CRITERIA FOR THE IRREDUCIBILITY OF A RECIPROCAL EQUATION.

BY PROFESSOR L. E. DICKSON.

(Read before the Chicago Section of the American Mathematical Society, April 17, 1908.)

1. A reciprocal equation \( f(x) = x^m + \cdots = 0 \) is one for which

\[
x^m f(1/x) = cf(x).
\]

Replacing \( x \) by \( 1/x \), we see that \( f = c^2 f \), \( c = \pm 1 \). Now \( f(x) \) has the factor \( x \pm 1 \) and hence is reducible, unless \( m \) is even and \( c = +1 \). Further discussion may therefore be limited to equations

\[
(1) \quad F(x) = x^{2n} + c_2 x^{2n-2} + \cdots + c_2 x^2 + c_1 x + 1 = 0
\]

of even degree and having

\[
(2) \quad x^{2n} F(1/x) = F(x).
\]

Let \( R \) be a domain of rationality containing the \( c \)'s.

Under the substitution

\[
(3) \quad x + 1/x = y,
\]

\( x^{2n} F(x) \) becomes a polynomial in \( y \),

\[
(4) \quad \phi(y) = y^n + k_1 y^{n-1} + \cdots + k_n,
\]

with coefficients in \( R \). By a suitable choice of the \( c \)'s, the \( k \)'s may be made equal to any assigned values.

We shall establish in §§ 2–7 the following:

**Theorem.** Necessary and sufficient conditions for the irreducibility of \( F(x) \) in the domain \( R \) are

(I) \( \phi(y) \) must be irreducible in \( R \).

(II) \( F(x) \) must not equal a product of two distinct irreducible functions of degree \( n \).

The second condition is discussed in §§ 8–10.

2. The irreducibility of \( F(x) \) in \( R \) implies that of \( \phi(y) \). For, if
\[ \phi(y) = (y + \cdots)(y^m + \cdots), \quad (l + m = n, \quad l > 0, \quad m > 0), \]
then would
\[ F'(x) = x'^{(x + 1/x)' + \cdots} \cdot x^m{(x + 1/x)^m + \cdots}. \]

3. If \( F'(x) \) has in \( R \) an irreducible factor
\[ A(x) = x^{2r+1} + a_{2r}x^{2r} + \cdots + a_2x + a_{2r+1} \]
of odd degree, then \( F'(x) \) has the irreducible factor.
\[ B(x) = \frac{x^{2r+1}}{a_{2r+1}} A\left(\frac{1}{x}\right) = x^{2r+1} + \frac{a_{2r}}{a_{2r+1}} x^{2r} + \cdots + \frac{a_1}{a_{2r+1}} x + \frac{1}{a_{2r+1}}, \]
ot identical with \( A(x) \). For, from \( F \equiv A Q \) and (2) follows
\[ F(x) \equiv B(x)Q', \quad Q' \equiv a_{2r+1}x^{2n-2r-1} Q(1/x). \]
Next, if \( B \equiv A \), then
\[ A = x^{2r+1} \pm 1 + a_r(x^{2r-1} \pm 1) + a_{2r}x^{2r-3} \pm 1 + \cdots + a_{2r}x(x \pm 1), \]
so that \( A \) would have the factor \( x \pm 1 \) and be reducible.

4. If \( \phi(y) \) is irreducible in \( R \), \( F(x) \) has in \( R \) no irreducible factor \( A \) of odd degree < \( n \). For, if so, \( P \equiv AB \), where \( B \) is given in § 3, would be a self-reciprocal factor of \( F(x) \). In fact,
\[ P(1/x) = A(1/x)A(x)x^{2r+1}a_{2r+1}, \quad x^{2(2r+1)} P(1/x) = AB = P(x). \]
Hence, in view of (3), \( x^{-(2r+1)} P(x) \) would equal a factor of degree \( 2r + 1 \) of \( \phi(y) \).

