AN APPLICATION OF THE NOTIONS OF "GENERAL ANALYSIS" TO A PROBLEM OF THE CALCULUS OF VARIATIONS.

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The object of the following note is to give an illustration of the unifying power of Professor E. H. Moore's methods of "General Analysis"* by showing that a certain theorem of the calculus of variations and a certain theorem of analytic geometry are special cases of one and the same theorem of general analysis.

The theorem of the calculus of variations is the so-called fundamental lemma for isoperimetric problems,† viz.,

**Theorem I.** "If

\begin{equation}
\mu_0(\eta) = \int_{x_1}^{x_2} \left[ M_0(x)\eta(x) + N_0(x)\eta'(x) \right] dx = 0
\end{equation}

for all functions \( \eta(x) \) which are (a) of class \( C' \) on \([x_1, x_2] \), (b) vanish at \( x_1 \) and \( x_2 \), and (c) satisfy the \( m \) conditions

\begin{equation}
\mu_i(\eta) = \int_{x_1}^{x_2} \left[ M_i(x)\eta(x) + N_i(x)\eta'(x) \right] dx = 0
\end{equation}

\((i = 1, 2, \ldots, m)\),

then there exist \( m \) constants \( c_1, c_2, \ldots, c_m \) such that

\begin{equation}
\mu_0(\eta) + c_1\mu_1(\eta) + c_2\mu_2(\eta) + \cdots + c_m\mu_m(\eta) = 0
\end{equation}

for all functions \( \eta(x) \) satisfying conditions (a) and (b).

The functions \( M(x), N(x) \) are supposed to be continuous on \([x_1, x_2]\).

The theorem of analytic geometry is the well known

†Compare for instance Bolza, Vorlesungen über Variationsrechnung, p. 462, footnote 1, and the references given there.
THEOREM II. "If, in a plane and in homogeneous coordinates,
\[ U_0 \equiv A_0 x + B_0 y + C_0 z = 0 \]
is the equation of a straight line passing through the point of intersection of the two non-coinciding* lines
\[ (1') \]
\[ U_1 \equiv A_1 x + B_1 y + C_1 z = 0, \quad U_2 \equiv A_2 x + B_2 y + C_2 z = 0, \]
then there exist two constants \( \lambda_1, \lambda_2 \) such that
\[ U_0 \equiv \lambda_1 U_1 + \lambda_2 U_2. \]

§ 1. The General Theorem.

Let \( \mathcal{P} \) be a general parameter \( \dagger \) ranging over a set \( \mathcal{Q} \) of elements; these elements may be any mathematical entities whatever: real or complex numbers, pairs, triples, etc., of such numbers, even infinite sets of numbers; functions of one or several variables; systems of functions; points, curves, surfaces; etc., etc.

Along with the set \( \mathcal{P} \) we consider the set \( \mathcal{Q} \) of all possible systems \( (a_1, a_2; p_1, p_2) \) of a pair of real numbers \( a_1, a_2 \) and a pair of elements \( p_1, p_2 \) of \( \mathcal{P} \), and we suppose that a correspondence has been established by which to every element of \( \mathcal{Q} \) corresponds a unique element of \( \mathcal{Q} \) which we denote by \( F(a_1, a_2; p_1, p_2) \).

We shall then say that a real single-valued function \( \mu(p) \) defined on \( \mathcal{Q} \) is "linear as to \( F, \)" if
\[ \mu[F(a_1, a_2; p_1, p_2)] = a_1 \mu(p_1) + a_2 \mu(p_2) \quad \text{on } \mathcal{Q}, \]
i.e., for every combination \( (a_1, a_2; p_1, p_2) \) of \( \mathcal{Q} \).

Then the following theorem holds: ||

THEOREM III. If
\[ \mu_0(p), \mu_1(p), \ldots, \mu_m(p) \]

* We may omit the word "non-coinciding" if we replace "point of intersection of" by "point or points common to."

\( \dagger \) Compare Moore, "Introduction etc.," § 1; I use throughout this section Moore's notation.

\( \ddagger \) In Moore's terminology \( F \) is a "function on \( \mathcal{Q} \) to \( \mathcal{P}, \)" "Introduction etc.," § 4.

§ Compare Moore, "Introduction etc.," § 5; if \( \mathcal{P} \) denotes the set of all real numbers, \( \mu(p) \) is in Moore's terminology a "function on \( \mathcal{P} \) to \( \mathcal{Q}, \)"

|| This generalization of Theorem I has been suggested to me by a remark in § 177 of Hadamard's Leçons sur le calcul des variations, Paris, 1910.
are $m + 1$ real single-valued functions of $p$, defined on $\mathcal{P}$, which satisfy the following two conditions:

A) they are linear as to $F$,

B) the equation

$$(1'') \quad \mu_0(p) = 0$$

holds for every element of $\mathcal{P}$ which satisfies simultaneously the $m$ equations

$$(2'') \quad \mu_i(p) = 0, \quad i = 1, 2, \ldots, m,$$

then there exist $m$ real numbers $c_1, c_2, \ldots, c_m$, independent of $p$, such that

$$(3'') \quad \mu_i(p) + c_1\mu_1(p) + \cdots + c_m\mu_m(p) = 0 \quad \text{on } \mathcal{P},$$

i. e., for every element of $\mathcal{P}$.

