from (16), on eliminating $h$ or $k$, that $h = \mu h_1$, $k = \lambda k_1$, where $h_1$ and $k_1$ are integral, and we get

$$h_1a_1 + k_1b_1 \equiv 0 \pmod{\varepsilon_1}, \quad h_1c_1 + k_1d_1 \equiv 0 \pmod{\varepsilon_1}.$$ 

The nature of the singularities on the sides of the triangle $ABC$ is readily determined. For instance, suppose in (6) $c > a > 0$. Then (6) gives an expansion for $t$ in ascending powers of $x^{1/a}$, and thence we get for $y$ an expansion of the form

$$y = x^{c/a}(\alpha + \beta x^{1/a} + \gamma x^{2/a} + \cdots)$$

in general, fixing the nature of the singularity for which $t$ is zero.

Bedford College, University of London

SURFACES WITH ORTHOGONAL LOCI OF THE CENTERS OF GEODESIC CURVATURE OF AN ORTHOGONAL SYSTEM*

BY MALCOLM FOSTER

We consider a surface $S$ referred to any orthogonal system. Let $G_1$ and $G_2$ be the centers of geodesic curvature of the curves $u = \text{const.}$ and $v = \text{const.}$ respectively, through any point $M$ of $S$. As $M$ is displaced over the entire surface the loci of $G_1$ and $G_2$ will in general be two surfaces $S_1$ and $S_2$, corresponding elements of which are those which result from a common displacement of $M$. We ask: What are the surfaces $S$ for which the surfaces $S_1$ and $S_2$ correspond with orthogonality of linear elements?

The condition that the displacements of $G_1$ and $G_2$ be orthogonal for every displacement of $M$, is that the absolute displacements of these points in the directions of the axes of the moving trihedral at $M$ satisfy the relation

$$\sum \delta x_1 \delta x_2 = 0,$$

* Presented to the Society, April 28, 1923.
for all values of \( \frac{dv}{du} \). The \( x \)-axis of the trihedral is chosen tangent to the curve \( v = \text{const} \). The radii of geodesic curvature of the curves \( u = \text{const} \) and \( v = \text{const} \), we denote by \( q_{uv} \) and \( q_{gu} \) respectively. The relation (1) becomes*

\[
q_{uv} \left( -\frac{\partial q_{uv}}{\partial u} du - \frac{\partial q_{uv}}{\partial v} dv + \xi du \right) dv 
+ q_{vr} \left( \frac{\partial q_{vr}}{\partial u} du + \frac{\partial q_{vr}}{\partial v} dv + \eta_1 dv \right) du 
- q_{uv} q_{uv} (pdu + p_1 dv)(qdu + q_1 dv) = 0.
\]

Hence, setting the coefficients of \( du^2 \), \( dudv \), and \( dv^2 \) equal to zero, we get

\[
q_{uv} \left( q_{uv} q_1 - r \frac{\partial q_{uv}}{\partial u} \right) = 0,
q_{uv} \left( \xi + \frac{\partial q_{uv}}{\partial u} \right) + q_{uv} \left( \eta_1 + \frac{\partial q_{uv}}{\partial v} \right) 
- q_{uv} q_{uv} (p_1 q_1 + p_1 q) = 0,
q_{gu} \left( q_{gu} q_1 + r_1 \frac{\partial q_{gu}}{\partial v} \right) = 0.
\]

Now \( q_{uv} = \frac{\eta_1}{r_1} \) and \( q_{gu} = \frac{\xi}{r} \); using these values, and the relations between the fundamental quantities for the surface, the equations (2) reduce to

\[
q_{uv} q_1 - r \frac{\partial q_{uv}}{\partial u} = 0,
q_{uv} q_1 + r \frac{\partial q_{uv}}{\partial v} = 0,
\]

since \( q_{uv} \), \( q_{gu} \) \( \neq \) 0.

Since \( p = D'/\eta_1 \) and \( q_1 = -D'/\xi \),\( \xi \) we see from the second member of (3) that both \( p \) and \( q_1 \), are zero, and that the parametric curves must be the lines of curvature.

† Eisenhart, p. 132, formula (47), and p. 167, formulas (45).
‡ Eisenhart, p. 168 and p. 170.
§ Eisenhart, p. 174, formulas (73).
Consequently the first and third members of (3) reduce to *
\[ \frac{\partial e_{gu}}{\partial u} = 0, \]
\[ \frac{\partial e_{gv}}{\partial v} = 0. \]
Hence

\[ (4) \quad e_{gv} = U, \quad e_{gu} = V, \]
where \( U \) and \( V \) are functions of \( u \) and \( v \) alone respectively. The parametric curves therefore have constant geodesic curvature and the system is isothermal.† The surface is therefore isothermal.

