We proceed to prove that I' does not involve any operator whose order exceeds p^{m_1} whenever G does not involve more than p invariants which are equal to the same number. In fact, this result follows almost immediately from a method which has been employed to find the orders of operators in the group of isomorphisms of an abelian group.* According to this method the pth power of an operator of I' transforms every independent generator of G into itself multiplied by an operator whose order is less than the order of this independent generator. Hence there results the theorem:

If an abelian group of order p^m does not involve more than p invariants which are equal to each other, the group of isomorphisms of this abelian group involves no operator whose order is a power of p and exceeds the largest invariant of this abelian group.

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THEORY OF PRIME NUMBERS.

Handbuch der Lehre von der Verteilung der Primzahlen. Von Dr. EDMUND LANDAU, ordentlichem Professor der Mathematik an der königlichen Georg-August-Universität zu Göttingen. Vol. I, pp. xviii + 1–564; Vol. II, pp. ix + 565–961. Leipzig und Berlin, B. G. Teubner, 1909.

THE central problem in the prime number theory consists in proving that the number $\pi(x)$ of primes less than or equal to x may be represented asymptotically by the expression $x/\log x$, or

$$\lim_{x=\infty} \frac{\pi(x)\log x}{x} = 1. \tag{1}$$

This formula was conjectured by Legendre and Gauss at the end of the eighteenth century, but the first definite step toward a proof was taken by Tchebychef, who showed in 1851–52 that for sufficiently large values of x the quotient $\pi(x)\log x/x$ lies between two positive boundaries, one of them less and the other greater than unity. The next great advance is marked by Riemann's paper of 1859 "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse." His point of departure is the equation established by Euler

* G. A. Miller, BULLETIN, vol. 7 (1901), p. 351.

$$(2) \qquad \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}} \qquad \text{for } s \text{ real and } > 1,$$

where the right-hand product extends over all prime numbers p. The occurrence of all prime numbers in (2) suggests an intimate relation between the prime number problem and the function $\zeta(s)$ (the Riemann Zeta function) defined for $s > 1$ by the left-hand member of (2). Riemann's fundamental idea consists in the introduction of complex values of s and the investigation of the properties of the analytic function $\zeta(s)$ of the complex variable s; in this direction, he gives a number of important and profound theorems, part of which, however, are incompletely proved or even stated as mere conjectures. While Riemann does not arrive at a proof of (1), his function-theoretic point of view is the one which has made further progress possible. The difficulties presented by the unproven statements of Riemann were, however, so great that no further advance was made until 1893, when Hadamard, by his theory of entire functions of finite rank, created methods capable of proving several of Riemann's assertions and prepared the way for the proof of (1), which was accomplished independently by Hadamard and de la Vallée Poussin in 1895. Since then, the prime number theory has been enriched in results, as well as simplified in methods, by an ever increasing number of mathematicians, until the literature on the subject has now grown to such an extent as to necessitate a systematic presentation of its main results.

None could be more competent to meet this need than Professor Landau, to whom the greater part of recent progress in the theory of primes is due. In his exposition of a theory presenting as many open questions as the present one, Professor Landau has followed what is probably the most commendable course, that of dividing the results, according to the methods required to establish them, into four large groups, namely those obtainable, *first,* by quite elementary methods, *second,* by analysis in a real variable, *third,* by elementary processes in the theory of analytic functions of a complex variable, and *fourth,* by Hadamard's theory of entire functions of finite rank. This treatment of the subject has two great advantages: it enables the author to determine as nearly as possible just how far each of these four methods will lead, which is of invaluable service to those engaged in research work, and furthermore, it affords an opportunity of introducing all simplifications of which the frequently long and difficult original investigations have shown themselves capable, without slighting the historical aspect of the subject. Instead of burdening the text with footnotes, the author has assembled all bibliographical references at the end of the second volume under the heading "Quellenangaben," pages 883–907, followed by a very complete "Literaturverzeichnis," pages 908–961.

The only prerequisite for the study of the work under review is a good working knowledge of the elements of the theory of functions of a complex variable, and particularly of Cauchy's theorem; whatever results from the elementary theory of numbers or Hadamard's theory of entire functions are needed, are presented fully and in a very accessible form in the text.

After an introduction on the history of the prime number problem (pages 3–55), the work is divided into five books and twenty-seven sections.

