series practically as useful as the converging series, perhaps even more so, for it is very frequent that the greater the ultimate divergence, the greater also is the primitive tendency towards convergence."

The theorem that "the first term neglected is a superior limit of the error of approximation," though, as De Morgan says, not universally true, is true, he says, of large classes of alternating series, including the series $\phi(x) - \phi(x+1) + \phi(x+2) - \cdots$ "for all cases in which $\phi(x)$ can be the expressed by $\int_\alpha^\beta e^{nvx}X_v dv$, X_v being always positive between limits."

In the development of the modern theories of divergent series, Augustus De Morgan deserves to be ranked as a pioneer.

On December 23, 1857, Sir William R. Hamilton* wrote to De Morgan: "About diverging series, you know a great deal more than I do. In fact you are aware that I early conceived a sort of prejudice against them, in consequence of some of Poisson's remarks. Counter-remarks of yours had staggered me, but had not been carefully weighed. At last (and, I regret to say it, without having yet found the Papers by you and Stokes on *such* series, for Stokes, or Adams for him, sent me about a month ago a duplicate of his memoir on the numerical calculation of the values of certain definite integrals, having a great affinity to my last Paper) I am become a convert to those Divergents; so far at least as to be satisfied that in an extensive class of cases, and with suitable limitations, they may be safely and advantageously used."

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RUSSELL'S INTRODUCTION TO MATHEMATICAL PHILOSOPHY.

Introduction to Mathematical Philosophy. By BERTRAND RUSSELL. (The Library of Philosophy.) London, Allen and Unwin, and New York, The Macmillan Company, 1919. 8vo. viii + 208 pp. $3.00.

THIS book, called an introduction to mathematical philosophy, is an excellent introduction to that field and, more

* R. P. Graves, Life of Sir William Rowan Hamilton, vol. 3, 1899, p. 538.

particularly, to mathematical logic. In the preface the author brings out the fact that mathematical logic is relevant to philosophy and "for this reason, as well as on account of the intrinsic importance of the subject, some purpose may be served by a succinct account of the main results of mathematical logic in a form requiring neither a knowledge of mathematics nor an aptitude for mathematical symbolism."

The first chapter is concerned with the logical basis of the series of natural numbers. The system of postulates of Peano is discussed in some detail. The postulates used in "arithmetization" are indefinite and there is an increase in definiteness produced by "logicizing" mathematics. We cannot, by Peano's method, explain what we mean by the undefined terms 0, number, and successor in terms of simpler concepts although we may know what we mean by them. Russell says: "It is quite legitimate to say this (the last statement) when we must, and at *some* point we all must; but it is the object of mathematical philosophy to put off saying it as long as possible. By the logical theory of arithmetic we are able to put it off for a very long time."

In Chapters II and III the logical theory of the natural numbers is developed. Chapter II contains an exposition of the definition of cardinal number given by Frege, i.e., the cardinal number of a class is the class of all those classes that are similar to it. The following chapter is headed "Finitude and Mathematical Induction." There the definitions of 0, successor, hereditary property, hereditary class, inductive property, inductive class and the posterity of a natural number are given in terms of elemental logical concepts. The "natural numbers" are defined as the posterity of 0 with respect to the relation "immediate predecessor." The idea back of this procedure is that of mathematical induction. Russell emphasizes the fact that mathematical induction is a definition and not a principle. "There are some numbers to which it can be applied and there are others to which it cannot be applied. We *define* the 'natural numbers' as those to which proofs by mathematical induction can be applied, i.e., as those that possess all inductive properties." For this reason the author prefers the term "inductive numbers" to "natural numbers." Of course, this point of view is legitimate on the basis of the procedure above outlined. With respect to another setting up of the natural numbers mathematical induction might well be a principle.