5. If \( F(x) \) has in \( R \) an irreducible factor
\[ A(x) = x^{2r} + a_{2r}x^{2r-1} + \cdots + a_{2r-1} x + a_2 \]
of even degree, then \( F(x) \) has the irreducible factor
\[ B(x) = \frac{1}{a_{2r}} x^{2r} A\left(\frac{1}{x}\right). \]
If \( B(x) \equiv A(x) \), \( A \) is self-reciprocal, viz.,
\[ A(x) = x^{2r} + 1 + a_1(x^{2r-1} + x) + \cdots + a_{r-1}(x^{r+1} + x^{r-1}) + a_rx. \]
In fact, the conditions for $B = A$ are 

$$a_{2r} = \pm 1, \quad a_{2r-1} = \pm a_1, \quad a_{2r-2} = \pm a_2, \ldots.$$ 

For the lower signs, $A$ has the factor $x^2 - 1$, contrary to hypothesis.

6. If $\phi(y)$ is irreducible in $R$, $F(x)$ has in $R$ no irreducible factor $A$ of even degree $< n$. For, by § 5, either $B$ is distinct from $A$ so that $AB$ is a self-reciprocal factor of $F(x)$, or else $A$ itself is a self-reciprocal factor. In either case $\phi(y)$ would have in $R$ a factor of degree $< n$.

7. It follows from §§ 4, 6 that, when $\phi(y)$ is irreducible in $R$, $F(x)$ has no irreducible factor of degree $< n$. Further, by §§ 3, 5, an irreducible factor $A(x)$ of degree $n$ implies a second irreducible factor $x^nA(1/x)$, algebraically distinct from $A(x)$. The theorem of § 1 is therefore proved.

8. It remains to consider the case $F' = AB$, 

$$A = x^n + a_1x^{n-1} + \cdots + a_n,$$

$$B = x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \cdots + \frac{a_1}{a_n}x + \frac{1}{a_n},$$

where $A$ and $B$ are distinct irreducible functions in $R$. To determine the $a_i$ we have $n$ distinct relations

$$(5) \quad a_1 + \frac{a_{n-1}}{a_n} = c_1, \quad a_2 + \frac{(a_1a_{n-1} + a_{n-2})}{a_n} = c_2, \ldots$$

We may eliminate $a_1, \ldots, a_{n-1}$ and obtain an equation for $a_n$. As shown in § 9, this equation is of degree $2^n$. Except for certain sets of values of the $c_i$, we may express $a_1, \ldots, a_{n-1}$ rationally in terms of $a_n$; the problem is then reduced to the consideration of the rationality of a root of the equation of degree $2^n$. This equation for $a_n$ is a reciprocal equation. In fact, if we set

$$A_1 = a_{n-1}/a_n, \quad \ldots, \quad A_{n-1} = a_1/a_n, \quad A_n = 1/a_n,$$

equations $(5)$ become

$$\begin{align*}
(5') A_1 + A_{n-1}/A_n &= c_1, \\
A_2 + (A_1A_{n-1} + A_{n-2})/A_n &= c_2, \ldots
\end{align*}$$

That the equations $(5')$ are throughout of the same form as equations $(5)$ is evident from the fact that we have merely interchanged the rôles of the factors $A$ and $B$ of $F'$. Hence the equation in $a_n$, obtained by eliminating $a_1, \ldots, a_{n-1}$ from
is identical with the equation in $A_n = 1/a_n$, obtained from $(5')$.

9. Denote the roots of $F = 0$ by

$$(6) \quad a_1, \; a_1^{-1}, \; a_2, \; a_2^{-1}, \; \ldots, \; a_n, \; a_n^{-1}.$$ 

A factorization $F = AB$, of the kind considered in § 8, corresponds uniquely to a separation of the roots (6) into two sets each of $n$ roots, such that reciprocal roots belong to different sets. Hence the roots of the first set may be selected in

$$\frac{2n(2n-2)(2n-4) \cdots 2}{n!} = 2^n$$

ways. The number of factors $A$ is thus $2^n$.

