ON POINCARÉ'S CORRECTION TO BRUNS' THEOREM.

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THE differential equations of motion for the problem of three bodies were first set up by Clairaut, and were published by him with the remark, "Let anyone integrate them who can." Clairaut himself had found ten of the eighteen integrals necessary for the complete solution of the equations, but in despair gave up the hope of finding any more, contenting himself with methods of approximation for those cases which were presented by our solar system, particularly, the motion of the moon. The solutions of these equations have engaged the attention of nearly all of the great mathematicians from Clairaut down to the present time, but no more integrals have been forthcoming. This universal failure has given rise, naturally, to a suspicion that there are no more integrals of a simple type, and this suspicion has been strengthened by the researches of Bruns and of Poincaré. In 1887 Bruns published his famous theorem* that the equations of motion of the problem of n bodies ($n > 2$) do not admit any integral which is algebraic in the rectangular coördinates and in the time, other than the ten classical integrals which were found by Clairaut. Bruns' theorem was soon followed by another† due to Poincaré. According to Poincaré's theorem the equations of motion of the problem of n bodies ($n > 2$) do not admit any uniform transcendental integral for values of the masses sufficiently small, other than the ten classical integrals. Comparing his own theorem with that of Bruns, Poincaré has said:‡ "The theorem which precedes is more general, in a sense, than that of M. Bruns, since I have shown not only that there does not exist any algebraic integral but that there does not exist even a uniform transcendental integral, and not only that an integral cannot

* *Acta Mathematica*, vol. 11 (1887).
† *Acta Mathematica*, vol. 13.
‡ Les Méthodes nouvelles de la Mécanique céleste, vol. 1, p. 253.

be uniform for all values of the variables but that it cannot remain uniform in a domain restricted as above. But, in another sense, the theorem of M. Bruns is more general than mine; I have established only, in effect, that there cannot exist algebraic integrals for sufficiently small values of the masses; and M. Bruns has shown that they do not exist for any system of values of the masses."

The demonstration of his theorem which was given by Bruns contained an error which was pointed out by Poincaré* in 1896, and the proper correction indicated by him. In order to see the nature of this correction it will be necessary to have an outline of the method by which Bruns achieved his demonstration.

Let the rectangular coördinates of the bodies be denoted by x_i, and let $dx_i/dt = y_i$. Bruns first observes that the differential equations

$$dx_i/dt = y_i, \quad dy_i/dt = f_i(x_i)$$

are algebraic in the variables x_i, y_i; and if a single irrationality s be introduced, the differential equations will be not only algebraic but also rational in the variables x_i, y_i, and s. The variable s is defined as a root of a certain algebraic equation

$$F(s;\ x_1,\ \cdots,\ x_n) = 0.$$

Considering first integrals which do not contain the time explicitly, it is shown that every integral must contain some of the variables y_i, and this is followed by the proof that the assumed algebraic integral can be built up of integrals which are rational functions of the variables x_i, y_i, and s, and consequently it is necessary to consider only integrals which are rational in these variables, e. g.,

$$\frac{G_1(x_i,\ y_i,\ s)}{G_2(x_i,\ y_i,\ s)} = \text{constant},$$

where G_1 and G_2 are polynomials in the arguments indicated and have certain homogeneity properties. It is then shown that the polynomials G_1 and G_2 satisfy the same differential equation

$$dG/dt = \omega G,$$

* *Comptes Rendus*, vol. 123, p. 1224.

where

$$\omega = \omega_1 y_1 + \omega_2 y_2 + \cdots + \omega_n y_n,$$

and the ω_i are rational in x_i and s, homogeneous of degree -1, and do not contain the variables y_i.

The polynomial G is now arranged according to powers of the y_i, thus

$$G = \psi_0 + \psi_2 + \cdots,$$

where ψ_0 contains the terms of highest degree in the y_i, ψ_2 is the ensemble of the terms of the next highest degree, etc. It is found that ψ_0, which is a homogeneous polynomial in the y_i, and also a homogeneous polynomial in the x_i and s, must satisfy the partial differential equation

$$(1) \qquad \Sigma y_i \partial\psi_0/\partial x_i = (\omega_1 y_1 + \omega_2 y_2 + \cdots)\psi_0.$$

If ψ_0 does not contain the irrationality s, there exists a multiplier $m(x_i)$ which is a rational function of the x_i alone such that

$$(2) \qquad m \cdot G = \varphi_0 + \varphi_2 + \cdots = \text{const.}$$

is an integral, where $\varphi_0 = m\psi_0$, etc., and φ_0 satisfies the partial differential equation

$$(3) \qquad \Sigma y_i \frac{\partial \varphi_0}{\partial x_i} = 0.$$

If ψ_0 contains s, Bruns considered the product

$$\Psi = \prod_j \psi_0^{(j)},$$

where $\psi_0^{(j)}$ is the same as ψ_0 except that s is replaced by one of the other roots of $F(s, x_i) = 0$. Since the product Ψ is symmetrical in all the roots of $F = 0$, it is a rational function of the x_i. Consequently there exists a multiplier $H(x_i)$ such that $\Phi = H\Psi$ satisfies the equation

$$(4) \qquad \Sigma y_i \partial\Phi/\partial x_i = 0.$$

From the character of the solutions of this equation, Bruns inferred that $\Sigma\omega_i dx_i$ of (1) was an exact differential even when ψ_0 contains s. It is necessary therefore only to consider integrals of the form (2).

