A THEOREM CONCERNING CONTINUOUS CURVES.

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In this paper I propose to show that every continuous curve has the simple property stated below in Theorem 1. Though my proof is worded for the case of a plane curve, it is clear that with a slight change in phraseology it would apply to a curve in a space of any number of dimensions.

**Lemma.** If $S_1, S_2, S_3, \ldots$ is a countable sequence of connected,* bounded point sets such that, for every $n$, $S_n$ contains $S_{n+1},$† then the set of all points that are common to $S_1, S_2, S_3, \ldots$ is closed and connected.

For a proof of this lemma see my paper “On the foundations of plane analysis situs,” Transactions of the American Mathematical Society, volume 17 (1916), page 137. Cf. also S. Janiszewski and E. Mazurkiewicz, Comptes Rendus, volume 151 (1910), pages 199 and 297 respectively.

**Theorem 1.** Every two points of a continuous curve are the extremities of at least one simple continuous arc that lies entirely on that curve.

**Proof.** Suppose $A$ and $B$ are two points belonging to the continuous plane curve $C$. Hahn has shown‡ that the curve $C$ is connected “im kleinen,” i.e., that if $P$ is a point of $C$, $\epsilon$ is a positive number and $K$ is a circle, of radius $1/\epsilon$, with center at $P$, then there exists, within $K$ and with center at $P$, another circle $K_{\epsilon P}$ such that if $X$ is a point within $K_{\epsilon P}$, and belonging to $C$, then $X$ and $P$ lie together in some connected subset of $C$ that lies entirely within $K$. Let $K_{\epsilon P}$ denote the set of all points $[Y]$ belonging to $C$ such that $Y$ and $P$ lie together in some connected subset of $C$ that lies entirely within $K$. Clearly $K_{\epsilon P}$ contains $K_{\epsilon P}$.

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* A set of points is said to be connected if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one.
† If $S$ is a point set, $S'$ denotes the set of points composed of $S$ together with all its limit points.
and indeed if $Z$ is any point of $K_e$ then there exists about $Z$ a circle such that every point of $C$ within this circle belongs to $K_e$. Let $e = 1$ and for each point $P$ of $C$ construct the corresponding $K_{1P}$. Let $G_1$ denote the set of all such $K_{1P}$'s. I will proceed to show that there exists a simple chain* from $A$ to $B$ every link of which is a point set of the set $G_1$. Suppose that no such simple chain exists. Then the points of $C$ fall into two classes, $S_A$ and $S_B$ where $S_A$ is the set of all points $[P]$, belonging to $C$, such that $P$ can be joined to $A$ by a simple chain every link of which belongs to the set $G_1$, while $S_B$ is composed of all the remaining points of $C$. Since $C$ is connected, one of the sets $S_A$ and $S_B$ contains a point $X$ which is a limit point of the other one. There exists a circle $k$ with center at $X$ such that every point of $C$ that lies within $k$ lies also in $K_{1X}$. But the interior of $k$ contains a point $A_1$ belonging to $S_A$ and a point $B_1$ belonging to $S_B$ where $A_1$ is $X$ or $B_1$ is $X$ according as $X$ belongs to $S_A$ or to $S_B$. The point $A$ can be joined to $A_1$ by a simple chain $R_1R_2R_3 \cdots R_n$ every link of which is a point set of the set $G_1$. Let $R_k$ be the first link of this chain that has a point in common with $K_{1X}$. Then $R_1R_2R_3 \cdots R_kK_{1P}$ is a simple chain from $A$ to $B_1$ every link of which belongs to $G_1$. Thus the supposition that $A$ can not be joined to $B$ by a simple chain every link of which is a point set belonging to $G_1$ leads to a contradiction. It follows that $A$ can be joined to $B$ by at least one such chain $R_{11}R_{12} \cdots R_{1m_1}$. Call this chain $C_1$. For each $i$ $(1 \leq i < m_1)$ select a point $P_{1i}$ common to $R_{1i}$ and $R_{1i+1}$ Let $P_{10} = A$ and $P_{1m_1} = B$. If $0 \leq i < m_1$ then $P_{1i}$ can be joined to $P_{1i+1}$ by a simple chain $C_{1i+1}$ each link of which is a $K_{1P}$ for some point $P$ of $C$ and some $r \geq 2$ and lies with all its limit points entirely in $K_{1P}$. If any link of the chain $C_{11}$, except the last one, has a point in common with any link of $C_2$, then omit from $C_{11}$ every link that follows $T_{11}$, where $T_{11}$ is the first link of $C_{11}$ that has a point in common with a link of $C_{12}$; also omit