Chapter IV is devoted to order relations and Chapters V and VI to relations in general. In the treatment of order the three kinds of relations, asymmetrical, transitive and connected, are defined preliminary to giving the following definition: A series or a serial relation is a relation which is asymmetrical, transitive and connected. On the basis of these definitions it is shown how the "natural numbers" can be ordered serially. Other examples of series are also given. Relations which do not have the three characteristic properties of serial relations are discussed and the chapter closes with a brief account of series for which the defining relation is between more than two terms. The relation "between" is discussed in some detail. Chapter V treats in a general way of relations. Neither here nor anywhere else in the book is "relation" defined. A clear cut definition of relation, say as a correspondence or as the underlying propositional function, and its discussion would seem to be essential to a treatment of relations such as given in this book. The author does not make this omission in the Principia (volume 1) and there seems to be no good reason for making it here. The necessary material is right at hand. The material of the present chapter is largely a repetition of matter which appeared in previous chapters, but its importance warrants repetition. Relations of the following kinds are considered: asymmetrical, transitive, connected, ancestral, one-one, one-many, many-one, many-many. In discussing the similarity of relations in Chapter VI the following two definitions are fundamental: "A relation S is said to be a correlator of two relations P and Q if S is one-one, has the field of Q for its converse domain and is such that P is the relative product of S and Q and the converse of S." "Two relations P and Q are similar if there exists at least one correlator of P and Q." When two relations are similar they share all properties which do not depend upon the actual terms in their fields. In this connection the question arises: "Given some statement in a language of which we know the grammar and the syntax, but not the vocabulary, what are the possible meanings of such a statement, and what are the meanings of the unknown words that would make it true?" This question is important because "it represents, much more nearly than might be supposed, the state of our knowledge of nature."

The notion of similarity leads to the concept "the relation number of a given relation": the class of all those "relations that are similar to the given relation." The ordinal numbers are special cases of relational numbers. The chapter closes with an interesting application to the philosophical speculation concerning a comparison between an objective and a subjective world.

In Chapter VII the idea of number is extended by supplying logical definitions of rational, real and complex numbers. The author remarks that the discovery of correct definitions in this field was delayed by the common idea that each extension of number included the previous sorts as special cases. The definition of positive and negative integers, which is here given, is: "If m is any inductive number (natural number) then $+ m$ is the relation of $n + m$ to n for any n (a cardinal number) and $- m$ is the converse relation, i.e., the relation of n to $n + m$." According to this definition "$+ m$ is every bit as distinct from m as $- m$ is." A definition of a similar sort is given for positive and negative ratios. Definitions of the following terms are then given: "upper limit (lower limit) of a class α with respect to a relation P"; "maximum (minimum) of a class α with respect to a relation P"; "upper (lower) boundary of a set α." A "real number" is a segment of the series of ratios in order of magnitude. An "irrational number" is a segment of the series of ratios which has no boundary. A "rational number" is a segment of the series of ratios which has a boundary. In these definitions a segment is that class of the two determined by a Dedekind cut which contains the smaller numbers. A complex number is defined as an ordered pair of real numbers. The various arithmetical operations are defined and discussed for each particular class of numbers. The extensions in this chapter do not involve infinity.

In the next two chapters the notion of number is applied to infinite collections. On the basis of the assumption that no two inductive numbers have the same successor (given by Peano) it is shown that the number of inductive numbers is a new number not possessing all inductive properties. To quote the author again: "The difficulties that so long delayed the theory of infinite numbers were due largely to the fact that some, at least, of the inductive properties were wrongly judged to be such as *must* belong to all numbers; indeed it was thought that they could not be denied without contradiction. The first step in understanding infinite numbers consists in realizing the mistakenness of this view." The course of the discussion of this chapter leads naturally to the definitions: A reflexive class is one which is similar to a proper part of itself; a reflexive cardinal number is the cardinal number of a reflexive class. In order to give a definition of the number of inductive numbers the following definition of a progression is given: A progression is a one-one relation such that there is just one term belonging to the domain but not to the converse domain and the domain is identical with the posterity of this one term. The number of inductive numbers, $\aleph_0$, is the set of all domains of progressions. Some properties of $\aleph_0$ are developed. A finite class or cardinal is defined as one which is inductive and an infinite class or cardinal is one which is not inductive. The statement is made without proof that all reflexive classes are infinite (non-inductive). The reader is referred to a later chapter for the connection between the theorem that all infinite classes are reflexive and the multiplicative axiom. The higher transfinite cardinals and ordinals are briefly discussed and Chapter IX closes with a review of the formal laws obeyed by the transfinite cardinals and ordinals. A general definition of a transfinite ordinal is not given. "The importance of ordinals, though by no means small, is distinctly less than that of cardinals, and is very largely merged in that of the more general conception of relation-numbers."

