CONDITION THAT A TENSOR BE THE CURL OF A VECTOR *

BY L. P. EISENHART

It is the purpose of this note to establish the following theorem.

THEOREM. A necessary and sufficient condition that a co-

variant skew-symmetric tensor $A_{ij}$ in a space of any order $n$ be expressible in terms of $n$ functions $\varphi_i$ in the form

(1) $A_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}$

is that

(2) $\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} = 0, \quad (i, j, k = 1, \ldots, n)$.

Consider first the case of 3-space. If $\varphi_2$ and $\varphi_3$ are any two functions such that

$A_{23} = \frac{\partial \varphi_2}{\partial x^3} - \frac{\partial \varphi_3}{\partial x^2}$,

the conditions of integrability of

$\frac{\partial \varphi_1}{\partial x^2} = \frac{\partial \varphi_1}{\partial x^1} + A_{12}, \quad \frac{\partial \varphi_1}{\partial x^3} = \frac{\partial \varphi_3}{\partial x^1} + A_{13}$

are satisfied in consequence of (2), and the theorem is established for 3-space.

Now we show that, if the theorem is true for $n$-space, it is true for $(n + 1)$-space. On this assumption equations (1) hold for $i, j = 1, \ldots, n$. For a particular $i$ and $j$ and for $k = n + 1$, equation (2) may be written in the form

$\frac{\partial}{\partial x^i} \left( A_{jn+1} - \frac{\partial \varphi_j}{\partial x^{n+1}} \right) = \frac{\partial}{\partial x^j} \left( A_{in+1} - \frac{\partial \varphi_i}{\partial x^{n+1}} \right)$.

Hence a function $\varphi_{n+1}$ is defined by the equations

(3) $A_{in+1} = \frac{\partial \varphi_i}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^i}, \quad A_{jn+1} = \frac{\partial \varphi_j}{\partial x^{n+1}} - \frac{\partial \varphi_{n+1}}{\partial x^j}$.

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Replacing $j$ in (2) by $l (= 1, \cdots, n; \neq j)$, we have, by (3),

\[
\frac{\partial A_{i_{n+1}}}{\partial x^i} = \frac{\partial}{\partial x^i} \left( \frac{\partial \varphi_i}{\partial x^{i_{n+1}}} - \frac{\partial \varphi_{n+1}}{\partial x^i} \right).
\]

Consequently (1) holds for $i, j = 1, \cdots, n+1$, and the theorem is established. It should be remarked that one of the functions $\varphi_i$ may be chosen arbitrarily, or what is equivalent, that the functions $\varphi_i$ are determined to within additive functions $\partial \psi / \partial x^i$, where $\psi$ is an arbitrary function of the $x$'s.

Thus far we have made no use of the fact that $A_{ij}$ are the components of a covariant tensor. If $A'_{\alpha \beta}$ denote the components of the tensor in terms of coordinates $x'$, then

\[
(4) \quad A'_{\alpha \beta} = A_{ij} \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}}.
\]

If $\Gamma^i_k$ and $\Gamma'^i_\beta$ denote the Christoffel symbols of the second kind for the respective systems of coordinates $x$ and $x'$ of a Riemannian geometry, then*

\[
\frac{\partial^2 x^p}{\partial x'^i \partial x'^j} = \Gamma^i_{ij} \frac{\partial x^p}{\partial x'^i} - \Gamma^p_i \frac{\partial x^a}{\partial x'^i} \frac{\partial x^r}{\partial x'^j}.
\]

The same equations obtain in the more general case of a geometry of paths, where the functions $\Gamma^i_{\alpha \beta}$ and $\Gamma'^i_\beta$ are the coefficients of the equations of the paths in the two systems of coordinates.† By means of these equations we show that, if the functions $A_{ij}$ satisfy (2), so also do $A'_{\alpha \beta}$ defined by (4). In consequence of the above theorem equation (4) may be replaced by the equation

\[
\frac{\partial \varphi'_{\alpha}}{\partial x'^{\beta}} - \frac{\partial \varphi'_{\beta}}{\partial x'^{\alpha}} = \left( \frac{\partial \varphi_{\alpha}}{\partial x^i} - \frac{\partial \varphi_{\beta}}{\partial x^i} \right) \frac{\partial x^i}{\partial x'^{\alpha}} \frac{\partial x^j}{\partial x'^{\beta}} = \frac{\partial \varphi_{\alpha}}{\partial x^i} \frac{\partial x^j}{\partial x'^{\beta}} - \frac{\partial \varphi_{\beta}}{\partial x^i} \frac{\partial x^j}{\partial x'^{\alpha}}
\]

\[
= \frac{\partial}{\partial x'^{\beta}} \left( \varphi_i \frac{\partial x^i}{\partial x'^{\alpha}} \right) - \frac{\partial}{\partial x'^{\alpha}} \left( \varphi_j \frac{\partial x^j}{\partial x'^{\beta}} \right).
\]

Hence

\[
(5) \quad \varphi'_{\alpha} = \varphi_{\alpha} \frac{\partial x^i}{\partial x'^{\alpha}} + \frac{\partial \psi}{\partial x'^{\alpha}},
\]

* Bianchi, Lezioni, vol. 1, p. 64.
where $\psi$ is an arbitrary function.

From (5) it is evident that if $A_{ij}$ are defined as the components of the curl of covariant vector, then (2) are necessarily satisfied; but (2) is not a sufficient condition. That this condition is not sufficient was overlooked by me in a recent paper,* and my conclusions in § 5 are not correct. In fact, the skew-symmetric tensor there defined by $S_{ij}$ is given by

$$S_{ij} = \frac{\partial \Gamma_{\alpha j}}{\partial x^i} - \frac{\partial \Gamma_{\alpha i}}{\partial x^j},$$

and the functions $\Gamma_{\alpha j}$ and $\Gamma_{\alpha i}$ in two sets of coordinates are in the relation

$$\Gamma'^{\alpha ^i}_{\alpha j} = \Gamma_{\alpha j} \frac{\partial x^j}{\partial x'^i} + \frac{\partial}{\partial x'^i} \log \Delta,$$

where $\Delta$ is the Jacobian of the transformation.

A NEW GENERALIZATION OF TCHEBYCHEFF'S STATISTICAL INEQUALITY

BY B. H. CAMP

1. Introduction. If $f(x)$ is any frequency distribution, and $s$ its standard deviation, the symbol $P(\lambda s)$ may be used to represent the probability that a datum drawn from this distribution will differ from the mean value by as much as $\lambda s$, numerically. For the solution of various statistical problems it is desirable to have a formula which will measure $P(\lambda s)$ when $f(x)$ is only partially known. A case of practical importance occurs when $f(x)$ represents the distribution of values of a statistical constant determined by sampling from a known distribution, such a constant as, for example, a mean value, or a coefficient of correlation. In such cases it is usually difficult or impossible to find the complete distribution $f(x)$, but quite feasible to find its lower moments. Tchebycheff's well known inequality is: $P(\lambda s) \leq 1/\lambda^2$. It has been general-

* Proceedings of the National Academy, vol. 8 (1922), p. 236.