The first book (sections 1–4) treats of the prime number problem, i. e., the asymptotic expression of $\pi(x)$. Section 1 (pages 59–102) deals with the results obtainable by quite elementary methods. First the following very convenient notations are introduced, x being positive but not necessarily an integer:

$$f(x) = O(g(x))$$

signifies that two positive constants ξ and A exist such that $|f(x)| < Ag(x)$ for $x \geqq \xi$ (examples: $\sqrt{x} = O(x)$, $e^{ix} = O(1)$);

$$f(x) = o(g(x))$$

means that

$$\lim_{x=+\infty} \frac{f(x)}{g(x)} = 0 \quad \text{(examples: } \log x = o(\sqrt{x}), \frac{1}{x} = o(1));$$

denoting by $[x]$ the greatest integer contained in x, the sum

$$\sum_{n=v}^{w} f(n)$$

stands for a sum extended over all integers n from $[v]$ inclusive to $[w]$ inclusive; and finally

$$\sum_{p \leqq x} f(p), \quad \prod_{p \leqq x} f(p)$$

signify that the sum, or product, is extended over all primes p not exceeding x.

Now the functions

$$(3)\qquad \begin{aligned} T(x) &= \sum_{n=1}^{x} \log n, \quad \vartheta(x) = \sum_{p \leqq x} \log p, \\ \psi(x) &= \vartheta(x) + \vartheta(\sqrt{x}) + \vartheta(\sqrt[3]{x}) + \cdots = \sum_{p^m \leqq x} \log p \end{aligned}$$

are introduced, and the Tchebychef-de Polignac identity

$$(4)\qquad T(x) = \sum_{n=1}^{x} \psi\left(\frac{x}{n}\right)$$

is proved. From the definition of $T(x)$, it follows that

$$(5)\qquad T(x) = x \log x - x + O(\log x);$$

the combination of (4) and (5) shows that

$$\limsup_{x=\infty} \frac{\vartheta(x)}{x} = \limsup_{x=\infty} \frac{\psi(x)}{x} \leqq 2 \log 2,$$

$$\liminf_{x=\infty} \frac{\vartheta(x)}{x} = \liminf_{x=\infty} \frac{\psi(x)}{x} \geqq \log 2,$$

whence the conclusion

$$(6)\qquad \frac{\pi(x) \log x}{x} - \frac{\vartheta(x)}{x} = O\left(\frac{1}{\log x}\right)$$

is drawn, showing that both terms on the left side have the same lim. sup. and the same lim. inf. These upper and lower boundaries are now narrowed down step by step, and the interval between them is shown to contain unity as an interior point. Tchebychef's proof of Bertrand's celebrated postulate (for $x \geqq 7$ there exists at least one prime p such that $\frac{1}{2}x < p \leqq x - 2$) is given, as well as two asymptotic formulas due to Mertens:

$$(7)\qquad \begin{aligned} &\sum_{\leqq x} \frac{\log p}{p} = \log x + O(1), \\ &\sum_{p \leqq x} \frac{1}{p} = \log \log x + B + O\left(\frac{1}{\log x}\right), \end{aligned}$$

B being a numerical constant.

In section two (pages 103–150), the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{8}$$

are introduced, and some of their elementary properties as functions of the *real* variable s are given. The application of these results to the special Dirichlet series for the functions $\zeta(s)$ and $\zeta'(s)/\zeta(s)$ gives new and simpler proofs for the theorems of the first section, but does not lead much farther; in particular, these methods are insufficient to prove the asymptotic relation (1). For this problem, the introduction of complex values of the variable $s = \sigma + ti$ in section three (pages 151–269) proves more efficient. After defining $\zeta(s)$ as the analytic function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for } \sigma > 1,$$

it is shown that equation (2) is true for these values of s, and that consequently $\zeta(s)$ has no zeros in the half plane $\sigma > 1$. Two different proofs are given for the fact that $\zeta(s) - 1/(s-1)$ is holomorphic for $\sigma > 0$. By an ingenious artifice due to Mertens, it is shown that $\zeta(s)$ has no zeros to the right of the curve defined by

$$\sigma = 1 - \frac{1}{c(\log|t|)^9} \text{ for } |t| \geqq 3, \quad \sigma = 1 - \frac{1}{c(\log 3)^9} \text{ for } |t| \leqq 3, \tag{9}$$

and that, to the right of this curve and for $|t| \geqq 3$,

$$\begin{gathered} \frac{1}{c_2(\log|t|)^7} < |\zeta(s)| < c_1 \log|t|, \\ \left|\frac{\zeta'(s)}{\zeta(s)}\right| < c_3(\log|t|)^9, \end{gathered} \tag{10}$$

c, c_1, c_2 being positive constants, as well as all c's with subscripts that are introduced later.

