NOTE ON THE DIVISION OF A PLANE BY A POINT SET*

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A plane set of points $K$ is said to divide a plane $S$ if the set $S - K$ is composed of two mutually exclusive domains $S_1$, $S_2$, of which $K$ is a common boundary, where by domain is meant a connected open set. The condition that $K$ be a simple closed curve or an open curve has been stated by J. R. Kline† in terms of the concept "connected im kleinen." In proving that the set $K$ is a connected set, Kline employs the condition "connected im kleinen."

If we assume that $K$ is bounded and that we have at our disposal the parallel and perpendicular straight lines of a number plane, then the connectedness of $K$ is established by other writers, for example Hausdorff, Grundzüge der Mengenlehre, page 346, Theorem XII. Hausdorff calls attention, in a footnote to page 342, to the difficulty of extending his argument to the case of unbounded sets.

It seems in view of the importance of the theory of open curves as indicated by R. L. Moore‡ and of the importance in general of the fundamental theorems of plane analysis situs that it is of interest to show that the set $K$ is connected, whether bounded or not, without the use of the restriction employed by Kline or of the properties of straight lines and rectangles. The present note is concerned, therefore, with the proof of the following theorem.

THEOREM. Let $K$ be a plane point set, $S$, the set of all points of the plane, and denote by $S_1$, $S_2$ two mutually exclusive domains such that

$$S - K = S_1 + S_2.$$  

Then if every point of $K$ is a limit point of both $S_1$ and $S_2$, the set $K$ is closed and connected.

* Presented to the Society Nov. 26, 1921.
‡ R. L. Moore, On the foundations of plane analysis situs. Transactions of this Society, vol. 17 (1916), pp. 131–164. This paper will be referred to as "Foundations."
Proof. The set $K$ is closed. For any limit point of $K$ is evidently a common limit point of $S_1$ and $S_2$. But no limit point of $S_1$ can belong to the domain $S_2$, and likewise no limit point of $S_2$ can belong to $S_1$. Such a point must belong to $K$.

Assume that the set $K$ is not connected. Then there exist two mutually exclusive closed subsets $K_1$ and $K_2$ of $K$ such that $K = K_1 + K_2$. Let $P_i$ denote a point of $K_i$ ($i = 1, 2$). We may enclose $P_i$ in a region $R_i$ which contains no point of $K_{i+1}$.* Let $J_i$ be a simple closed curve lying in $R_i$ and containing $P_i$ as an interior point. Let $P_{ij}$ be a point of $S_j$ lying within $J_i$. Since $S_j$ is a domain, there is an arc

$$P_{1j} X P_{2j}$$

lying entirely in $S_j$. Let $A_{1j}$ be the last point which this arc has in common with $J_1$ and let $A_{2j}$ be the first point which the arc has in common with $J_2$ after $A_{1j}$ on $P_{1j} X P_{2j}$.

Since the point $A_{ij}$ lies on the boundary of $J_i$, it may be connected with $P_i$ by an arc $P_i X A_{ij}$ which, except for the end-point $A_{ij}$, lies entirely within $J_i$. Furthermore the arcs $P_i X A_{1i}$ and $P_i X A_{2i}$ may be constructed so that they have no common point besides $P_i$.

From the arcs so defined we construct a simple closed curve $J$,

$$J: P_1 A_{11} A_{21} P_2 A_{22} A_{12} P_1$$

Let $H_1$ denote the set of all points of $K_1$ which are interior to $J$ but not interior to $J_1$. The set $H_1$ is closed.

Case I. No point of $H_1$ lies on $J_1$. Then the points of $J_1$ which lie in $J$ lie in $S_1 + S_2$. There exist points of $J_1$ within $J$, since by Theorem 40 of the *Foundations* it is possible to join $P_1$ and $P_2$ by an arc $P_1 X P_2$ such that $P_1 X P_2$† lies entirely within $J$. This arc must meet $J_1$ in at least one point. It follows readily that there is an arc $A_{11} X A_{12}$ of $J_1$ which lies in $J$ and therefore in $S_1 + S_2$. This is contrary to Lemma A of the paper of J. R. Kline.‡

* The subscripts are to be reduced modulo 2.
† The symbol $A X B$ denotes the set $A X B - A - B$, that is, the set of all points of the arc $A X B$ except its end-points.
‡ Loc. cit., p. 452. Every arc joining a point of $S_1$ to a point of $S_2$ contains a point of $K$. 
CASE II. The curve \( J_1 \) contains a point of \( H_1 \). From the Heine-Borel property we may assume a finite set of regions

\[(1) \quad \overline{R}_1, \overline{R}_2, \ldots, \overline{R}_m,\]

covering \( H_1 \). We may without loss of generality assume that the boundary of each region \( \overline{R}_k \) \((k = 1, 2, \ldots, m)\) is a simple closed curve \( \overline{J}_k \). We will also assume that no point of \( J, J_2, \) or \( K_2 \) lies in or on the boundary of any of the regions \( \overline{R}_k \).

By hypothesis some of the regions \( \overline{R}_k \) have points in common with the interior of \( J_1 \). Let

\[(2) \quad \overline{R}_1, \overline{R}_2, \ldots, \overline{R}_p, \quad (p \leq m),\]

be a subset of the regions (1) such that \( \overline{R}_1 + \cdots + \overline{R}_p \) forms, with the interior of \( J_1 \), a connected set. Then the curves \( J_1, \overline{J}_1, \overline{J}_2, \ldots, \overline{J}_p \) form a finite family \( G \) of closed curves whose interiors form a connected set. By theorem 42 of the Foundations there is a simple closed curve \( \overline{J} \) which satisfies these conditions: every point of \( \overline{J} \) belongs to some curve of \( G \); the interior of \( \overline{J} \) contains the interiors of all the curves of the set \( G \).

The curve \( \overline{J} \) meets \( J \) in the points \( A_{11} \) and in no other points. We may show as in Case I that there is a point \( X \) on \( \overline{J} \) which is interior to \( J \), and that the arc \( A_{11}X \) of the curve \( \overline{J} \) lies within \( J \).

We will show that no point of \( A_{11}X \) is in \( K \). By construction no point of \( K_2 \) lies on any curve of the set \( G \). Suppose \( Q \), a point of \( K_1 \), lies on the arc \( A_{11}X \). Then \( Q \) must lie on one of the curves of the set \( G \). If \( Q \) is on \( J_1 \), it must belong to \( H_1 \) since \( Q \) is interior to \( J \). Consequently \( Q \) is in \( H_1 \) in any case and must lie in some region \( \overline{R}_k \) \((k \leq m)\) of the regions (1). Suppose that \( Q \) lies on \( \overline{J}_q \) \((q \leq p)\). Then \( \overline{R}_k \) must contain an interior point of \( \overline{J}_q \). But the interior of \( \overline{J}_q \) is connected with \( R_1 \). Consequently \( \overline{R}_k \) is connected with \( R_1 \) and is interior to \( \overline{J} \). It follows that \( Q \) is an interior point of \( \overline{J} \), contrary to hypothesis.

Since no point of \( K \) lies on \( A_{11}X \), we have obtained an arc connecting a point of \( S_1 \) with a point of \( S_2 \) which contains no point of \( K \). This again contradicts Lemma A.

The proof that the set \( K \) is connected is completed.