Making use of (4), we see that the elements \( ds_1^2 \) and \( ds_2^2 \) of the loci of \( G_1 \) and \( G_2 \) respectively, are

\[ (5) \]
\[
\begin{align*}
    ds_1^2 &= \sum \delta x_1^2 = \eta_1 \left[ \frac{(\partial r_1^2)}{\partial u} + r_1^2 + q^2 \right] du^2, \\
    ds_2^2 &= \sum \delta x_2^2 = \xi_2 \left[ \frac{(\partial r_2^2)}{\partial v} + r_2^2 + p_2^5 \right] dv^2.
\end{align*}
\]
Hence the loci of \( G_1 \) and \( G_2 \) are curves and not surfaces. As the vertex of the trihedral describes a curve \( u = \text{const.} \), the point \( G_1 \) remains fixed, and as the vertex of the trihedral describes a curve \( v = \text{const.} \), the point \( G_2 \) remains fixed. The lines of curvature are therefore spherical in both systems; they lie on spheres whose centers lie on the loci of \( G_1 \) and \( G_2 \), and which are mutually orthogonal with \( S \) at every point.

We denote the curves which are the loci of \( G_1 \) and \( G_2 \) by \( \Gamma_1 \) and \( \Gamma_2 \) respectively. The curve \( \Gamma_1 \) is described by \( G_1 \) as the vertex of the trihedral describes every curve \( v = \text{const.} \), and \( \Gamma_2 \) is described by \( G_2 \) as the vertex of

* We exclude the cases where either \( r = 0 \), or \( r_t = 0 \), since in either case the curves in one family are geodesics, and one of the points \( G_1 \) and \( G_2 \) is at infinity.
† Eisenhart, p. 137.
the trihedral describes every curve \( u = \text{const.} \). Suppose that the vertex of the trihedral describes a definite curve \( u = \text{const.} \). The point \( G_1 \) remains fixed at some point on \( \Gamma_1 \) while \( G_2 \) describes \( \Gamma_2 \), and every tangent to \( \Gamma_2 \) will be perpendicular to the fixed tangent to \( \Gamma_1 \) at \( G_1 \). The curve \( \Gamma_2 \) is therefore either a plane curve whose plane is perpendicular to the fixed tangent to \( \Gamma_1 \), or a straight line perpendicular to this fixed tangent. Suppose now that the vertex of the trihedral describe a second curve \( u = \text{const.} \). Then \( G_2 \) remains fixed at some second point of \( \Gamma_1 \) while \( G_2 \) describes \( \Gamma_2 \), and every tangent to \( \Gamma_2 \) will be perpendicular to the fixed tangent to \( \Gamma_1 \) at the second position of \( G_1 \). Consequently if \( \Gamma_2 \) be a plane curve, the locus of \( G_1 \), namely \( \Gamma_1 \), must be a straight line. We obtain similar conclusions if we consider the vertex of the trihedral to describe two different curves \( v = \text{const.} \). Hence the locus of at least one of the points \( G_1 \) and \( G_2 \) is a straight line.

We suppose that it is the locus of \( G_1 \) which is a straight line. Consider the absolute displacements of the point \( G_1 \) in the directions of the axes of the trihedral \( T_u \) of a curve \( v = \text{const.} \). We have

\[
(6) \quad \frac{\delta x_1}{ds} = \frac{dx_1}{ds} + 1, \quad \frac{\delta y_1}{ds} = \frac{x_1}{\rho}, \quad \frac{\delta z_1}{ds} = 0,
\]

where \( ds \) is the element of arc of the curve \( v = \text{const.} \), and \( \rho \) is the radius of first curvature. From the third member of (6), we see that the line \( \Gamma_1 \) which is the locus of \( G_1 \), lies in the osculating plane of the curve \( v = \text{const.} \) at every point. Now

\[
x_1 = - \rho_{uv} = - \frac{\eta_1}{\rho_1}, \quad ds = \sqrt{E} \, du = \xi \, du;
\]

using these relations, together with the relations between the fundamental quantities for the surface, equations (6) become

* Eisenhart, p. 32.

† It is necessary that \( \rho_{uv} \) be measured in the opposite direction to that in which the parameter \( u \) increases. Cf. Darboux, vol. II, p. 359.