Defining $\Lambda(n)$ as the numerical function which is zero for $n = 1$ and for all values of n which are not powers of a prime, but equals unity when n is a power of a prime, the formula

$$\sum_{n=1}^{x} \Lambda(n) \log \frac{x}{n} = -\frac{1}{2\pi i} \int_{2-x^{2i}}^{2+x^{2i}} \frac{x^s}{s^2} \frac{\zeta'(s)}{\zeta(s)} ds + O(1)$$

is established, and by the application of Cauchy's theorem to the integral

$$\int \frac{x^s}{s^2} \frac{\zeta'(s)}{\zeta(s)} ds \tag{11}$$

taken over the closed contour bounded by the curve (9) and the three straight lines $t = x^2$, $\sigma = 2$ and $t = -x^2$, it follows that

$$\sum_{n=1}^{x} \Lambda(n) \log \frac{x}{n} = x + O(xe^{-\sqrt[11]{\log x}}). \tag{12}$$

By elementary transformations of (12), it is now shown that

$$\begin{aligned} \vartheta(x) &= x + O(xe^{-\sqrt[13]{\log x}}), \\ \pi(x) &= \int_2^x \frac{du}{\log u} + O(xe^{-\sqrt[14]{\log x}}), \end{aligned} \tag{13}$$

where the second formula includes the prime number theorem (1) as a very special case.

The formulas (13) furnish the means for obtaining asymptotic expressions for various sums of the type $\Sigma_{p \leqq x} F(p, x)$, where the function $F(p, x)$ is subject to certain very general conditions. It is shown that there are more primes between 1 and x than between x and $2x$ for x sufficiently large; the formulas

$$\liminf_{x=\infty} \frac{\varphi(x)}{\dfrac{x}{\log \log x}} = e^{-C}, \qquad \limsup_{x=\infty} \frac{\log \tau(x)}{\dfrac{\log x}{\log \log x}} = \log 2,$$

are proved for integral values of x, $\varphi(x)$ being the number of integers $\leqq x$ and relative primes to x, C Euler's constant, and $\tau(x)$ the number of divisors of x; various other applications of (13) are also given.

Section four (pages 270–388) goes deeper into the properties of the analytic function $\zeta(s)$. It is shown according to Riemann that $\zeta(s) - \dfrac{1}{s-1}$ is an entire function of s, and that

$$\xi(s) = \frac{s(s-1)}{2} \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s)$$

satisfies the functional equation

$$(14) \qquad \xi(1 - s) = \xi(s).$$

The principal results of Hadamard's theory of entire functions of finite rank are now exposed in the particular case where the rank equals zero, and it is shown that, writing $\xi(s) = g(x)$ where $s = \frac{1}{2} + i\sqrt{x}$, $g(x)$ is an entire function of rank zero, so that, $\xi_1, \xi_2, \cdots$ being its zeros, the relation

$$g(x) = g(0)\prod_{\nu=1}^{\infty}\left(1 - \frac{x}{\xi_\nu}\right)$$

subsists. From this formula it follows that

$$(15) \qquad (s - 1)\zeta(s) = \tfrac{1}{2}e^{bs}\frac{1}{\Gamma\left(\dfrac{s}{2} + 1\right)}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{s/\rho},$$

b being a numerical constant and the ρ's being all those zeros of $\zeta(s)$ which have their real parts between 0 and 1. In consequence of (15), it is shown that the curve (9), to the right of which $\zeta(s)$ has no zeros, may be replaced by the curve

$$(16) \quad \sigma = 1 - \frac{1}{a\log|t|} \text{ for } |t| \geqq 2, \quad \sigma = 1 - \frac{1}{a\log 2} \text{ for } |t| \leqq 2,$$

a being a positive constant, and an application of Cauchy's theorem similar to the one leading to (13) now gives

$$(17) \qquad \pi(x) = \int_2^x \frac{du}{\log u} + O(xe^{-\alpha\sqrt{\log x}}),$$

where $\alpha \geqq \frac{1}{5}$. Furthermore, two conjectures of Riemann are proved: one concerning a certain series involving the ρ's, and another stating that the number of zeros $\rho = \beta + \gamma i$ satisfying the condition $0 < \gamma \leqq T$ is equal to

$$(18) \qquad \frac{1}{2\pi}T\log T - \frac{1 + \log(2\pi)}{2\pi}T + O(\log T).$$

