Let us see what the effect will be when the value of \( z \) as obtained from \( q = 0 \) is substituted in equations (1). Suppose that the substitution has been made in \( X \) and \( Z \). It is easy to see that \( X_z \) and \( Z_z \) are equal to zero, and that to differentiate \( X \) completely with respect to \( x \), it is necessary to differentiate with respect to \( x \) and then to use the function of a function rule, thus \( X_x + X_z(\partial z/\partial x) \), and similarly for the other letters. Thus using the fact that \( q = 0 \), we may write the equations (5) in the form

\[
(P + \frac{\partial z}{\partial p} P_z)(X_z + pZ_z) - (Z_p + \frac{\partial z}{\partial p} Z_z)(X_x + pX_z) = 0,
\]

\[
(P + \frac{\partial z}{\partial p} P_z)(X_x + pX_z) - (X_p + \frac{\partial z}{\partial p} X_z)(P_x + pP_z) = 0,
\]

\[
(P + \frac{\partial z}{\partial p} P_z)(Z_x + pZ_z) - (Z_p + \frac{\partial z}{\partial p} Z_z)(P_x + pP_z) = 0.
\]

It is very easy to see that these equations are now the expanded form of the determinants of the matrix (15). Hence the theorem is proved.

*The University of Missouri*

**INTEGRO-DIFFERENTIAL INVARIANTS OF ONE-PARAMETER GROUPS OF FREDHOLM TRANSFORMATIONS**

**BY A. D. MICHAL**

1. **Statement of the Problem.** The author has already considered functionals of the form \( f[y(x), y'(x)] \) (depending only on a function \( y(x) \) and its derivative \( y'(x) \) between 0 and 1) which are invariant under an arbitrary Volterra one-parameter group of continuous transformations. The
calculation of the invariants in question was effected in the case of a large class of functionals known as analytic functionals.

The purpose of this note is to consider the problem of finding analytic functionals \( f[y(x), y'(x)] \) invariant under a Fredholm group of transformations

\[
y_1(x) = y(x) + \int_0^1 K(x, s \mid a)y(s)ds,
\]

where \( a \) is the parameter of this continuous one-parameter group of transformations, and where \( a = 0 \) corresponds to the identical transformation.

We restrict ourselves to transformations (1) for which the \( ^{(r)}'s \) exist and are continuous in the interval \( I: 0 \leq r \leq 1 \); \( K(x, s \mid a) \) and \( \partial K/\partial x \) are continuous in \( x \) and \( s \) in the square \( S: 0 \leq x \leq 1, \ 0 \leq s \leq 1 \); and \( \partial K/\partial x \) is not identically zero when \( a \neq 0 \).

The infinitesimal transformation corresponding to (1) will be of the form

\[
\delta y(x) = \left[ \int_0^1 H(x, s)y(s)ds \right] \delta a
\]

with

\[
\delta y'(x) = \left[ \int_0^1 H_1(x, s)y(s)ds \right] \delta a,
\]

where

\[
H_1(x, s) = \frac{\partial H(x, s)}{\partial x},
\]

as the extended group of infinitesimal transformations.

Here follow the well known relations* between the kernel \( K(x, s \mid a) \) of the Fredholm finite transformation (1) and the kernel \( H(x, s) \) of the corresponding infinitesimal transformation (2):

* Gerhard Kowalewski, \( \text{"Uber Funktionenräume}, \) \( \text{Wiener Sitzungsberichte}, 1911, \) \text{vol. 120}, II A.
(4) \[ K(x, s | a) = \sum_{i=1}^{\infty} \frac{a^i}{i!} H^i, \]

(5) \[ H(x, s) = \left[ \frac{\partial K(x, s | a)}{\partial a} \right]_{a=0} = \frac{1}{a} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{K^i}{i}, \quad a \neq 0, \]

where \( H^i \) and \( K^i \) are to be interpreted according to Volterra’s symbolic multiplication.

By methods similar to those employed in proving the lemma of Part I of I. D. I. V., we can prove* without difficulty the following lemma.

**Lemma.** Necessary and sufficient conditions that \( H(x, s) \) and \( \frac{\partial H}{\partial x} \) be continuous in \( x \) and \( s \) and that \( \frac{\partial H}{\partial x} \) be not identically zero are that \( K(x, s | a) \) and \( \frac{\partial K}{\partial x} \) be continuous in \( x \) and \( s \) and that \( \frac{\partial K}{\partial x} \) be not identically zero, when \( a \neq 0 \).