‡ Eisenhart, p. 168 and p. 170.
(7) \[ \frac{\delta x_1}{ds} = \frac{\eta_1}{\xi r_1^a}, \quad \frac{\delta y_1}{ds} = -\frac{\eta_1}{\psi r_1}, \quad \frac{\delta z_1}{ds} = 0. \]

Hence

\[ ds_1 = \frac{\eta_1 \left[ q^a \left( \frac{\partial r_1}{\partial u} \right)^a + \xi^a r_1^a \right]^{\frac{1}{a}}}{\xi r_1^a}, \]

where \( ds_1 \) is the element of the line \( I_1 \). The direction-co
cosines of \( I_1 \) relative to the trihedral \( T_u \) are therefore

\[ \alpha_1 = \frac{\xi}{\sqrt{q^a \left( \frac{\partial r_1}{\partial u} \right)^a + \xi^a r_1^a}} , \quad \beta_1 = \frac{-\eta}{\sqrt{q^a \left( \frac{\partial r_1}{\partial u} \right)^a + \xi^a r_1^a}}, \quad \gamma_1 = 0. \]

Since \( I_1 \) is a straight line fixed in space, we must have

\[ \frac{\delta \alpha_1}{ds} = \frac{\delta \beta_1}{ds} = \frac{\delta \gamma_1}{ds} = 0. \]

These equations become on using (9),

\[ \left\{ \begin{array}{l}
\frac{\delta \alpha_1}{ds} = \frac{d \alpha_1}{ds} - \frac{\beta_1}{q} \\
\frac{\delta \beta_1}{ds} = \frac{d \beta_1}{ds} + \frac{\alpha_1}{q} \\
\frac{\delta \gamma_1}{ds} = \frac{d \gamma_1}{ds} - \frac{\alpha_1}{\xi}
\end{array} \right. 
\]

\[ = q^a \left[ q^a \left( \frac{\partial r_1}{\partial u} \right)^a + \xi^a r_1^a \right]^{\frac{1}{a}} = 0, \]

\[ \left\{ \begin{array}{l}
\frac{\delta \alpha_1}{ds} = \frac{d \alpha_1}{ds} - \frac{\beta_1}{q} \\
\frac{\delta \beta_1}{ds} = \frac{d \beta_1}{ds} + \frac{\alpha_1}{q} \\
\frac{\delta \gamma_1}{ds} = \frac{d \gamma_1}{ds} - \frac{\alpha_1}{\xi}
\end{array} \right. 
\]

\[ = \xi \left[ q^a \left( \frac{\partial r_1}{\partial u} \right)^a + \xi^a r_1^a \right]^{\frac{1}{a}} = 0, \]

\[ \left\{ \begin{array}{l}
\frac{\delta \alpha_1}{ds} = \frac{d \alpha_1}{ds} - \frac{\beta_1}{q} \\
\frac{\delta \beta_1}{ds} = \frac{d \beta_1}{ds} + \frac{\alpha_1}{q} \\
\frac{\delta \gamma_1}{ds} = \frac{d \gamma_1}{ds} - \frac{\alpha_1}{\xi}
\end{array} \right. 
\]

\[ = \xi \eta_1 r_1 \left[ q^a \left( \frac{\partial r_1}{\partial u} \right)^a + \xi^a r_1^a \right]^{\frac{1}{a}} = 0, \]

\* Eisenhart, p. 32.
where \( r \) is the radius of second curvature of the curve \( v = \text{const.} \). Hence we must have

\[
\left \{ \begin{align*}
q^2 \left( \frac{q^2 r_1 \partial r}{\partial u^2} + \frac{\partial \varphi}{\partial u} \frac{\partial r_1}{\partial u} \right) - q^2 \frac{\partial r_1}{\partial u} \frac{\partial \varphi}{\partial u} + \varphi^2 r_1 = 0, \\
\frac{1}{r} = 0.
\end{align*} \right.
\] (11)

The curves \( v = \text{const.} \) are therefore plane; and since they are spherical, they are circles. Consequently \( q = \text{const.} \), and the first member of (11) reduces to

\[
q^2 \left( \frac{\partial^2 r_1}{\partial u^2} - \frac{\partial \varphi}{\partial u} \frac{\partial r_1}{\partial u} \right) + \varphi^2 r_1 = 0.
\] (12)

The relation (12) may also be written in the form

\[
q^2 \frac{\partial}{\partial u} \left( \frac{\partial r_1}{\partial u} \right) + \frac{\partial \varphi}{\partial u} = 0.
\] (13)

The line \( I_1 \) therefore lies in the plane of every curve \( v = \text{const.} \), and consequently the surfaces have plane lines of curvature in one system for which all the planes pass through the straight line \( I_1 \); such surfaces are called surfaces of Joachimsthal.*

Finally, the surfaces considered are isothermal surfaces of Joachimsthal for which the lines of curvature which lie in coaxial planes are circles, and for which either the relation (12) holds, or the corresponding relation is satisfied with reference to the curves \( u = \text{const.} \).

YALE UNIVERSITY

* Eisenhart, pp. 308–310.