The last of Riemann's conjectures is that all the ρ's have their real part $\beta = \frac{1}{2}$, and the question of proving (or disproving) it is the most important unsolved problem in the theory of primes. It is intimately connected with the question of reducing the order of the remainder term in (17). Section four ends with the demonstration that, if this conjecture of Riemann is true, (17) may be replaced by

$$\pi(x) = \int_2^x \frac{du}{\log u} + O(\sqrt{x}\log x).$$

Book two (sections 5 to 8, pages 391–564) deals with the prime numbers contained in an arithmetical progression $ky + l$ ($y = 0, 1, 2, \cdots$), where l is any one of the $h = \varphi(k)$ integers not exceeding k and relative primes to it. In section five, the properties of the Dirichlet characters $\chi_k(n)$ ($k = 1, 2, \cdots, h$) (which are either zero or certain roots of unity, depending upon the nature of the decomposition of n in prime factors) are developed, and the Dirichlet series

$$L_k(s) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s} \quad (k = 1, 2, \cdots, h) \tag{19}$$

are introduced, which generalize the Zeta function to the present case. The consideration of the series (19) for real values of s leads to Dirichlet's celebrated theorem: every arithmetical progression $ky + l$, where l and k are relative primes, contains an infinity of prime numbers.

In sections six and seven, the introduction of a complex variable s in (19) leads to results quite similar to those obtained for $\zeta(s)$ in book one, and culminating in the theorem: the number $\Pi(x)$ of primes of the form $ky + l$ and not exceeding x is asymptotically

$$\Pi(x) = \frac{1}{h}\int_2^x \frac{du}{\log u} + O(xe^{-a\sqrt{\log x}}). \tag{20}$$

In section eight, various applications of this result are given, for instance the theorem: every positive integer above a certain limit may be decomposed into a sum of not more than eight positive cubes.

The second volume begins with book three entitled "the function $\mu(n)$ and the distribution of numbers without quadratic divisors" (sections 9–13, pages 567–621). The Möbius function $\mu(n)$ equals unity for $n = 1$, zero when n has a quadratic divisor, and $(-1)^\rho$ when n is the product of ρ different primes. In this book, some new properties of the Zeta function are deduced, which lead to the formulas

$$(21)\qquad \begin{gathered} \sum_{n=1}^{x}\mu(n) = O(xe^{-\alpha\sqrt{\log x}}), \qquad \sum_{n=1}^{x}\frac{\mu(n)}{n} = O(e^{-\alpha\sqrt{\log x}}), \\ \sum_{n=1}^{x}\frac{\mu(n)\log n}{n} = -1 + O(e^{-\alpha\sqrt{\log x}}). \end{gathered}$$

Book four (sections 14–16, pp. 625–639) treats similar problems concerning the numbers without quadratic divisors contained in an arithmetic progression, and book five (sections 17–20, pages 641–719) contains convergence proofs for various types of series connected with prime numbers, as well as some theorems on $\pi(x)$, of which the following is an example: above any finite limit however large, there exist numbers x for which

$$\pi(x) - \int_2^x \frac{du}{\log u} + \frac{1}{2}\int_2^{\sqrt{x}} \frac{du}{\log u} > \frac{1}{c}\frac{\sqrt{x}}{\log x},$$

as well as numbers x for which the left-hand member is less than $-\dfrac{1}{c}\dfrac{\sqrt{x}}{\log x}$.

Finally book six (sections 21–27, pages 723–882) contains an elaborate investigation, with many applications to the Zeta function, of functions of the complex variable s defined by a generalized Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

where the λ_n are real and increase monotonely toward infinity with n.

In concluding this review of Professor Landau's monumental work, it should be stated that the exposition is a model of clearness and rigor. There are surprisingly few misprints, none of them serious, and the make-up of the book even surpasses the usual high standard of the Teubners.

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