2. A Sufficient Condition for Invariance.

**Theorem 1.** Let \( f[y(x_0), y'(x_0)] \) be an analytic functional of \( y(x) \) and \( y'(x) \), i.e., developable in a Volterra expansion†

\[
\begin{align*}
\sum_{j=1}^{\infty} \frac{1}{j!} \int_0^1 \cdots \int_0^1 \left[ \sum_{k=0}^{j} \frac{j!}{k!} f_{j-k,k}(t_1, \ldots, t_{j-k}; t_{j-k+1}, \ldots, t_j) \right] \left[ \prod_{i=1}^{j-k} y(t_i) \right] \left[ \prod_{i=j-k+1}^{j} y'(t_i) \right] dt_1 dt_2 \cdots dt_j.
\end{align*}
\]

We shall assume that \( f_{j-k,k} \) is continuous in its \( j \) arguments, symmetric separately in the sets of arguments \( t_1, t_2, \ldots, t_{j-k} \) and \( t_{j-k+1}, \ldots, t_j \) respectively; and for convenience we assume also that

\[
|f_{j-k,k}| < \gamma, \quad |y| < \epsilon_1, \quad |y'| < \epsilon_2,
\]

where \( \gamma, \epsilon_1, \epsilon_2 \) are positive constants. Then a sufficient condition that \( f[y(x_0), y'(x_0)] \) be invariant under a given group of transformations (1) is that it satisfy the relation

* The proof comes by a direct calculation of the series involved.
† This is a generalization of Taylor’s series given by Volterra. See for example his *Leçons sur les Équations Intégrales*, 1913.
(8) \[ \int_0^1 H(t, t_i + k)f_i, \ldots, t_i; t_i + 1, \ldots, t_i + k - 1, t) dt = -\int_0^1 H(t, t_i + k)f_{i+1, k-1}(t_i, t_i + 1, \ldots, t_i + k - 1) dt. \]

The necessary and sufficient condition that \( f[y(r_0), y'(r_0)] \) be invariant under (1) is that under (2)

(9) \[ \delta f[y(r_0), y'(r_0)] \equiv 0 \text{ in } y \text{ and } y'. \]

Since the analyticity of our functionals insures the validity of a Volterra variation, we may use Volterra's\(^*\) form of the variation of a functional. Then condition (9) becomes

(10) \[ \int_0^1 f_y(t)\delta y(t) dt + \int_0^1 f_y'(t)\delta y'(t) dt = 0 \]

in \( y \) and \( y' \), where \( f_y(t) \) and \( f_y'(t) \) are the partial functional derivatives of \( f[y(r_0), y'(r_0)] \) with respect to \( y(r) \) and \( y'(r) \), respectively, both taken at the point \( t \).

Substituting in (10) the values of \( \delta y(t) \) and \( \delta y'(t) \) as given by (2) and (3), respectively, rearranging and dividing through by \( \delta a \), we get

(11) \[ \int_0^1 y(s) \left[ \int_0^1 f_y(t) H(t, s) dt + \int_0^1 f_y'(t) H_1(t, s) dt \right] ds \equiv 0 \]

in \( y \). We may now apply Lemma 2 of I. D. I. V.; doing so, we find

(12) \[ \int_0^1 f_y'(t) H_1(t, s) dt = -\int_0^1 f_y(t) H(t, s) dt. \]

Such operations as functional differentiations term by term are valid since the series involved are uniformly convergent under our hypotheses.\(^\dagger\) Calculating the partial functional derivatives \( f_y(t) \) and \( f_y'(t) \), respectively, and substituting them in (12), we get by an easy reduction\(^\ddagger\)

\(^*\) A more general expression for \( \delta f \) would be in the form of Stieltjes integrals.


\(^\ddagger\) Cf. similar reductions of I. D. I. V.
\[ \sum_{j=1}^{\infty} \frac{1}{(j-1)!} \int_0^1 \int_0^1 \cdots \int_0^1 \left[ \sum_{k=0}^{j-1} \binom{j-1}{k} \right] H(t, s) f_{j-1-l} \cdot t_{i+1}(t_1, \ldots, t_{j-1-i}, t_{j-1}, t) \times \prod_{i=1}^{j-1-l} y(t_i) \prod_{i=j-l}^{j-1} y'(t_i) \, dt_1 \cdots dt_{j-1} \, dt \]

(13)

\[ \equiv - \sum_{j=1}^{\infty} \frac{1}{(j-1)!} \int_0^1 \int_0^1 \cdots \int_0^1 \left[ \sum_{k=0}^{j-1} \binom{j-1}{k} \right] H(t, s) f_{j-k} \cdot k(t, t_1, \ldots, t_{j-k}, t) \times \prod_{i=1}^{j-k-1} y(t_i) \prod_{i=j-k}^{j-1} y'(t_i) \, dt_1 \cdots dt_{j-1} \, dt \]

where \( k = i + 1 \). Equating coefficients of similar terms in \( y \) and \( y' \), we find

\[ \int_0^1 H(t, s) f_{j-i} \cdot k(t, t_1, \ldots, t_{j-1-i}, t_{j-1}, \ldots, t_{j-1}, t) \, dt \]

\[ \equiv - \int_0^1 H(t, s) f_{j-i} \cdot k(t, t_1, \ldots, t_{j-1-i}, t_{j-1}, \ldots, t_{j-1}, t) \, dt, \]

which can be written in the form (8).

3. Calculation of the Invariants \( f[y(t_0), y'(t_0)] \). In order that \( f[y(t_0), y'(t_0)] \) be invariant under (1) it is sufficient that the following recurrence formula hold

\[ f_{i+k}(t_1, \ldots, t_i; t_{i+1}, \ldots, t_{i+k-1}, t) \]

\[ \equiv - \frac{H(t, t_{i+k})}{H(t, t_{i+k})} f_{i+1, k-1}(t, t_1, \ldots, t_i; t_{i+1}, \ldots, t_{i+k-1}). \]

We shall now prove the following theorem.