Limits and continuity are the topics discussed in the next two chapters. The ordinal character of the notion of limit is emphasized. In the first of these chapters the notion of the limit of a set of elements and such related notions as minima, maxima, sequents, precedents, upper limits, lower limits and boundaries of a class with respect to a given relation are defined. A brief treatment of the Dedekind and Cantor definitions of continuous series is given at the end of the chapter. The other chapter is devoted to limits and continuity of functions and is more technical. The ordinary definitions of $\lim_{x \to a} f(x)$ and continuous function are given, though sometimes in a more abstract form.

The rest of the book is devoted to the logic (proper) of mathematics, the various topics treated becoming more and more elemental as the end of the book is reached. In Chapter

XII a very clear discussion of the multiplicative axiom is given. It is shown how this axiom or a weaker form is needed to prove such theorems as these: that any class can be well-ordered; that the sum of $\aleph_0$ classes of $\aleph_0$ members each has $\aleph_0$ members; that a non-inductive class is reflexive. The author's reaction to questions in this chapter is contained in the closing paragraph: "It is not improbable that there is much to be discovered in regard to the topics discussed in the present chapter. Cases may be found where propositions which seem to involve the multiplicative axiom can be proved without it. It is conceivable that the multiplicative axiom in its general form may be shown to be false. From this point of view, Zermelo's theorem offers the best hope: the continuum or some still more dense series *might* be proved to be incapable of having its terms well ordered, which would prove the multiplicative axiom false, in virtue of Zermelo's theorem. But so far, no method of obtaining such results has been discovered, and the subject remains wrapped in obscurity."

The subject of the next chapter is "The Axiom of Infinity and Logical Types." One form of the axiom of infinity is "If n be any inductive cardinal number, there is at least one class of individuals having n terms." Without the axiom of infinity or its equivalent the theory of real numbers and the theory of transfinite numbers would not exist. Russell spends some time showing that the axiom of infinity cannot be proved after postulating a class of individuals by forming the complete set of individuals, classes, classes of classes, etc. This kind of reasoning leads to such contradictions as the existence of the greatest cardinal number, the class of all classes, etc. The fallacy of the reasoning consists in the formation of a "class which is not pure as to type." At this point a little is said about the theory of types but the mention is too brief to be satisfying. A good brief exposition of the theory of types is probably impossible at this time. Some pertinent remarks are: "Classes are logical fictions and a statement which appears to be about a class will only be significant if it is capable of translation into a form in which no mention is made of the class." "If there are n individuals in the world and 2^n classes of individuals we cannot form a new class, consisting of both individuals and classes and having $n + 2^n$ members." The author does not pretend to have explained the doctrine of types, but his object is to indicate why there is need for such a doctrine. Other "proofs," more or less metaphysical, of the axiom of infinity are briefly examined.

The next four chapters are the most fundamental of the book. Their object is a critique of the notion of *class*. The topics of the first three of these chapters, viz: (1) the theory of deductions and incompatibility, (2) propositional functions, (3) descriptions, although very important in themselves, are introductory to the study of the theory of classes given in the last of these chapters.

In the chapter on the theory of deduction Russell restates his thesis that "what can be known, in mathematics and by mathematical methods, is what can be deduced from pure logic." The essential part of the chapter consists of the definitions of the "truth-functions": "not-p" (negation); "p or q" (disjunction); "p and q" (conjunction); "p and q are not both true" (incompatibility); "not-p or q" (implication). All five truth-functions are not independent. Two, negation and disjunction, were chosen in the Principia Mathematica as fundamental and the others defined in terms of these. Sheffer has shown that one primitive idea is sufficient for that purpose. It is here shown that the single primitive idea of incompatibility is sufficient. An analysis of deduction is made on the basis of the five formal principles of deduction given in the Principia. A formal principle of deduction (e.g., "p or p implies p") has a double use: to serve as the premise of an inference or as a rule of deduction. A proof that the five formal principles can be reduced to one is given in detail. This single formal principle which is much more complicated, at least in statement, than any of the five is due to M. Nicod. This formal principle and two non-formal principles furnish the apparatus from which the whole theory of deduction follows "except in so far as we are concerned with deduction from or to the existence or the universal truth of propositional functions," which are studied in the next chapter. The chapter closes with an argument in support of the author's views on implication as against those of C. I. Lewis.