Up to this point Bruns has used only the most general properties of the differential equations, viz., homogeneity and rationality, and from this point on the differential equations play a more important rôle. The next step consists in showing that if φ_2 [(equation (2)] is to be free from transcendental functions of the x_i, and to be a polynomial in the y_i, φ_0 must be a function of the ten classical integrals only. The assumed integral, mG = constant, is therefore compounded of the ten classical integrals plus another integral K which is of degree two less than mG in the y_i. The discussion of the integral K does not differ from that of mG. Its leading term must be built up from the ten classical integrals plus another integral K_1, and so on to the conclusion that mG is built up entirely of the ten classical integrals.

The case in which the assumed integral contains the time explicitly can be reduced to that in which the time does not occur explicitly.

The error committed by Bruns was in the character of the solutions of (4). The function Φ is a homogeneous polynomial in the y_i, and homogeneous in the x_i. Removing all factors from Φ which contain only the y_i and then taking $y_3 = y_4 = \cdots = 0$, Bruns arrived at a function Φ_{02} which satisfies the equation

$$y_1 \frac{\partial \Phi_{02}}{\partial x_1} + y_2 \frac{\partial \Phi_{02}}{\partial x_2} = 0.$$

The function Φ_{02} is homogeneous in y_1 and y_2. Bruns supposed it was also homogeneous in x_1 and x_2, while as a matter of fact it is homogeneous in $x_1, \cdots, x_n$. The conclusion drawn by Bruns that $\Sigma \omega_i dx_i$ must be an exact differential is not correct, and Poincaré gave an example in which it is not verified. But Poincaré remedied this defect by showing that while in general there exist functions ϕ_0 which satisfy the conditions imposed upon it and which satisfy equation (1), without satisfying the condition $\Sigma \omega_i dx_i$ = an exact differential, such functions cannot arise from the astronomical problem. Poincaré did not give the details of his analysis and sketched his proof only in its broadest outlines. The details of this proof have been given by Forsyth,* but the proof given by Forsyth is open to the objection that while the y_i are constants so far as the x_i are concerned in the partial differential equation

* Theory of Differential Equations, vol. 3, p. 351 et seq.

(1), they have not been consistently regarded as such by Forsyth. An excellent exposition of this proof has been given by Whittaker,* but while Whittaker has avoided the errors of Forsyth he has committed one of his own.

The function $\Phi = \varphi_0(s_1) \cdot \varphi_0(s_2) \cdot \varphi_0(s_3) \cdots$ is a rational homogeneous function of the x_i and a homogeneous polynomial in the y_i. The factors $\varphi_0(s_j)$ differ from one another only in the roots s_j; consequently, two factors become equal when two of the s_j become equal. Suppose $\varphi_0(s_1) = \varphi_0(s_2)$; then $s_1 = s_2$ defines a relation between the x_i, say

$$f(x_1, \cdots, x_n) = 0. \tag{5}$$

For values of the x_i lying on $f = 0$, the factor $\varphi(s_1) = \varphi(s_2)$ and consequently Φ has a double factor. Let us think of $y_2, \cdots, y_n$ as fixed, or given arbitrarily; then y_1 can be determined so as to satisfy the two equations

$$\Phi = 0, \quad \partial\Phi/\partial y_1 = 0, \tag{6}$$

and consequently also† $\partial\Phi/\partial x_1 = 0$. In fact, the partial derivative with respect to any of the variables will vanish if equations (6) are satisfied.

Since (5) is a condition of equal roots of $\Phi = 0$ it follows that $f(x_1, \cdots, x_n) = 0$ is the eliminant of (6), or a factor of the eliminant. Consequently, by the theory of elimination, there exist multipliers, A and B, such that

$$f \equiv A\Phi + B\,\partial\Phi/\partial y_1. \tag{7}$$

On differentiating (7) with respect to x_i it is found that

$$\frac{\partial f}{\partial x_i} \equiv A\frac{\partial \Phi}{\partial x_i} + B\frac{\partial^2\Phi}{\partial y_1 \partial x_i} + \Phi\frac{\partial A}{\partial x_i} + \frac{\partial \Phi}{\partial y_1}\cdot\frac{\partial B}{\partial x_i}.$$

Multiplying through by y_i and adding with respect to i, there results

$$\sum_i y_i\frac{\partial f}{\partial x_i} \equiv A\sum y_i\frac{\partial \Phi}{\partial x_i} + B\sum y_i\frac{\partial^2\Phi}{\partial y_1\partial x_i} + \Phi\sum y_i\frac{\partial A}{\partial x_i} + \frac{\partial\Phi}{\partial y_1}\sum y_i\frac{\partial B}{\partial x_i}. \tag{8}$$

* Analytical Dynamics.

