THE RATIONAL PLANE CUBIC.

THE PROJECTION OF A LINE SECTION UPON THE RATIONAL PLANE CUBIC CURVE.

BY PROFESSOR JOSEPH EUGENE ROWE.

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Introduction.

The rational plane curve of the third order, which we shall refer to as the $R^3$, is of the fourth class; that is, from an arbitrary point of the plane four tangents can be drawn to the curve. But if the point is selected on the $R^3$ itself, the tangent at the point accounts for two of these tangents, and, therefore, from such a point only two additional tangents can be drawn to the curve. A line section yields three points of the $R^3$ and these, in the manner just described, determine three pairs of additional tangents. An investigation of the points of a line and the six tangents so determined shows that the relations which exist among these are interesting as well as of a fundamental character.

We shall let

\[ (1) \quad x_i = a_i \xi^3 + 3b_i \xi^2 + 3c_i \xi + d_i \quad (i = 0, 1, 2) \]

be the parametric equations of the points of the $R^3$, and it has been found convenient to use the following abbreviations:

\[ (2) \quad \alpha = \begin{vmatrix} a \\ b \\ c \end{vmatrix}, \quad \beta = \begin{vmatrix} a \\ b \\ d \end{vmatrix}, \quad \beta' = \begin{vmatrix} a \\ c \\ d \end{vmatrix}, \quad \alpha' = \begin{vmatrix} b \\ c \\ d \end{vmatrix}. \]

Also, it may be verified that the identities

\[ (3) \quad a_i \alpha' - b_i \beta' + c_i \beta - d_i \alpha = 0 \]

exist among the coefficients in (1) and the Greek letters of (2).

The Choice of a Line Section.

As the parameters 0 and $\infty$ may be assigned to any two elements in a one-dimensional space, we select the line determined by the points of the $R^3$ whose parameters are 0 and $\infty$. From (1) it follows that the coordinates of these points are $d_i$ and $a_i$, respectively; hence the equation of the line determined by them is $|adx| = 0$, and the parameter of the third point
of the $R^3$ (found by substituting from equations (1) in $|adx| = 0$) collinear with $a_i$ and $d_i$ is the root of

$$\beta t + \beta' = 0.$$  

By substituting $t = -\beta/\beta'$ in (1) we obtain for the coordinates of the point (4)

$$x_i = -a_i\beta^a + 3b_i\beta^a\beta - 3c_i\beta'\beta^2 + d_i\beta^3 \quad (i=0, 1, 2).$$

**The Projections of the Three Points upon the $R^3$.**

The projection of a point $x_i$ upon (1) is

$$|abx| t^4 + 2|acx| t^3 + (|adx| + 3|bcx|)t^2 + 2|bdx| t + |cdx| = 0.$$  

That is, if the coordinates of a point $x_i$ are substituted in (6), the result is a quartic in $t$ whose roots are the parameters of the points of contact of the four tangents that can be drawn from $x_i$ to the $R^3$ of (1).

By substituting $a_i$, $d_i$, and the coordinates (5) in (6) for $x_i$ we obtain

$$3\alpha t^2 + 2\beta t + \beta' = 0,$$

$$\beta t^2 + 2\beta' t + 3\alpha' = 0,$$

$$(\beta^2 - 3\alpha\beta')t^2 + (3\alpha'\beta - \beta'^2) = 0,$$

whose roots are the parameters of the points of contact of the tangents to $R^3$ drawn from the points whose parameters are $\infty$, 0, and $-\beta'/\beta$, respectively.

The form of equations (7)–(9), if properly interpreted, conveys a great deal of information. Evidently the roots of (9)† are harmonic‡ (apolar) to 0 and $\infty$; the roots of (8) are harmonic to $\infty$ and $-\beta'/\beta$; and the roots of (7) are harmonic to 0 and $-\beta'/\beta$. These results we summarize in

**Theorem I.** The parameters of any two of three collinear points on the $R^3$ are harmonic to the parameters of the points of contact of the two additional tangents that can be drawn to the $R^3$ from the third point.

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† Observe that the order of statement in this sentence is not without purpose.
‡ Salmon, Higher Algebra, fourth edition, p. 179.
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Also, the determinant of equations (7)-(9) vanishes, as may be seen at once from the fact that (9) may be obtained by subtracting \( \beta' \) times (7) from \( \beta \) times (8). Hence* we have

**Theorem II.** The parameters of the points of contact of the three pairs of tangents that can be drawn to the \( R^3 \) from three collinear points of the \( R^3 \) are harmonic to the same quadratic, or form a set in involution.

Another result which may be derived as a corollary of Theorem I we shall state as

**Theorem III.** Lines on a point \( P \) of an \( R^3 \) cut the \( R^3 \) in pairs of residual points whose parameters are harmonic to the parameters of the points of contact of the two additional tangents drawn to \( R^3 \) from \( P \).

Although Theorem III may be regarded a corollary of Theorem I, it may be established independently. Thus: Let \( P(d_0, d_1, d_2) \) be the point and \( (k\alpha) = k_0\alpha_0 + k_1\alpha_1 + k_2\alpha_2 = 0 \) any line on \( P \). Then \( (kd) = 0 \). The parameters of the residual points cut out of (1) by \( (k\alpha) = 0 \) are the roots of

\[
(ka)\beta^2 + 3(kb)t + 3(kc) = 0
\]

and (10) is apolar to (8), for

\[
3(kc)\beta + 3(ka)\alpha' - 3(kb)\beta' = 0,
\]
as may be shown from (3) and the fact that \( (kd) = 0 \).

**Pennsylvania State College,**

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**EXAMPLES OF A REMARKABLE CLASS OF SERIES.**

**BY PROFESSOR R. D. CARMICHAEL.**

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**Two-Fold and One-Fold Expression of the Properties of Functions.**

1. In the development of analysis during the past generation it has frequently happened that functions have arisen which are analytic in a sector of the complex plane and in