TWO RECENT BOOKS ON ALGEBRAS

Algebras and their Arithmetics. By Leonard Eugene Dickson. University of Chicago Press, 1923. ix + 241 pp.

Corpi Numerici e Algebre. By Gaetano Scorza. Messina, Giuseppe Principato, 1921. ix + 462 pp.

On first picking up Professor Dickson's book, the reader's attention is caught by the fact that the author assumes no knowledge beyond the most elementary parts of a first course in the theory of equations. As the reviewer turned from one page to another, there was a certain ease and simplicity in the style which called to mind some of those science primers of the days of our grandfathers. One felt once more the half-forgotten glamor of the rainy day long ago when one discovered several of these "primers", during one's secret investigation of a pompous old book-case on the landing of the stair.

In the first chapter, he begins with the definition of a field of numbers and the elementary notions of a linear transformation and then easily makes the transition to matrices. Having pointed out that the set of all complex numbers, $a+bi$, is an algebra over the field of all real numbers and that the set of all p-rowed square matrices with elements in any field, F, is an algebra over F, in which multiplication is usually not commutative and in which division may fail, the author defines the general algebra over any field by five postulates which differ from those used in Scorza's book in one respect. After definitions of a few elementary notions in connection with algebras, he returns to matrices and shows that the matric algebra consisting of all two-rowed square matrices whose elements are complex numbers is carried by a linear transformation of the units into the familiar quaternions. Thus, at the very beginning, he gives an illustration of one of the most interesting and important results in the general theory of algebras, to wit, that any associative division algebra (an algebra in which division is always uniquely possible when the divisor is not zero) is equivalent, in a suitably enlarged field, to a matric algebra.

The second chapter is a brief one giving several elementary but useful theorems about linear sets, which are the same as algebras except that they are not necessarily closed under multiplication. Although the third chapter is also rather brief, it presents, in that simple, direct manner so characteristic of most of this author's work, about a dozen theorems and corollaries grouped around the fundamental notions of subalgebra, invariant subalgebra (analogous to self-conjugate subgroup in the theory of groups), reducible algebras and simple algebras. Of these the most significant is the one which asserts that any reducible algebra with a modulus (i. e., a number which behaves under multiplication like unity) can be expressed as a direct sum of irreducible algebras each with a modulus in one and essentially only one way. Together, these two chapters contain many of the theorems used in the later proof of the fundamental theorem of linear algebras due to Wedderburn.

Chapter 4 is concerned primarily with nilpotent algebras and idempotent elements. A nilpotent element is one such that some power of it is zero, and an analogous definition applies to nilpotent algebras. Although such algebras may seem scarcely more than figments of the imagination, yet it is very easy to exhibit one such, to wit, the algebra consisting of a single unit, e_{12}, a square matrix of order two having unity in the first row and second column and zeros elsewhere, for $(e_{12})^2 = 0$. In fact, one wishes that the author had taken a few lines to give a simple example of this type to remove the scruples of the materialistic soul who insists on having everything in mathematics securely anchored to something tangible. After defining idempotent elements (i. e., numbers u such that $u^2 = u$) and properly nilpotent elements (Scorza's *elementi eccezionali)*, he proves Peirce's theorem about the decomposition of an algebra relative to an idempotent element. The last part of the chapter is concerned with semi-simple algebras—those containing no nilpotent invariant subalgebra. In addition to proving theorems which show that, under certain conditions, a subalgebra of a given algebra, A, is necessarily semi-simple, he shows that any semi-simple algebra which is not simple is the direct sum of simple algebras, and conversely. Although the results are necessarily technical, as are all results in mathematics, yet the author manages to keep the number of technical terms down to a minimum, so that, if the reader is willing to do any thinking, he will find that the text is most disarming in its simplicity.