Theorem II. A necessary and sufficient condition on (1) that an analytic functional \( f[y(t_0), y'(t_0)] \) be invariant under (1) when (14) holds is that the kernel \( H(x, s) \) of the infinitesimal transformation be of the form

\[ H(x, s) \equiv \psi(s)e^{\psi e^c}, \]

where \( \psi \) is a suitable function.
where $\psi(s)$ is an arbitrary function of $s$, and where $c$ is a constant.\footnote{That is, if $H(x, s) = \psi(s)e^{x/c}$, we may assert that invariant analytic functionals $f(y(x_0), y'(x_0))$ always exist.}

It is evident from (14) that

$$\frac{H(t, t_i+k)}{H_1(t, t_i+k)}$$

must be independent of $t_i+k$, and hence it is necessary that it be a function of $t$ alone, say $\varphi(t)$. On applying (14) until $f_{i,k}$ is written in terms of $f$'s with second index zero, we get the recurrence formula

$$f_{i,k}(t_1, t_2, \ldots, t_i; t_i+1, \ldots, t_{i+k-1}, t) = (-1)^k \varphi(t)^k f_{i+k,0}(t, t_1, \ldots, t_i, t_{i+1}, \ldots, t_{i+k-1}).$$

By hypothesis $f_{i+k,0}$ is symmetric in all its arguments. Therefore, interchanging $t_1$ and $t_{i+1}$ leaves the right-hand side of (16) unchanged. Hence if (16) is to hold, $f_{i,k}$ must be symmetric with respect to $t_1$ and $t_{i+1}$, and therefore it must be symmetric in all its arguments. On interchanging $t$ and any $t_j$ in (16), we see at once that $\varphi(t)$ must be a constant, say $c$; i.e., $H(x, s)$ must satisfy the equation

$$c\frac{\partial H(x, s)}{\partial x} H(x, s) = 0,$$

whose most general solution is (15).

We may now remark that the arbitrariness of the coefficients $f_{i+k,0}$, in terms of which all the other $f_{i,k}$'s can be evaluated, on making use of the recurrence formula, enables us to state immediately the following theorem.

**Theorem III.** Let the kernel $H(x, s)$ of the infinitesimal transformation (2) be of the form $\psi(s)e^{x/c}$, and let us take an analytic functional $f[y(x_0), y'(x_0)]$, all of whose $f_{i,k}$'s are symmetric in all their arguments, and assign arbitrarily for initial conditions the coefficients $f_{i+k,0}$ in its Volterra expansion; that is, take an arbitrary $F[y(x_0)]$ such that $F[y(x_0)] \equiv f[y(x_0), y'(x_0)]$, and for convenience take $y_0(x) \equiv 0$. Then, if the $f_{i,k}$'s are calculated by the recurrence formula
(18) \( f_i, k(t_1, \ldots, t_i + k) = (-1)^k c^k f_{i+k, 0}(t_1, \ldots, t_i + k) \),
we shall have an analytic functional \( f[y(\tau_0), y'(\tau_0)] \) which
will be invariant under a transformation (1) whose kernel
\( K(x, s | a) \) is given by
\[
K(x, s | a) = \sum_{i=1}^\infty \frac{a^i}{i!} \psi(s) e^{\alpha x} \int_0^1 \int_{0(t-1)}^{t-1} \cdots \int_0^1 \frac{1}{e^{x}} \sum_{j=1}^{i-1} \psi(t_i) dt_i \cdots dt_{i-1}.
\]

4. Example. We here give an easy example in which
the direct verification by means of the finite transformation
is very simple. Let us suppose that
\[
\delta y(x) = \left[ e^x \int_0^t sy(s) ds \right] \delta a
\]
is the given infinitesimal transformation, i.e., that \( H(x, s) = se^x \). By means of an easy calculation, the finite trans­formation may be written in the form
\[
y(x) = y(x) + (e^a - 1)e^x \int_0^1 sy(s) ds,
\]
i.e., \( K(x, s | a) = (e^a - 1)e^x s \). Let us take for initial condition
\[
f[y(\tau_0), 0] = F[y(\tau_0)]
\]
\[
= f_{00} + \int_0^1 f_{10}(t_1) y(t_1) dt_1 + \frac{1}{2!} \int_0^1 dt_2 \int_0^1 f_{20}(t_1, t_2) y(t_1) y(t_2) dt_1.
\]
Then the functional \( f[y(\tau_0), y'(\tau_0)] \) given by
\[
f[y(\tau_0), y'(\tau_0)] = f_{00} + \int_0^1 f_{10}(t_1)[y(t_1) - y'(t_1)] dt_1
\]
\[
+ \frac{1}{2!} \left[ \int_0^1 dt_2 \int_0^1 f_{20}(t_1, t_2)[y(t_1)y(t_2) - 2y(t_1)y'(t_2) + y'(t_1)y'(t_2)] \right] dt
\]
is invariant under (20).

The Rice Institute