THE JACOBIAN
OF A CONTACT TRANSFORMATION*

BY E. F. ALLEN

The equations

(1) \[ x_1 = X(x, z, p), \quad z_1 = Z(x, z, p), \quad p_1 = P(x, z, p), \]

where \( X, Z, \) and \( P \) are functions of class \( C'' \), represent a transformation of line-elements in the \( xz \) plane to line-elements in the \( x_1 z_1 \) plane. With Lie we shall define every transformation in \( x, z, p \), which leaves the Pfaff differential equation

(2) \[ dz - pdx = 0 \]

invariant, as a contact transformation of the \( xz \) plane to the \( x_1 z_1 \) plane. Hence the equations (1) must satisfy an identity of the form

(3) \[ dz_1 - p_1 dx_1 = \phi(dz - pdx), \]

where \( \phi \) is a function of \( x, z, \) and \( p \) alone.

The following relations connecting \( X, Z, P, \) and their partial derivatives are easily obtained:

(4) \[
\begin{align*}
Z_x - PX_x &= -p\phi, \\
Z_z - PX_z &= \phi, \\
Z_p - PX_p &= 0;
\end{align*}
\]

(5) \[
\begin{align*}
[XZ] &= X_p(Z_x + pZ_z) - Z_p(X_x + pX_z) = 0, \\
[PX] &= \phi, \text{ and } [PZ] = \phi P.
\end{align*}
\]

The jacobian of transformation (1) is

(6) \[
J = \begin{vmatrix}
X_x & X_x & X_x \\
Z_x & Z_z & Z_p \\
P_x & P_z & P_p
\end{vmatrix}.
\]

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† Lie und Scheffers, *Geometrie der Berührungstransformationen*, p. 68, Chap. 3.
We shall show that this jacobian is equal to $q^2$. Let us multiply the first row by $P$ and subtract the product from the second row; then

$$
\begin{vmatrix}
X_x & X_z & X_p \\
Z_x - PX_x & Z_z - PX_z & Z_p - PX_p \\
P_x & P_z & P_p
\end{vmatrix}
$$

(7)

Hence, using equations (4), we find

$$
\begin{vmatrix}
X_x & X_z & X_p \\
-pq & q & 0 \\
P_x & P_z & P_p
\end{vmatrix}
$$

(8)

This reduces to

$$
\begin{vmatrix}
X_x + pX_z & X_z & X_p \\
0 & q & 0 \\
P_x + pP_z & P_z & P_p
\end{vmatrix}
$$

(9)

when the second column is multiplied by $p$ and the sum is added to the first column. Evaluating this determinant, we have

$$
J = q[p(X_x + pX_z) - X_p(P_x + pP_z)].
$$

(10)

Therefore, by equation (5), we may write

$$
J = q^2.
$$

(11)

The equations of a contact transformation may be regarded as the equations of a point transformation, which transforms points in $xz_p$ space to points in $x_1z_1p_1$ space. In general a surface in $xzp$ space, represented by the equation $F_1(x, z, p) = 0$, will be transformed into a surface in $x_1z_1p_1$ space, represented by the equation $F_2(x_1, z_1, p_1) = 0$. Or if we regard equations (1) as the equations of a transformation of line-elements, it will transform a differential equation in $x, z, p$, into one in $x_1, z_1, p_1$, and also the solutions of the first differential equation into the solutions of the second.

Now if we set $q$ equal to zero,* we will have the equa-

* In some cases there are no values of the variables that will make $p$ equal to zero. The following theory does not apply to such cases.
tion of a surface in $xzp$ space, or we might say that we have a differential equation in $xz$ space. Let us see into what this surface or into what this differential equation is transformed when it is subjected to the transformation (1).

A few examples result in obtaining curves in $x_1z_1p_1$ space or in obtaining differential equations free from $p_1$. This leads to the following theorem.

**Theorem.** The surface $Q = 0$ is transformed into a curve in space by the transformation (1).

If the partial derivative of $Q$ with respect to $z$ is not identically equal to zero, the equation $Q = 0$ may be solved for $z$.\(^*\) Assuming that this is true, when the value thus obtained for $z$ is substituted in $X, Z,$ and $P$, they become functions of $x$ and $p$ alone. Regarding $x$ and $p$ as parameters, the equations (1) are the parametric equations of a surface. A necessary and sufficient condition\(^\dagger\) that

\[
\begin{align*}
  x_1 &= f(x, y), \quad y_1 = g(x, y), \quad z_1 = h(x, y)
\end{align*}
\]

define a curve in space and not a surface is that

\[
EG - F^2 = A^2 + B^2 + C^2 \equiv 0,
\]

where

\[
A = \frac{\partial(y_1, z_1)}{\partial(x, y)}, \quad B = \frac{\partial(z_1, x_1)}{\partial(x, y)}, \quad C = \frac{\partial(x_1, y_1)}{\partial(x, y)}.
\]

To prove our theorem it is necessary and sufficient to show that the $A, B,$ and $C$ connected with equations (1) are identically equal to zero. That is, it is sufficient to show that all the determinants of the following matrix vanish identically:

\[
\begin{vmatrix}
  X_x + pX_z & X_p + \frac{\partial z}{\partial p}X_z \\
  Z_x + pZ_z & Z_p + \frac{\partial z}{\partial p}Z_z \\
  P_x + pP_z & P_p + \frac{\partial z}{\partial p}P_z
\end{vmatrix}
\]

\(^*\) If $\partial p/\partial z \equiv 0$ we will be able to solve for either $x$ or $p$ if $p \neq \text{const.}$

\(^\dagger\) Eisenhart, *Differential Geometry*, p. 71.
Let us see what the effect will be when the value of \( z \) as obtained from \( q = 0 \) is substituted in equations (1). Suppose that the substitution has been made in \( X \) and \( Z \). It is easy to see that \( X_z \) and \( Z_z \) are equal to zero, and that to differentiate \( X \) completely with respect to \( x \), it is necessary to differentiate with respect to \( x \) and then to use the function of a function rule, thus \( X_x + X_z(\partial z / \partial x) \), and similarly for the other letters. Thus using the fact that \( q = 0 \), we may write the equations (5) in the form

\[
\begin{align*}
\left( \frac{\partial^2 z}{\partial p} X_z \right) (Z_x + p Z_z) - \left( \frac{\partial^2 z}{\partial p} Z_z \right) (X_x + p X_z) &= 0, \\
\left( \frac{\partial^2 z}{\partial p} P_z \right) (X_x + p X_z) - \left( \frac{\partial^2 z}{\partial p} X_z \right) (P_x + p P_z) &= 0, \\
\left( \frac{\partial^2 z}{\partial p} P_z \right) (Z_x + p Z_z) - \left( \frac{\partial^2 z}{\partial p} Z_z \right) (P_x + p P_z) &= 0.
\end{align*}
\]

It is very easy to see that these equations are now the expanded form of the determinants of the matrix (15). Hence the theorem is proved.

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**INTEGRO-DIFFERENTIAL INVARIENTS OF ONE-PARAMETER GROUPS OF FREDHOLM TRANSFORMATIONS**

**BY A. D. MICHAL**

1. *Statement of the Problem.* The author† has already considered functionals of the form \( f[y(x_0), y'(x_0)] \) (depending only on a function \( y(x) \) and its derivative \( y'(x) \) between 0 and 1) which are invariant under an arbitrary Volterra one-parameter group of continuous transformations. The

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† Cf. *Integro-differential expressions invariant under Volterra's group of transformations* in a forthcoming issue of the *Annals of Mathematics.* This paper will be referred to as "I.D.I.V."