A THEOREM ON THE VARIATION OF A FUNCTION.

BY DR. PAUL R. RIDER.

The following is a well known theorem of differential geometry:

The differential quotient \( \frac{d\phi}{ds} \) (\( ds \) is the element of arc) of a function \( \phi(u, v) \) at a point on a surface varies in value with the direction from the point. It equals zero in the direction tangent to the curve \( \phi = c \), and attains its greatest absolute value in the direction normal to this curve.*

This theorem admits of a generalization if we use a more comprehensive definition of length, a definition sometimes employed in the calculus of variations. Let then

\[
S = \int_{t_0}^{t_1} F(x, y, x', y') dt
\]

be the generalized length of arc along a curve

\[(C) \quad x = x(t), \quad y = y(t).\]

By reason of homogeneity conditions†

\[
S = \int_{t_0}^{t_1} F(x, y, \cos \theta, \sin \theta) \sqrt{x'^2 + y'^2} dt
\]

\[
= \int_{s_0}^{s_1} F(x, y, \cos \theta, \sin \theta) ds,
\]

in which

\[
\cos \theta = \frac{x'}{\sqrt{x'^2 + y'^2}}, \quad \sin \theta = \frac{y'}{\sqrt{x'^2 + y'^2}}.
\]

Then

\[
A = \left| \frac{d\phi}{dS} \right| = e \frac{\phi_x dx + \phi_y dy}{F(x, y, \cos \theta, \sin \theta) ds}
\]

\[
= e \frac{\phi_x \cos \theta + \phi_y \sin \theta}{F(x, y, \cos \theta, \sin \theta)},
\]

* See Eisenhart, Differential Geometry, pp. 82–83.
† See Bolza, Vorlesungen über Variationsrechnung, p. 194.
where subscripts indicate partial differentiation, and where $e$ is chosen equal to $\pm 1$ so as to make $A$ positive. Differentiating $A$ with respect to $\theta$, and setting the result equal to zero, we get

$$F(-\phi_x \sin \theta + \phi_y \cos \theta)$$

$$- (\phi_x \cos \theta + \phi_y \sin \theta)(-F_x' \sin \theta + F_y' \cos \theta) = 0,$$

$F_x', F_y'$ denoting partial derivatives of $F$ with respect to its third and fourth arguments respectively. Since

$$F = F_x' \cos \theta + F_y' \sin \theta,$$

equation (1) reduces to

$$\phi_y(x, y)F_x'(x, y, \cos \theta, \sin \theta)$$

$$- \phi_x(x, y)F_y'(x, y, \cos \theta, \sin \theta) = 0,$$

and if we define direction on the curve $\phi = c$ by means of the angle $\tilde{\theta} = \arctan (-\phi_x/\phi_y)$, (2) becomes

$$F_x'(x, y, \cos \theta, \sin \theta) \cos \tilde{\theta} - F_y'(x, y, \cos \theta, \sin \theta) \sin \tilde{\theta} = 0.$$

But this equation determines the value of $\theta$ to which the curve $\phi = c$ is transversal.†

Therefore the differential quotient $d\phi/dS$ is equal to zero in the direction tangent to the curve $\phi = c$ and has its maximum absolute value in the direction to which the curve $\phi = c$ is transversal.

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TANGENTIAL INTERPOLATION OF ORDINATES
AMONG AREAS.

BY DR. C. H. FORSYTH.

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If we wish to interpolate several values in each interval between the successive ordinates $u_0, u_1, u_2, \cdots, u_n$ by finite differences, only a low order of differences can with propriety be used, since high orders based on ordinary statistical data

* See Bolza, loc. cit., p. 196.
† See Bolza, loc. cit., p. 303.