Proof: We notice first that there always exist elements of $\mathcal{P}$ which do satisfy the $m$ equations (2''); for $F(0, 0; p_1, p_2)$ is an element of $\mathcal{P}$ for any two elements $p_1, p_2$ of $\mathcal{P}$, and on account of A)

$$\mu_i[F(0, 0; p_1, p_2)] = 0, \quad (i = 1, 2, \ldots, m).$$

Further we observe that if we define

$$F[1, a_3; F(a_1, a_2; p_1, p_2), p_3] = F(a_1, a_2, a_3; p_1, p_2, p_3)$$

and generally

$$(5) \quad F[1, a_n; F(a_1, a_2, \ldots, a_{n-1}; p_1, p_2, \ldots, p_{n-1}), p_n] = F(a_1, a_2, \ldots, a_n; p_1, p_2, \ldots, p_n),$$

then $F(a_1, a_2, \ldots, a_n; p_1, p_2, \ldots, p_n)$ is again an element of $\mathcal{P}$, and, if (4) is satisfied, then also

$$(6) \quad \mu[F(a_1, a_2, \ldots, a_n; p_1, p_2, \ldots, p_n)] = a_1\mu(p_1) + a_2\mu(p_2) + \cdots + a_n\mu(p_n).$$

After these preliminary remarks we distinguish two cases:

Case I: The $m$ equations (2'') are satisfied for every $p$ of $\mathcal{P}$.

Then according to B)

$$\mu_0(p) = 0 \quad \text{on } \mathcal{P}.$$

Hence we may write

$$\mu_0(p) + 0 \cdot \mu_1(p) + 0 \cdot \mu_2(p) + \cdots + 0 \cdot \mu_m(p) = 0 \quad \text{on } \mathcal{P},$$
and the theorem is proved with the particular values \( c_1 = 0, c_2 = 0, \ldots, c_m = 0 \).

**Case II:** The \( m \) equations (2") are not all satisfied for every \( p \) of \( \mathfrak{P} \).

Then there exists a definite integer \( n \) (1 \( \leq \) \( n \) \( \leq \) \( m \)) such that in the determinant

\[
\Delta = |\mu_i(p_h)| \quad (i, k = 1, 2, \ldots, m)
\]

at least one minor of degree \( n \) is different from zero for some special system \( p_1, p_2, \ldots, p_m \), whereas (for \( n < m \)) all minors of degree \( n + 1 \) vanish identically, that is, for every choice of the \( m \) elements \( p_1, p_2, \ldots, p_m \).

In order to fix the ideas we suppose that the minor

\[
\Delta_0 = |\mu_j(p_h)| \neq 0 \quad (g, h = 1, 2, \ldots, n).
\]

Let now \( p \) be any element of \( \mathfrak{P} \) and \( p_1, p_2, \ldots, p_n \) the \( n \) special elements for which \( \Delta_0 \neq 0 \); then

\[
q = F(1, a_1, a_2, \ldots, a_n; p, p_1, p_2, \ldots, p_n)
\]

is an element of \( \mathfrak{P} \), and according to \( A \)

\[
\mu_j(q) = \mu_j(p) + a_1\mu_j(p_1) + \cdots + a_n\mu_j(p_n) \quad (j = 0, 1, 2, \ldots, m).
\]

On account of (7) we can so determine \( a_1, a_2, \ldots, a_n \) that

\[
\mu_j(q) = 0, \quad \mu_j(q) = 0, \quad \ldots, \quad \mu_n(q) = 0.
\]

If \( n < m \), it follows from the identical vanishing of the minors of degree \( n + 1 \) of the determinant \( \Delta, p \) taking the place of \( p_{n+1} \), that also

\[
\mu_{n+1}(q) = 0, \quad \mu_{n+2}(q) = 0, \quad \ldots, \quad \mu_n(q) = 0.
\]

Hence for \( n < m \) as well as for \( n = m \), \( q \) is an element of \( \mathfrak{P} \) which satisfies the \( m \) equations (2") and therefore it satisfies according to \( B \) also the equation

\[
\mu_0(q) = 0.
\]

But from the \( n + 1 \) equations (9) and (11) it follows, if we write the \( \mu_j(q) \)'s in their explicit form (8), that the determinant

\[
|\mu_j(p), \mu_j(p_1), \ldots, \mu_j(p_n)| = 0 \quad (j = 0, 1, 2, \ldots, n).
\]
If now we expand this determinant according to the elements of the first column, the coefficient of \( \mu_0(p) \) is the determinant \( \Delta_0 \) and therefore different from zero, and this determinant as well as the remaining coefficients of the expansion is independent of \( p \). Hence if we divide by \( \Delta_0 \), we obtain equation (3") with \( c_{n+1} = 0, c_{n+2} = 0, \cdots, c_m = 0 \), and this equation holds on \( \mathcal{B} \), since \( p \) was any element of \( \mathcal{B} \). Thus our theorem is proved.*

§ 2. Theorems I and II as Special Cases of Theorem III.