One of the most fascinating chapters in the book is Chapter 5. This is devoted entirely to division algebras, which might be described as fields that are not necessarily commutative, if we use the word "field" (Körper) in Hurwitz' more general sense. In view of Wedderburn's theorem (proved in the next chapter) that every simple algebra over a field F can be expressed as the direct product of a division algebra over F and a simple matric algebra over F, a study of division algebras is exceedingly important. After eight elementary theorems and corollaries, each of which is proved in a very few lines, the author then devotes the remainder of the chapter to some important results on the determination of all division algebras. He first gives a simplification of his own elementary and neat proof of the famous theorem due to Frobenius and C. S. Peirce, to wit, that the only division algebras over the field of reals are (1) the field of reals, (2) the field of complex numbers and (3) the algebra of real quaternions. The text then defines the type of division algebra D of order n^2 over a field F which were discovered by the author and called Dickson algebras by Wedderburn. For the purpose of this review it will suffice to say that they are natural generalizations of real quaternions and are intimately connected with the theory of abelian equations. If the reader is curious, we can but refer him to the text, where the author gives in simplified form the most important results of his very interesting memoir, *Linear associative algebras and abelian equations,* in the TRANSACTIONS for 1914, together with other results. At the end of the chapter is a brief summary in which he tantalizingly refers the reader to Appendices I and II for further results in the general theory of division algebras. These results were proved by Wedderburn in two articles in the TRANSACTIONS for 1914 and 1921 and demonstrate still further the importance of Dickson algebras.

While Chapter 6 may not appeal to the casual reader quite so much as the one on division algebras, yet to any one who has read the preceding chapters thoughtfully, the present chapter on the structure of the general algebra will appear as a thing of beauty. Here will be found Wedderburn's important theorem that essentially every simple algebra can be expressed as the direct product of a division algebra and a simple matric algebra; and conversely. By this theorem are determined all simple algebras over the field of reals. Finally, he proves a vital theorem which, when shorn of technicalities, essentially reduces the problem of the determination of all algebras to the determination of certain special types.

In the next chapter, the author views algebras from an entirely different standpoint. Hitherto, he has proved every theorem by the aid of Wedderburn's calculus of linear sets; but now he recurs to the older method by which an algebra is thought of as given by a multiplication-table. Here he proves Peirce's theorem that every associative algebra is equivalent to a subalgebra of a matric algebra. This is followed by the orthodox set of theorems on characteristic matrices, determinants and equations, and the rank equation.

Although the final chapter of the first part of the book is brief, it is of vital importance. The author picks up the thread which he momentarily dropped at the end of Chapter 6 and then, by using certain principles of Chapter 7, he proceeds to prove the principal theorem which asserts that any associative algebra over a non-modular field, F, can be expressed as the sum of its maximal nilpotent invariant subalgebra and a semi-simple algebra. Thus, in one hundred and twenty-seven pages, he gives the essence (the triple extract, as it were) of all that is interesting and important in the general theory of algebras.

While the first part of the book is largely Wedderburn's work recast by Dickson, the second part (on the theory of arithmetics) is entirely due to Dickson. Just as the theory of complex integers, due to Gauss, is in some respects equivalent to the representation of all substitutions, with rational coefficients, which leave unaltered a sum of two squares, so similarly the transformations which leave unaltered a sum of three squares lead to quaternions. Lipschitz, by the consideration of such substitutions having rational coefficients, was lead to consider integral quaternions and then irreducible or prime quaternions.

In his book,* he defines integral quaternions as those having all coordinates rational integers and then develops a theory of such quaternions by using a rather complicated theory of congruences. But, although his definition of an integral quaternion was in some ways a natural one, it led to a theory with strange contradictions. From this definition he proves that an integral quaternion has, in general, a set of irreducible factors which is, in a certain sense, essentially unique; but if the norm of the quaternion is divisible by 4, then the quaternion has not merely one but twenty-four sets of irreducible factors which are essentially distinct in any understanding of the term.

To Hurwitz belongs the credit of the discovery that the reason for this and other anomalies lay in the definition of integral quaternion used by Lipschitz. In his memoir in the GÖTTINGER NACHRICHTEN for 1896, Hurwitz gave a new definition which enlarged the set of integral quaternions and thus removed the above contradictions. But Du Pasquier, in a series of papers beginning in 1909, noticed that Hurwitz's definition is not satisfactory for every linear algebra and suggested other definitions. But these, in turn, are not suitable for every linear algebra, as they may either be vacuous or lead to insurmountable difficulties.

During the last few years, Professor Dickson has been interested in this problem, and his more important results are given in the second part of this book. In Chapter 9 he gives, in a simple and brief manner, the barest essentials of the theory of integral algebraic numbers; similarly, in Chapter 11, he gives the elements of the theory of fields, so that the reader who is unacquainted with these branches of mathematics can read the long and important Chapter 10, which gives Dickson's new work, with both understanding and appreciation.