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*A simple chain from $A$ to $B$ is a finite sequence of point sets $R_1$, $R_2$, $R_3$, $\cdots$, $R_n$ such that (1) $R_1$ contains $A$ if and only if $i = 1$, (2) $R_i$ contains $B$ if and only if $i = n$, (3) if $1 \leq i \leq n$, $1 \leq j \leq n$, $i < j$, then $R_i$ has a point in common with $R_j$ if and only if $j = i + 1$. The point set $R_k$ $(1 \leq k \leq n)$ is said to be the $k$th link of the chain $R_1R_2R_3 \cdots R_n$ and the chain $R_1R_2R_3 \cdots R_n$ is said to join $A$ to $B$. Cf. my paper "On the foundations of plane analysis situs," loc. cit., page 134.
from $C_{12}$ every link (if there be any such) that precedes the last link that has a point in common with $T_{11}$. These omissions having been made, the remaining links of the chains $C_{11}$ and $C_{12}$ form a simple chain $\overline{C}_{12}$ from $A$ to $P_{12}$. In a similar manner it may be shown that there exists a simple chain $\overline{C}_{13}$ from $A$ to $P_{13}$ such that each link of $\overline{C}_{13}$ is a link of either $\overline{C}_{12}$ or $C_{13}$. This process may be continued. It follows that there exists a simple chain $C_2$ from $A$ to $B$ such that each link of $C_2$ is a link of some $C_{1i}$ (1 ≤ $i$ ≤ $m_1$). The chain $C_2$ has the property that each one of its links lies wholly in some single link of the chain $C_1$ and if a link $x$ of $C_2$ lies in a link $y$ of $C_1$ then every link that follows $x$ in $C_2$ lies either in $y$ or in some link that follows $y$ in $C_1$. Similarly there exists a chain $C_3$ having a relation to $C_2$ analogous to the above indicated relation of $C_2$ to $C_1$ and such that every link of $C_3$ is a $\overline{K}_{rp}$ for some point $P$ of $C$ and some $r \geq 3$. This process may be continued. Thus there exists an infinite sequence of simple chains $C_1$, $C_2$, $C_3$, · · · such that (1) each link of the chain $C_{n+1}$ lies, together with all its limit points, wholly in some single link of $C_n$; (2) if a link $x$ of $C_{n+1}$ lies in a link $y$ of $C_n$ then each link that follows $x$ in $C_{n+1}$ lies either in $y$ or in some link that follows $y$ in $C_n$. Similarly there exists a chain $C_3$ having a relation to $C_2$ analogous to the above indicated relation of $C_2$ to $C_1$ and such that every link of $C_3$ is a $\overline{K}_{rp}$ for some point $P$ of $C$ and some $r \geq 3$. This process may be continued. Thus there exists an infinite sequence of simple chains $C_1$, $C_2$, $C_3$, · · · such that (1) each link of the chain $C_{n+1}$ lies, together with all its limit points, wholly in some single link of $C_n$; (2) if a link $x$ of $C_{n+1}$ lies in a link $y$ of $C_n$ then each link that follows $x$ in $C_{n+1}$ lies either in $y$ or in some link that follows $y$ in $C_n$. Similarly there exists a chain $C_3$ having a relation to $C_2$ analogous to the above indicated relation of $C_2$ to $C_1$ and such that every link of $C_3$ is a $\overline{K}_{rp}$ for some point $P$ of $C$ and some $r \geq 3$.