10. For $n = 2$, we set $a_1 = \alpha$, $a_2 = \beta$, and have

$$(7) \quad \alpha + \beta + \alpha^{-1} + \beta^{-1} = c_1, \quad 2 + \alpha \beta^{-1} + \beta \alpha^{-1} + \alpha \beta + \alpha^{-1} \beta^{-1} = c_2.$$ 

From these we derive

$$\alpha^2 + \beta^2 + \alpha^{-2} + \beta^{-2} = c_1^2 - 2c_2.$$ 

Hence $\alpha \beta + \alpha^{-1} \beta^{-1}$ and $\alpha \beta^{-1} + \alpha^{-1} \beta$ are the roots of

$$(8) \quad z^2 - (c_2 - 2)z + c_1^2 - 2c_2 = 0.$$ 

The quartic for $a_2(§8)$ is obtained by setting

$$(9) \quad z = a_2 + a_2^{-1}.$$ 

By (7), $\alpha + \beta$ is a rational function of $\alpha \beta$ and $c_1$ when $c_1 \neq 0,$ Hence, for $c_1 = 0$, the necessary and sufficient conditions for the factorization $F = AB$ in $R$ are that the roots

$$(10) \quad z_n = \frac{1}{2}(c_2 - 2) \pm [(1 + \frac{1}{2}c_2)^2 - c_1^2]^{\frac{1}{2}}$$

of (8) be rational and that one of the values $(z_n^2 - 4)^{\frac{1}{2}}$ be rational, so that (9) shall lead to a rational value of $a_2.$ Incorporating the condition that (4) shall be irreducible in $R$, we obtain the

**Theorem.** The necessary and sufficient conditions that

$$a^4 + c_1 a^3 + c_2 a^2 + c_1 a + 1 \quad (c_1 \neq 0)$$

*For other proofs by the writer, see Amer. Math. Monthly, vol. 10 (1903), p. 221; vol. 15 (1908), p. 75. The first paper cited also treats reciprocal sextic equations.*
shall be irreducible in a domain $R$ are that $(c_1^2 - 4c_2 + 8)^{\frac{1}{2}}$ be irrational, and that either $l = [(1 + \frac{1}{2}c_1)^2 - c_2^2]^{\frac{1}{2}}$ be irrational or else $l$ rational and $[\frac{1}{2}c_1^2 - c_2^2 - 2 \pm (c_2 - 2)l]^{\frac{1}{2}}$ both irrational.

11. The only linear fractional transformations which replace a reciprocal equation by a reciprocal equation are

$$x' = \pm \frac{ax + \beta}{\beta x + \alpha} \quad (a^2 \neq \beta^2).$$

Then $y$, given by (3), undergoes the transformation

$$y' = \pm \frac{(a^2 + \beta^2)y + 4\alpha \beta}{\alpha \beta y + a^2 + \beta^2}.$$

The transformation on $\frac{1}{2}y$ is the square of (11).

University of Chicago, March, 1908.

A NEW GRAPHICAL METHOD FOR QUATERNIONS.

By Professor James Byrnie Shaw.

(Read before the Southwestern Section of the American Mathematical Society, November 30, 1907.)

1. Any quaternion $q$ may be written in the form $q = (w + xi) + (y + zi)j$. For convenience let us represent numbers of the form $w + xi$ (practically equivalent to ordinary complex numbers save in their products by $j$) by Greek characters, so that $q$ may be written

$$q = \alpha + \beta j,$$

where for any number $\beta$ we have $\beta j = j\beta$, $\beta$ being the conjugate of $\beta$.

The tensor of $q$ is then the square root of the sum of the squares of the moduli of $\alpha$, $\beta$. Also the scalar of $q$ is $\frac{1}{2}(\alpha + \overline{\alpha})$, that is, the real part of $\alpha$.

2. The product of $q = \alpha + \beta j$ and $r = \gamma + \delta j$ is

$$qr = (\alpha \gamma - \beta \delta) + (\alpha \delta + \beta \gamma)j,$$

and also we have

$$rq = (\alpha \gamma - \beta \delta) + (\alpha \delta + \beta \gamma)j.$$