In this chapter he defines an integer for any (associative) algebra having a modulus, over the field of rational numbers, in such a way that, when applied to the special case of quaternions, the new definition yields Hurwitz' set of integral quaternions and yet, when applied to

* *Untersuchungen über die Summen von Quadraten,* Bonn, 1886; French translation in JOURNAL DE MATHÉMATIQUES, (4), vol. 2 (1886), pp. 393–439.

the general algebra, it leads to no contradiction or difficulty. He thus proves that the arithmetic of an algebra A is known as soon as we know that of a certain semi-simple subalgebra, S. Similarly, the arithmetic of a semi-simple algebra, S, is known when we know the arithmetic of each of the simple algebras, S_i, of which it is the direct sum. In turn, the integers of a simple algebra are known when we know the integers of a certain division algebra, D. Thus, under the new definition, the problem of arithmetics of all algebras reduces to the case of simple algebras and finally, in large measure, to the case of division algebras. Moreover, the arithmetic of certain simple algebras is treated by generalizing the classic theory of matrices whose elements are integers. As a direct application, he obtains all integral solutions of some Diophantine equations which had not been solved completely previous to Dickson's solution.*

Taking it all in all, this most recent book from the hand of this indefatigable writer will be a source of keen joy to any one who feels that a composition in mathematics, like one in poetry, should have a fine balance between nicety of detail and a broad sweep that gives inspiration. It approaches very close to the French dictum, "Nothing can be added and nothing removed to improve the effect".

Professor Scorza's book is of a very different type. It seems a fair philosophy, both for life in general and for book-reviewing in particular, to judge the attainment of an individual by his purpose. Since the author says, in his preface, that this book is intended for young students in the university, one is inclined to expect a little more organization and conscious orientation than in a volume written for specialists. But the phrase that seems best to describe this book is "an encyclopedic treatise on the theory of algebras", using the term "algebra" in the broad sense as including theory of numbers, matrices, and hypercomplex numbers. The book was to have consisted of three parts: (1) theory of number fields, (2) general theory of hypercomplex numbers or linear algebras, (3) application of (2). But, he confesses, the second part grew so large that he was forced to limit the third part to an appendix of fifty pages. It is to be regretted that he did not include more of the applications, especially those that would show to the mathematician who is inclined to be skeptical of the virtues of any

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* The only typographical errors noticed were the following:—

p. 53, label at top—Replace "§ 37" by "§ 39".

p. 60, l. 10—Replace "primitive element" by "primitive idempotent element".

p. 87, l. 5—Replace "$A-N$ is simple, also A is simple" by "A is simple, also $A-N$ is simple".

p. 178, equation (18)—Replace "$e_i = e_i$" by "$e_i^2 = e_i$".

abstract theory, the relation of hypercomplex numbers to quadric surfaces, abelian integrals, etc. Moreover, as will be seen below, the book would have gained both in power and charm if the second part had been compressed.

Part I, on the general theory of number fields (130 pages) is divided into three chapters. Of these, the first gives the definition of a number field via four postulates, the rudiments of the theory and the basic differences between finite and infinite fields. In this chapter he also introduces the notation for n-tuples and gives the elements of the theory of matrices and determinants. The second chapter is the longest of the three and is devoted to polynomials. After the usual definitions for polynomials in several variables, he gives the orthodox work on divisibility, rational fractions, the zeros of a polynomial and the solution of a system of linear equations together with Leibniz's formula for a general field (modular or otherwise), Galois' generalization of Fermat's theorem and Euclid's algorithm for the highest common factor. In the final chapter, he gives the fundamental properties of Galois fields.

The reader is soon struck by the fact that the author seems to want every definition and theorem expressed with a meticulous care that is rare. All through the text he makes precise statements similar to this: "Polynomials in the same field and in the same indeterminates that differ at most by null monomials are regarded as not distinct". At first, such precise care will please the cautious mathematician; but, after page upon page of such detailed precision, the reader begins to weary. In the case of some theorems, this results in his spreading over a page or more a proof that could be given in a few lines with complete rigor and greater simplicity. In fact, one is inclined to wonder if this style would not tend to dull the interest of the young student for whom the text is written.

Part II, on the general theory of linear algebras (273 pages), is divided into six chapters, of which the first gives in forty-seven pages not only the orthodox elementary work with matrices, but also much interesting material not usually found in a text-book (such as Kowalewski's *Determinantentheorie*) of which the most important are Cayley's theorem that every square matrix satisfies its characteristic equation, Frobenius' beautiful theorem that the equation of lowest degree satisfied by a square matrix is a factor of the characteristic determinant obtained in a certain definite manner and Hadamard's well known result for the maximum value of a determinant.

