THE DERIVATIVE OF A FUNCTIONAL.

BY PROFESSOR P. J. DANIELL.

In his book on Integral Equations* Volterra has given a
definition of the derivative of a functional and has stated
somewhat restricted conditions under which the variation can
be expressed as a linear integral. In the present paper it is
shown that, under more general conditions, the variation is a
linear functional in the sense of Riesz* and, therefore, a Stieltjes
integral. This theorem is assumed as a condition in a paper
by Fréchet.†

Let

$$F[f(x)]$$

denote a functional $F$ of a continuous function $f(x) (a \leq x \leq b)$. With Volterra we shall consider only continuous functions. Let us denote the first variation by

$$D(f; \varphi) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [F[f + \epsilon \varphi] - F[f]].$$

In place of Volterra’s four conditions we take the two follow­
ing:

I. $F[f]$ satisfies the Cauchy-Lipschitz condition, namely
that we can find a number $M$ such that

$$|F[f_1] - F[f_2]| \leq M \max |f_1(x) - f_2(x)|.$$

II. The first variation $D(f'; \varphi)$ exists for all continuous $\varphi$,
and all continuous $f'$ in the neighborhood of $f$; that is to say
that a number $\eta > 0$ can be found so that the variation exists
so long as

$$\max |f'(x) - f(x)| \leq \eta.$$

Under these conditions the variation is a linear functional,
and therefore a Stieltjes integral,

$$D(f; \varphi) = \int_a^b \varphi(x) d\alpha(x).$$

* V. Volterra, Equations Intégrales, p. 12 et seq. F. Riesz, Annales
In the first place

\[ D(f; l\varphi) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{ F[f + \epsilon\varphi] - F[f] \} \]

\[ = lD(f; \varphi). \tag{1} \]

If we choose \( \epsilon > 0 \) so small that

\[ \epsilon |l| \max |\varphi_1| + \epsilon |m| \max |\varphi_2| < \eta, \]

\( D(f + l\epsilon \varphi_2; \varphi_1) \) will exist by II, and

\[ F[f + l\epsilon \varphi_1 + l\epsilon \varphi_2] - F[f + l\epsilon \varphi_2] = \epsilon lD(f + l\epsilon \varphi_2; \varphi_1) + \epsilon \delta, \]

\[ F[f + l\epsilon \varphi_1] - F[f] = \epsilon lD(f; \varphi_1) + \epsilon \delta', \]

where \( \delta, \delta' \) approach 0 with \( \epsilon \). Then

\[ P(l\varphi_1, m\varphi_2) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{ F[f + l\epsilon \varphi_1 + l\epsilon \varphi_2] - F[f + l\epsilon \varphi_2] \}

\[ - F[f + l\epsilon \varphi_1 + F[f]] \}

\[ = l \lim_{\epsilon \to 0} [D(f + l\epsilon \varphi_2; \varphi_1) - D(f; \varphi_1)]. \tag{2} \]

Similarly

\[ P(l\varphi_1, m\varphi_2) = m \lim_{\epsilon \to 0} [D(f + l\epsilon \varphi_2; \varphi_2) - D(f; \varphi_2)]. \tag{3} \]

The expression (2) is the product of \( l \) and a function of \( m \) independent of \( l \), while (3) is the product of \( m \) and a function of \( l \) only, and they are equal. Each must be a product of \( lm \) and an expression \( K \) independent of \( l, m \).

\[ P(l\varphi_1, m\varphi_2) = lmK(\varphi_1, \varphi_2). \]

In this make \( l = 1 = m \); then

\[ P(\varphi_1, \varphi_2) = K(\varphi_1, \varphi_2). \]

Or

\[ P(l\varphi_1, m\varphi_2) = lmP(\varphi_1, \varphi_2). \]

Making \( m = l \),

\[ P(l\varphi_1, l\varphi_2) = l^2P(\varphi_1, \varphi_2). \]

But

\[ P(l\varphi_1, l\varphi_2) = l \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{ F[f + l\epsilon \varphi_1 + l\epsilon \varphi_2] \}

\[ - F[f + l\epsilon \varphi_1 + l\epsilon \varphi_2] - F[f + l\epsilon \varphi_2] + F[f] \}

\[ = lP(\varphi_1, \varphi_2). \]

\[ \therefore l^2P(\varphi_1, \varphi_2) = lP(\varphi_1, \varphi_2) \]

\[ P(\varphi_1, \varphi_2) = 0. \]
Or
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ F[f + \varepsilon \varphi_1 + \varepsilon \varphi_2] - F[f] - F[f + \varepsilon \varphi_1] + F[f] \right] = 0,
\]
(4) \[ D(f; \varphi_1 + \varphi_2) - D(f; \varphi_1) - D(f; \varphi_2) = 0. \]
Combining (1) and (4), we see that
\[ D(f; c_1 \varphi_1 + c_2 \varphi_2) = c_1 D(f; \varphi_1) + c_2 D(f; \varphi_2). \]
Thus the variation is distributive in \( \varphi \). Secondly, from condition I,
\[ |F[f + \varepsilon \varphi] - F[f]| \leq M \varepsilon \max |\varphi|, \]
or
\[ |D(f; \varphi)| = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |F[f + \varepsilon \varphi] - F[f]| \leq M \max |\varphi|. \]
The variation is also bounded, considered as an operation on \( \varphi \). This proves it to be a linear functional by Riesz’s definition. To find the integrating function \( \alpha(x) \) we may proceed as follows:
Let \( \varphi(x; c, d) \) denote the continuous function,
\[ \varphi = 1, \quad a \leq x \leq c, \]
\[ = 0, \quad d \leq x \leq b, \]
\( \varphi \) linear from \( c \) to \( d \).
Then
\[ \alpha(c) = \lim_{\varepsilon \to c} D(f; \varphi), \]
and in general for any continuous \( \varphi(x) \),
\[ D(f; \varphi) = \int_a^b \varphi(x) d\alpha(x). \]

Rice Institute,
Houston, Texas.