Let $S_n$ denote the point set which is the sum of all the links of the chain $C_n$. Let $S$ denote the set of all the points that the sets $S_1$, $S_2$, $S_3$, · · · have in common. It will be shown that $S$ satisfies Lennes’ definition* of a simple continuous arc from $A$ to $B$.

I. That $S$ is closed and connected follows easily with the help of the lemma on page 233. That it is bounded is evident.

II. To prove that $S$ contains no connected proper subset that contains both $A$ and $B$, let us first order the points of $S$. If $X_1$ and $X_2$ are two distinct points of $S$, then there exists $n$ such that if $r \geq n$ and $P$ is a point of $S$ then $X_1$ and $X_2$ do not both lie in $\overline{K}_{rp}$. But every link of $C_n$ is a $\overline{K}_{rp}$ for some point $P$ of $C$ and some $r \geq n$. Hence for every two distinct points $X_1$ and $X_2$ be-

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longing to $C$ there exists $n$ such that $X_1$ and $X_2$ do not belong to the same link of $C_n$. Furthermore, it is clear that if $X_1$ and $X_2$ do not lie in the same link of $C_n$ but $X_1$ lies in a link of $C_n$ that precedes one in which $X_2$ lies, then, if $m > n$, every link of $C_m$ that contains $X_1$ precedes every link of $C_m$ that contains $X_2$. The point $X_1$ is said to precede the point $X_2 (X_1 < X_2)$ if there exists $n$ such that every link of $C_n$ that contains $X_1$ precedes every link of $C_n$ that contains $X_2$. From facts observed above it follows that if $X_1$ and $X_2$ are distinct points of $C$ then either $X_1 < X_2$ or $X_2 < X_1$; while if $X_1 < X_2$ then it is not true that $X_2 < X_1$. Furthermore if $X_1 < X_2$ and $X_3 < X_3$ then $X_1 < X_3$. For there exist $n_1$ and $n_2$ such that $n_1$ and $n_2$ do not lie in the same link of $C_{n_1}$ and $X_2$ and $X_3$ do not lie in the same link of $C_{n_2}$. Hence every link of $C_{n_1 + n_2}$ that contains $X_1$ precedes every link of $C_{n_1 + n_2}$ that contains $X_2$ and every link of $C_{n_1 + n_2}$ that contains $X_3$ precedes every one that contains $X_3$. Hence every one that contains $X_1$ precedes every one that contains $X_3$. Therefore $X_1 < X_3$.

Suppose now that $H$ is a proper subset of $S$ that contains both $A$ and $B$. Then there exists a point $P$ belonging to $S$, but different from $A$ and from $B$, such that $H$ is a subset of $S - P$. Now $S - P = S_A + S_B$ where $S_A$ is the set of all points of $S$ that precede $P$ and $S_B$ is the set of all points of $S$ that follow $P$. It is clear that $S_A$ contains $A$ and $S_B$ contains $B$. Suppose that $P_A$ is a point of $S_A$. Then there exists $n$ such that every link of $C_n$ that contains $P_A$ precedes every one that contains $P$. Suppose that some link $y$ of the chain $C_n$ contains $P_A$ and also a point $P_B$ of the set $S_B$. Since $y$ precedes every link of $C_n$ that contains $P$, it follows that $P_B$ precedes $P$, which is contrary to hypothesis. Hence no link of $C_n$ that contains $P_A$ contains any point of $S$. But some link $l$ of $C_n$ does contain $P_A$. There exists, about $P_A$, a circle $t$ such that every point of $C$ that lies within $t$ is a point of $l$. It follows that there is no point of $S_B$ within the circle $t$. Therefore $P_A$ is not a limit point of $S_B$. Similarly no point of $S_B$ is a limit point of $S_A$. But $H$ contains a point $A$ that belongs to $S_A$ and a point $B$ that belongs to $S_B$. Moreover $H$ is a subset of $S_A + S_B$. It follows that $H$ is not connected.

It follows that $S$ is a simple continuous arc from $A$ to $B$. But clearly $S$ is a subset of $C$. The truth of Theorem 1 is therefore established.

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