Of a propositional function it might be said that it is true in all cases or that it is true in some cases. The importance of this use of propositional functions is clearly pointed out in Chapter XV. All the primitive propositions of logic as well as the principles of deduction consist of statements that certain propositional functions are always true. It is explained how the truth-functions as applied to propositions containing apparent variables can be defined in terms of the definitions and primitive ideas for propositions containing no apparent variables. For this it is found necessary to take as primitive ideas two of the following: "always," "sometimes," "not-ϕx sometimes" (or "always" as the case may be). The simpler forms of traditional formal logic really involve the assertion of all values or some values of a compound propositional function. For example: "all S is P" means "ϕx implies ψx is always true" where ϕx and ψx denote propositional functions. Russell's treatment of traditional logic leads to such results as the following: "if there are no S's then 'all S is P' and 'no S is P' will both be true, whatever P may be." Some reasons for preferring his treatment are given in convincing form.

The fundamental meaning of "existence" is contained in the following statement: If the propositional function ϕx is sometimes true we say that arguments satisfying ϕx *exist.* We may say men exist (here ϕx is x is a man) but it is nonsense to say John exists. Another instance of the use of propositional functions which we are considering is in the notions of "modality" (*necessary, possible* and *impossible*). An undetermined value of a propositional function ϕx is necessary if the function is always true, possible if sometimes true and impossible if it is never true. At the end of the chapter we have this sentence: "For clear thinking, in many diverse directions, the habit of keeping propositional functions sharply separated from propositions is of the utmost importance, and failure to do so in the past has been a disgrace to philosophy."

The theory of descriptions, treated in the next chapter, is very important from the point of view of logic and the theory of knowledge. Only those parts of the theory which are relevant to mathematics are here discussed. A proposition involving an indefinite description about "a so-and-so" is of the form "an object having the property ϕ has the property ψ" which means "The joint assertion of ϕx and ψx is not always false." It is an important point that such propositions contain no constituent represented by the phrase "a so-and-so." Thus such propositions can be significant when there is no such thing as "a so-and-so." This is the solution of the philosophical question of "unreality" which Russell gives. The definition of propositions involving definite descriptions is: "the term satisfying ϕx satisfies ψx" which means that there is a term c such that (1) ϕx is always equivalent to 'x is c,' (2) ψc is true." The extra condition of uniqueness is added in this case.

The theory of classes, which is taken up in Chapter XVII, is concerned with the word *the* in the plural while that of definite descriptions deals with the singular meaning of that word. Because of the paradoxes involving the notion of class the latter cannot be taken as a primitive idea. It is desired to find "a definition which will assign a meaning to propositions in whose verbal or symbolic expression words or symbols apparently representing classes occur but which will assign a meaning that altogether eliminates all mention of classes from a right analysis of such propositions." The theory here outlined reduces propositions nominally about classes to propositions about the propositional functions which define them. The theory is incomplete because it is thrown back, in part, upon the incomplete theory of types. Because of this incompleteness it is found necessary to assume the axiom of reducibility: there is a type τ such that if ϕ is a function which can take a given object a as argument, then there is a function ψ of the type τ which is formally equivalent to ϕ. The fundamental definition of the theory of classes is: if ϕ is a function which can take a given object a as argument, and τ the type mentioned in the above axiom, then to say that the class determined by ϕ has the property f is to say that there is a function of type τ, formally equivalent to ϕ, and having the property f.

The final chapter "Mathematics and Logic" opens with an assertion of the Russellian thesis that logic and mathematics are identical. The proof in all detail is not given, but one is referred to the Principia. The remainder of the chapter is devoted to a discussion of what is characteristic of mathematical (or logical) propositions. Logical propositions affirm that some propositional function is always true. Specific propositions whose truth depends upon something else than the form of the propositions do not belong to mathematics but to its applications. Mathematical propositions have the characteristic described, perhaps, by the word "tautology."