The remaining chapters of this part are devoted to the general theory of algebras and look much like a revision of the first part of the author's memoir *Le algebre di ordine qualunque e le matrici di Riemann* which appeared in the RENDICONTI DI PALERMO for 1921. In many sections, the point of view and the method are those of Wedderburn's beautiful memoir in the PROCEEDINGS OF THE LONDON SOCIETY for 1908, though Professor Scorza has polished a number of the proofs—some to a slight degree and some to a greater degree, notably the proof of the theorem on the structure of simple algebras.

Although Chapter 2 is, with one exception, the longest in the book, it contains scarcely more than a long array of definitions and minor results in the general theory of linear algebras. To one already familiar with the essentials of the theory, the significance of this chapter will, of course, be clear; but the reviewer wonders if, when these multitudinous minor results are presented at the beginning of the theory, in a terminology unnecessarily complicated, they may not confuse or discourage some students. Chapters 3 and 4 are brief and discuss properties of an algebra connected with its invariant subalgebras and idempotent elements, respectively. Of these, the most important is the theorem that any algebra is the sum of its maximum nilpotent subalgebra and a semi-simple algebra (page 255). Chapter 5 is also comparatively brief and is concerned with the coordinates of an algebra, characteristic determinants, etc. The last chapter is the longest, occupying nearly one fifth of the entire book. In this he proves the set of theorems on the structure of the different types of algebras: simple algebras, matric algebras, division algebras; but ignores the very interesting and important work on division algebras by Dickson and Wedderburn referred to above in connection with Dickson's book.

Although the subject-matter of these five chapters is essentially the same as that of the first half of Dickson's book, it would be scarcely possible to imagine two mathematical books more different in their treatment. This difference is partly due to the unnecessary detail in Scorza's presentation, and partly due to the unfortunate fact that Scorza uses the older terminology which has been discarded. Also, it sometimes seems to the reader that he delights in multiplying the number of technical terms. For example, since his *algebra eccezionale* of an algebra A is simply the maximum nilpotent invariant subalgebra of A, why not call it such? The latter term is not only more descriptive, but is also the usual one.

At the end he has placed a bibliography of fifty-three books and memoirs on number fields, matrices and linear algebras. Although this is surely far better than none, it is not all that might be desired, since there are several errors, both of omission and commission. For example, why does it omit

Steinitz, Ernst, *Algebraische Theorie der Körper,* JOURNAL FÜR MATHEMATIK, vol. 137 (1909), pp. 167–309;

Dickson, L. E., *Linear associative algebras and abelian equations,* TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 31–46;

Wedderburn, J. H. M., *A type of primitive algebra,* ibid., pp. 162–166?

The last two articles (together with a brief but suggestive paper by Wedderburn in the TRANSACTIONS for 1921 which, presumably, appeared too late to be included in the bibliography) are, without doubt, the most interesting and the only important articles on the general theory of division algebras, and contain all that has been discovered about this difficult and extremely fascinating branch of linear algebras since Wedderburn's memoir in the PROCEEDINGS OF THE LONDON SOCIETY mentioned elsewhere in this review.

One notes with dismay that there is no index. This lack is an inconvenience in any book of this size and is only slightly ameliorated by putting in bold-face type the caption of every section and of every definition. In fact, one might almost say that the presence or lack of an index is a characteristic invariant which distinguishes Anglo-American texts from Continental ones.*

But altogether, the book contains a great deal of information not previously available outside of technical periodicals; and Professor Scorza is to be congratulated on the courage with which he attempted and the care with which he finished the task that he had set himself. As one reads, one can not but feel that the task has been to him a pleasant one and that when it was completed, he left it with a caress.

OLIVE C. HAZLETT

* In addition to the errors listed in the *errata* at the end of the volume, the reviewer noted only the following non-trivial mistakes:
p. 118, l. 20—Interchange "Γ_1" and "Γ".
p. 167, l. 20—Replace "$\| \xi_{j,1} - \delta_{j,1}\, \sigma \| = 0$" by "$| \xi_{j,1} - \delta_{j,1}\, \sigma | = 0$".
p. 374, l. 4,6—Place square brackets around each "u".

A CORRECTION

In line 16, page 4, volume 29 of this BULLETIN (Jan., 1923), insert the words *and contain only decomposable continua* between the word *points* and the word *then*. The desirability of this correction was called to my attention by Professor J. R. Kline.

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