† This relation seems to have been overlooked by Whittaker.

Consider now the various terms of the right member of (8): From (4) it is seen that $\Sigma y_i \partial\Phi/\partial x_i \equiv 0$; and from (6), $\Phi = 0$ and $\partial\Phi/\partial y_1 = 0$. On differentiating the identity $\Sigma y_i \partial\Phi/\partial x_i \equiv 0$ with respect to y_1 it is seen that

$$\frac{\partial\Phi}{\partial x_1} + \Sigma y_i \frac{\partial^2\Phi}{\partial y_1 \partial x_i} \equiv 0$$

and since $\partial\Phi/\partial x_1 = 0$ it follows that $\Sigma y_i \partial^2\Phi/\partial y_1 \partial x_i = 0$. Hence the right member of (8) vanishes and we have the result that values of the x_i and y_i which satisfy (5) and (6) also satisfy the equation

$$\Sigma y_i \frac{\partial f}{\partial x_i} = 0. \tag{9}$$

Since Φ satisfies the equation $\Sigma y_i \partial\Phi/\partial x_i \equiv 0$ it involves the x_i only through the expressions $x_i y_1 - x_1 y_i$, $(i = 2, \cdots, n)$. Let us define new variables v_i by the relations

$$v_i = x_i + y_i \tau \qquad (i = 1, \cdots, n). \tag{10}$$

The variables v_i can be regarded as the coördinates of a straight line in space of $3n$ dimensions, the line passing through the point whose coördinates are the x_i, the slopes of the line being defined by the y_i. One sees that

$$v_i y_1 - v_1 y_i = x_i y_1 - x_1 y_i,$$

and hence if the point x_i satisfies $\Phi = 0$, so also do all the points v_i as defined by (10); and this is the justification of Poincaré's remark that $\Phi = 0$ represents an aggregate of straight lines in space of $3n$ dimensions.

Consider now any line $v_i = x_i + y_i\tau$, where the x_i lie on the surface $f = 0$, and the y_i are such as to satisfy (6). Its intersections with the surface $f = 0$ are given by the equations

$$f(x_i + y_i\tau) = 0 = f(x_i) + \frac{\partial f}{\partial \tau} \cdot \tau + (\cdots)\tau^2, \tag{11}$$

or

$$0 = f(x_i) + \tau \Sigma \frac{\partial f_i}{\partial x_i} y_i + \tau^2(\cdots);$$

but since $f(x_i) = 0$ and $\Sigma y_i(\partial f/\partial x_i) = 0$ it is seen that $\tau = 0$ is a double root and consequently the line is tangent to the surface $f = 0$. Thus all the lines which belong to the double factor of Φ are tangent to the surface $f = 0$.

Now let the x_i and y_i have such values that they satisfy $\Phi = 0$ but the point x_i does not necessarily lie on $f = 0$, and consider the totality of lines $v_i = x_i + y_i\tau$ which are tangent to $f = 0$. They are given by the equations

$$(12) \qquad f(x_i + y_i\tau) = 0, \quad \frac{\partial}{\partial\tau}f(x_i + y_i\tau) = 0.$$

The τ-eliminant of equations (12) represents the totality of lines tangent to $f = 0$. Hence it includes the two or more factors of Φ which become equal when the x_i satisfy $f = 0$. Since the eliminant is rational and Φ is irreducible, the eliminant must be Φ itself or a multiple of it.

In the astronomical problem the equation $F = 0$ which defines the roots s_j is known. The surfaces $f = 0$ are therefore readily determined and all possible functions Φ can be found. To satisfy the conditions which Bruns has stated, Φ must be factorable into real factors which are polynomials in the y_i and rational in the x_i and s. It is found upon examination that there does not exist a Φ which satisfies all these conditions and consequently the original φ_0 with which we set out cannot contain s. Therefore Bruns' conclusion that we need consider only integrals of the type (2) was correct, even though his argument was wrong. The integrity of the theorem has been preserved by the penetrating insight of Poincaré.

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NOTE ON THE GROUPS FOR TRIPLE-SYSTEMS.

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THE method of "Triple-systems as transformations and their paths among triads," given by Professor White in the *Transactions*, volume 14 (1913), page 6, has been applied by me to the two following triple-systems on fifteen elements. The results obtained agree with the fact, which I had discovered previously by a different method of analysis, that two non-congruent triple-systems may have the same group.