \frac{\partial \Omega}{\partial \lambda} + \mu^{\frac{1}{2}} \frac{\partial \Omega}{\partial \mu} + \nu^{\frac{1}{2}} \frac{\partial \Omega}{\partial \nu} = 0,$$

where the line at infinity is $x + y + z = 0$ in point-coördinates, and the rational tangential equation of the circular points at infinity is $\Omega = 0$.

9. Find the equation of a normal at any point of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,$$

and deduce that normals meeting in (x_0, y_0, z_0) lie on the cone

$$\frac{x_0}{x-x_0}(b^2-c^2)+\frac{y_0}{y-y_0}(c^2-a^2)+\frac{z_0}{z-z_0}(a^2-b^2)=0.$$

If (x_0, y_0, z_0) lies on the ellipsoid and tangent planes be drawn to the ellipsoid at the five feet of normals other than (x_0, y_0, z_0), then the feet of the perpendiculars from the center on these tangent planes all lie in the tangent plane at $(-x_0, -y_0, -z_0)$, and are all situated on a rectangular hyperbola passing through the foot of the perpendicular from the center on the same plane.

10. Find the principal planes of the quadric

$$S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy-1=0.$$

If $X \equiv ax+hy+gz$, $X' \equiv a'x+h'y+g'z$, etc., show that the set of conjugate diametral planes which $S=0$, $S'=0$ have in common is determined by the equation

$$\frac{\partial(S, S', S'')}{\partial(x, y, z)}=0,$$

where S'' is given by either $(A, B, C, F, G, H)(X', Y', Z')^2$ or $(A', B', C', F', G', H')(X, Y, Z)^2$.

Part II. Divisions II, IV.

1. Prove that covariants of binary forms can be expressed in terms of symbolical factors of the type (ab) and a_x, and explain how to extend the result to ternary forms.

Prove that for binary forms the minimum weight of a perpetuant of degree four is seven, and explain how to extend the result to higher degrees.

2. A line l meets a conic S_1 in two conjugate imaginary points, P, Q. O is a point on a conic S_2 and OP, OQ meet S_2 in $P'Q'$; obtain a real construction for the line $P'Q'$.

One pair of opposite vertices of a quadrilateral are real and a second pair are conjugate and imaginary. Give a real construction for the third diagonal and the third pair of vertices.

State, in terms of cross ratios, the proposition that the angle at the center of a circle is double that at the circumference on the same arc, and give a direct proof of the statement.

3. Give an account of the properties of the twenty-seven

lines on a cubic surface, and explain how to deduce properties of the bitangents of a plane quartic curve.

Show that there exist sets of six bitangents which touch the same conic.

4. Prove the fundamental theorems relating to the parametric expressions for the coördinates of a point on a plane curve of deficiency 0 or 1.

In the case of a plane cubic find how many lines can be drawn such that, at each point where they meet the curve, a conic can be drawn having six-point contact with the curve.

Indicate briefly how the equation of any elliptic curve can be reduced to the form

$$y^2 = 4x^3 - g_2x - g_3.$$

VIRGIL SNYDER.

SHORTER NOTICES.

Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert. Bericht der Deutschen Mathematiker-Vereinigung, erstattet von MAX SIMON. Mit 28 Figuren im Text. Leipzig, B. G. Teubner, 1906. viii + 278 pp. Price, 8 marks.

Methodik der Elementaren Arithmetik in Verbindung mit Algebraischer Analysis. Von MAX SIMON. Mit 9 Textfiguren. Leipzig, B. G. Teubner, 1906. vi + 108 pp. Price, 3 marks 20 pf.

AMONG those occupying chairs of mathematics in the German universities there is no one who takes greater interest in the work of the secondary teacher than Professor Simon. He is earnest in his advocacy of reform, zealous in his application of the history of mathematics to the principles of teaching, and full of that good-humored argument that makes a man acceptable as a speaker in an assembly of teachers. Therefore it comes about that Professor Simon is able to command appreciative audiences for his addresses and a goodly circle of readers for his numerous literary efforts. It is rather in his work as a speaker, however, that he is most successful. His fund of enthusiasm, his genial countenance, and his action in address, all tend to lead his hearers to consider his arguments as wholes, without attending to minor inaccuracies of language or of state-