THE EINSTEIN SOLAR FIELD.

BY PROFESSOR LUTHER PFHAHLER EISENHART.

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The Schwarzschild form of the linear element of the Einstein field of gravitation of a mass $m$ at rest with respect to the space-time frame of reference is

$$ds^2 = \left(1 - \frac{2m}{u_1}\right) dt^2 - \frac{u_1}{u_1 - 2m} du_1^2 - u_1^2 (du_2^2 + \sin^2 u_2 du_3^2),$$

where $t$ is the coordinate of time, and $u_1$, $u_2$, $u_3$ are space coordinates. Since the coefficients in (1) are independent of $t$, the particle moves in the 3-space $S_3$ whose linear element is

$$ds_0^2 = \frac{u_1}{u_1 - 2m} du_1^2 + u_1^2 (du_2^2 + \sin^2 u_2 du_3^2).$$

If we put

$$x_1 = u_1 \sin u_2 \cos u_3, \quad x_2 = u_1 \sin u_2 \sin u_3,$$

$$x_3 = u_1 \cos u_2, \quad x_4 = 4m \sqrt{\frac{u_1}{2m} - 1},$$

we have

$$ds_0^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

Hence $S_3$ is immersed in the euclidean space of four dimensions, $S_4$, whose rectangular coordinates are $x_i (i = 1, \ldots, 4)$.* Moreover, as follows from (3), $S_3$ is the quartic variety defined by

$$x_1^2 + x_2^2 + x_3^2 = \left(\frac{x_4^2}{8m} + 2m\right).$$

We shall show that equations (3) define the only three-spread with the linear element (2) in euclidean four-space by making use of the following theorem of Bianchi:† In euclidean $n$-space ($n > 3$) every hypersurface is not deformable unless at least $n - 2$ of the principal radii of curvature are infinite.

† Lezioni, vol. 1, p. 467.
In fact we show that all of the principal radii of curvature of (5) are finite. If \( X_i \) denote the direction-cosines of the normal to (5), that is

\[
\sum_{i=1}^{4} X_i \frac{\partial x_i}{\partial u_j} = 0 \quad (j = 1, 2, 3), \quad \sum X_i^2 = 1,
\]

then

\[
X_1 = \sqrt{\frac{2m}{u_1}} \sin u_2 \cos u_3, \quad X_2 = \sqrt{\frac{2m}{u_1}} \sin u_2 \sin u_3,
\]

\[
X_3 = \sqrt{\frac{2m}{u_1}} \cos u_2, \quad X_4 = -\sqrt{1 - \frac{2m}{u_1}}.
\]

If we define functions \( \Omega_{rs} \) by

\[
\Omega_{rs} = -\sum \frac{\partial X_i}{\partial u_r} \frac{\partial x_i}{\partial u_s},
\]

we find \( \Omega_{rs} = 0 \quad (r \neq s) \) and

\[
\Omega_{11} = \frac{1}{u_1 - 2m} \sqrt{\frac{m}{2u_1}}, \quad \Omega_{22} = -\sqrt{2mu_1}, \quad \Omega_{33} = -\sqrt{2mu_1} \sin^2 u_2.
\]

The principal radii of curvature are given by*

\[
\frac{1}{R_1} = \frac{\Omega_{11}}{u_1} = \sqrt{\frac{m}{2u_1^3}}, \quad \frac{1}{R_2} = \frac{\Omega_{22}}{u_1^2} = -\sqrt{\frac{2m}{u_1^3}},
\]

\[
\frac{1}{R_3} = \frac{\Omega_{33}}{u_1^2 \sin^2 u_2} = \sqrt{\frac{2m}{u_1^3}},
\]

which are finite since \( u_1 \neq 0 \).

In accordance with the Einstein theory the world-line of a particle in the gravitational field is a geodesic of the space with the linear element (1), that is a curve along which \( \int ds \) is stationary; and the world-line of a ray of light is a curve for which \( ds = 0 \) and \( \int dt \) is stationary. In each case the frame of reference can be so chosen that a particular path satisfies the condition \( u_2 = \pi/2 \).† From (3) it follows that for this path \( x_3 = 0 \), and hence the path considered by astronomers is the projection upon the plane \( x_3 = 0 \) of a curve on the surface

\[
x_1^2 + x_2^2 = \left( \frac{x_4^2}{8m} + 2m \right)^2.
\]

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* Bianchi, l.c., pp. 368, 472.
This is the surface of revolution of a parabola of latus rectum 8m about its directrix. A similar result was obtained by Flamm* who considered the surface, in euclidean three-space, for which the linear element is given by (2) for $u_2 = \pi/2$.

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A COVARIANT OF THREE CIRCLES.

By Professor A. B. Coble.

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Dr. J. L. Walsh † has stated the following theorem.

Theorem. If the double ratio, $\left( \frac{z_1}{z_2} \mid \frac{z_3}{z} \right)$, of the four points $z_1, z_2, z_3, z$ in the complex plane is a real number $\lambda$, then as the points $z_1, z_2, z_3$ run over the circles $C_1, C_2, C_3$ (and their interiors) respectively, the locus of $z$ is a circle (and its interior).

This locus is evidently a covariant, under the inversive group, of the three given circles, which is rational in $\lambda$. We find in (8) its equation and incidentally prove the theorem.

In conjugate coordinates $z, \bar{z}$, a circle is

$$C_1(z) = a_1\bar{z}z + a_2z + \bar{a}_2\bar{z} + b_1 = 0,$$

where $a_1, b_1$ are real, and $a_2, \bar{a}_2$ are conjugate imaginary. The bilinear invariant of two circles $C_1(z), C_2(z)$ is

$$[C_1, C_2] = \alpha_1\bar{a}_2 + a_2\bar{a}_1 - a_1b_2 - a_2b_1.$$

It vanishes when the two circles are orthogonal. When they coincide it becomes $[C_1C_1] = 2(\alpha_1\bar{a}_1 - a_1b_1)$. This vanishes when $C_1$ is a point circle, i.e. one whose equation is

$$(1) \quad P_z(z) = (z - z_i)(\bar{z} - \bar{z}_i) = 0.$$

It is easily verified that

$$[C_1, P_z(z)] = -C_1(z_i); \quad [P_z(z), P_z(z)] = -P_z(z_i) = -P_z(z_k).$$

The two point circles of the pencil $C(z) + \mu K(z) = 0$ are determined by

$$[C + \mu K, C + \mu K] = [C, C] + 2\mu [CK] + \mu^2 [KK] = 0.$$