In order to obtain Theorem I as a special case of Theorem III, we identify the set \( \mathcal{B} \) with the totality of all functions \( \eta(x) \) of class \( C' \) on \([x, x_2]\) which vanish at \( x_1 \) and \( x_2 \), and define

\[
F(a_1, a_2; \eta_1, \eta_2) = a_1 \eta_1 + a_2 \eta_2.
\]

If \( a_1, a_2 \) are two constants and \( \eta_1(x), \eta_2(x) \) two functions of \( \mathcal{B} \), \( a_1 \eta_1(x) + a_2 \eta_2(x) \) again belongs to \( \mathcal{B} \) and the "functions"

\[
\mu_j(\eta) = \int_{x_1}^{x_2} [M_j(x)\eta(x) + N_j(x)\eta'(x)] dx \quad (j = 0, 1, \cdots, m)
\]

are "linear as to \( F \)" since

\[
\mu_j(a_1 \eta_1 + a_2 \eta_2) = a_1 \mu_j(\eta_1) + a_2 \mu_j(\eta_2).
\]

For this special choice of the set \( \mathcal{B} \), the operator \( F \) and the functions \( \mu_j \), Theorem III becomes identical with Theorem I.

More generally we may take for \( \mathcal{B} \) the totality of all functions \( \eta(x) \) of class \( C' \) on \([x, x_2]\) which satisfy any given system of conditions provided only that these conditions are linear, i.e., such that they are satisfied by \( a_1 \eta_1 + a_2 \eta_2 \) whenever they are satisfied by \( \eta_1 \) and \( \eta_2 \), two functions of class \( C' \) on \([x, x_2]\). We thus obtain a generalization of Theorem I indicated by Hadamard.†

On the other hand, to obtain Theorem II as a special case of Theorem III, we identify the set \( \mathcal{B} \) with the totality of all triples \( p = (x, y, z) \) formed with three independent variables \( x, y, z, \)

* I had originally thought it necessary to add to the assumptions \( A \) and \( B \) of the theorem the further assumption that \( \Delta = 0 \) for some system \( p_1, p_2, \cdots, p_m \); I am indebted to Professor Moore for calling my attention to the fact that this assumption may be omitted, as well as for other valuable suggestions.

† loc. cit., § 176.
Each ranging over all real values, and define, in Cayley's set notation,

\[ F(a_1, a_2; p_1, p_2) = a_1(x_1, y_1, z_1) + a_2(x_2, y_2, z_2), \text{ i. e.,} \]

\[ = (a_1 x_1 + a_2 x_2, a_1 y_1 + a_2 y_2, a_1 z_1 + a_2 z_2). \]

\( F(a_1, a_2; p_1, p_2) \) belongs again to \( \mathcal{E} \), however the numbers \( a_1, a_2 \) and the triples \( p_1 = (x_1, y_1, z_1) \) and \( p_2 = (x_2, y_2, z_2) \) may be chosen.

With this definition of \( F \), the functions

\[ \mu_j(p) = A_j x + B_j y + C_j z, \quad (j = 0, 1, 2) \]

are "linear as to \( F \)."

If \( n = 2 \), there exists at least one pair of triples \( (x_1, y_1, z_1), \)
\( (x_2, y_2, z_2) \) for which the determinant

\[
\begin{vmatrix}
A_1 x_1 + B_1 y_1 + C_1 z_1, & A_2 x_1 + B_2 y_1 + C_2 z_1 \\
A_1 x_2 + B_1 y_2 + C_1 z_2, & A_2 x_2 + B_2 y_2 + C_2 z_2
\end{vmatrix} \neq 0.
\]

This means geometrically, if we interpret \( x, y, z \) as homogeneous coordinates of a point in a plane, that the two lines

\[ A_1 x + B_1 y + C_1 z = 0, \quad A_2 x + B_2 y + C_2 z = 0 \]

do not coincide.

Theorem III then specializes into Theorem II.

The assumption \( n = 1 \) leads to the trivial case alluded to on page 403, footnote *.

In like manner the corresponding theorems on pencils and bundles of planes and their generalizations to spaces of higher dimensions follow immediately as special cases from Theorem III.

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