NOTE ON FOURIER'S CONSTANTS.

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A function $f(x)$ real and single valued, bounded, and in the sense of Riemann integrable from 0 to $2\pi$ gives rise to the Fourier's series

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \frac{1}{2} \sum_{k=-\infty}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos kx \, dx,$$

$$b_k = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin kx \, dx.$$  \[2\]  \[\text{(k = 0, ± 1, ...)}\]

Hurwitz * calls the constants $a_k, b_k$ the Fourier's constants of the integrable function $f(x)$, and for the realm of integrable functions he studies the theory of Fourier's constants instead of the theory of Fourier's series which are convergent only under conditions.

The realm of integrable functions is a realm of integrity, i.e., the sum, the difference, and the product of two integrable functions is integrable. The constants $a_k, b_k$ or in more explicit notation $a_{kr}, b_{kr}$, depend in a linear distributive way on the function $f = f(x)$. Further, the constants $a_{kr}, b_{kr}$ for the product $h(x) = f(x) g(x)$ of two functions $f(x), g(x)$ are determinable from the constants $a_{kr}, b_{kr}$ and $a_{kr}, b_{kr}$ of these functions. In fact, from (2) we have

$$a_{kr} = \frac{1}{\pi} \int_{0}^{2\pi} f(x) g(x) \cos kr \, dx,$$

$$b_{kr} = a_{kr, f(x) \cos kr} b_{kr} = a_{kr, f(x) \sin kr}.$$  \[3\]  \[4\]

so that

\[ a_{kh} = a_{0,fg} \cos kx, \quad b_{kh} = a_{0,fg} \sin kx. \]

These formulas in effect make the expression of \( a_{kh} \), \( b_{kh} \) depend simply upon the expression of \( a_{0h} \) in terms of the constants of \( f \) and \( g \). The fundamental theorem of the theory of Fourier's constants states that

\[ a_{0h} = \frac{1}{\pi} \int_0^{2\pi} f(x)g(x)dx = \frac{1}{2}a_{0g}a_{0g} + \sum_{k=1}^{\infty} (a_{kh}a_{kg} + b_{kh}b_{kg}). \]

If either \( f(x) \) or \( g(x) \) is expressible as a uniformly convergent Fourier series, the relation (6) follows by termwise integration. In its generality the fundamental theorem with important applications is due to de la Vallée Puissou (*1893), and it has been extended (1903) by Stekloff to all the types of Fourier constants frequently employed in analysis in connection with the representation of arbitrary functions of one or of several variables.

Hurwitz (loc. cit., volume 57, page 437) expresses the constants of \( h(x) = f(x)g(x) \) in terms of those of \( f(x), g(x) \), apart from notation, as follows:

\[ a_{0h} = \frac{1}{2}a_{0g}a_{0g} + \sum_{k=1}^{\infty} (a_{kh}a_{kg} + b_{kh}b_{kg}); \]

\[ a_{nh} = \frac{1}{2}a_{0g}a_{ng} + \frac{1}{2} \sum_{k=1}^{\infty} \{a_{kh}(a_{k+n} + a_{k-n}) + b_{kh}(b_{k+n} + b_{k-n})\}, \]

\[ b_{nh} = \frac{1}{2}a_{0g}b_{ng} + \frac{1}{2} \sum_{k=1}^{\infty} \{a_{kh}(b_{k+n} - b_{k-n}) - b_{kh}(a_{k+n} - a_{k-n})\}. \]

Hurwitz remarks further that these series converge absolutely, even when the terms at present collected into binomial expressions are taken as the constituent terms of the series. In the formulas (7) the rôles of \( f \) and \( g \) may of course be interchanged.

The series on the right of the equations (7) should however be expressed symmetrically with respect to \( f \) and \( g \).

Let us use as Fourier constants

\[ A_k' = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos kx \, dx, \quad B_k' = \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin kx \, dx, \]

* For references see the memoirs of Hurwitz
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viz., the halves of the constants according to Hurwitz. Then with use of the relations

\[(2^\circ)\]

\[A_{-k'} = A_{k'}, \quad B_{-k'} = -B_{k'} \]

one readily finds from (7)

\[
A_{nk} = \sum_{k+l=n} (A_{k'}A_{l'} - B_{k'}B_{l'}),
\]

\[
B_{nk} = \sum_{k+l=n} (A_{k'}B_{l'} + B_{k'}A_{l'}),
\]

These equations (7) have the desired symmetric form. The form suggests moreover the introduction of the complex Fourier constants

\[(2a)\]

\[C_{k'} = A_{k'} + jB_{k'} \quad (j^2 = -1),\]

in terms of which we have

\[(7a)\]

\[C_{nk} = \sum_{k+l=n} C_{k'}C_{l'} \quad (n = 0, \pm 1, \ldots).\]

This result may be expressed thus:

To every integrable function \(f(x)\) there corresponds a system of Fourier constants \(A_{k'}, B_{k'}, C_{k'} (2, 2a)\) and a Laurent series with coefficients \(C_{k'}\)

\[f(x) \sim \sum_{k=-\infty}^{+\infty} C_{k'} Z^k.\]

This Laurent series is to be considered formally. Then the Laurent series corresponding to the product of two functions is the formal product of the two Laurent series respectively to the two functions.

As thus expressed, the fundamental theorem of the theory of Fourier's constants points to the known expression of Fourier series for suitably conditioned functions as convergent Laurent series.

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