The relation (9) follows by comparing (6) with the identity
\[
J_0 J_3^2 = \left[ 1 + 2 \sum_{a=1}^{\infty} q^a (1-1)^a \right] \left[ 1 + \sum_{n=1}^{\infty} q^n \xi(n) \right],
\]
the second factor on the right being the algebraic expression of the well known theorem which gives the number of representations of \( n \) as a sum of two integer squares.

The University of Washington

NOTE ON A SPECIAL CONGRUENCE*

BY MALCOLM FOSTER

1. Introduction. Let \( S \) be any surface referred to its lines of curvature. With every point \( M \) of \( S \) we associate the trihedral of the surface, taking the \( x \)-axis of the trihedral tangent to the curve \( v = \text{const.} \) We consider the congruence of lines \( l \) parallel to the \( x \)-axis, the normal to \( S \), which pierce the \( xy \)-plane at the point \((\xi, \eta_1, 0)\).† The equations of \( l \) are \( x = \xi, \ y = \eta_1, \ z = t, \) and the coordinates of any point on \( l \) are
\[
(1) \quad x = \xi, \ y = \eta_1, \ z = t,
\]
where \( t \) is the distance on \( l \) measured from the point \((\xi, \eta_1, 0)\).

2. Condition for a Normal Congruence. If there be a surface normal to the congruence we must have
\[
(2) \quad \delta z = dz + p_1 y dv - qx du = 0, \quad \text{for all values of} \quad \frac{dv}{du}. \quad \text{Using} \quad (1) \quad \text{equation} \quad (2) \quad \text{becomes}
\]
\[
(3) \quad \frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv + p_1 \eta_1 dv - p_1 q_1 du = 0;
\]

hence
\[
(4) \quad \frac{\partial t}{\partial u} - q_1 \xi = 0, \quad \frac{\partial t}{\partial v} + p_1 \eta_1 = 0.
\]

* Presented to the Society, May 3, 1924.
† The notation used in this paper is the same as in Eisenhart's *Differential Geometry of Curves and Surfaces*; see pp. 166-176.
‡ Eisenhart, p. 170.
The condition of integrability is
\[ \frac{\partial}{\partial v} \left( \frac{\partial t}{\partial u} \right) = \frac{\partial}{\partial u} \left( \frac{\partial t}{\partial v} \right). \]

Hence from (4) we must have
\[ q \frac{\partial \xi}{\partial v} + \xi \frac{\partial q}{\partial v} = - \left( p_1 \frac{\partial \eta_1}{\partial u} + \eta_1 \frac{\partial p_1}{\partial u} \right), \]
which reduces to* \( q \eta_1 r - p_1 \xi r - p_1 \xi r_1 + q \eta_1 r_1 = 0. \) This may be written in the form
\[ (5) \quad (q \eta_1 - p_1 \xi)(r + r_1) = 0. \]

Conversely, when (5) is satisfied, the function \( t \) as given by (4) satisfies (3), and hence there exists a single infinity of parallel surfaces normal to the lines \( l \). If \( q \eta_1 - p_1 \xi = 0 \), the surface \( S \) is minimal.† Hence we have the following theorem.

**Theorem I.** *A necessary and sufficient condition that the congruence of lines \( l \) be normal is that \( S \) be a minimal surface, or else that \( r + r_1 = 0 \).*

3. **Equation of the Curves defining the Developables.** As \( M \), the vertex of the trihedral, is displaced along some curve \( C \) on \( S \) the locus of \( l \) is a ruled surface of the congruence; we seek the equation of the curves \( C \) for which the locus of \( l \) is developable. For \( l \) to generate a developable surface the displacement of some point on \( l \) must be in the direction of the line; hence for that point
\[ (6) \quad \delta x = \delta y = 0. \]

By means of (1), equations (6) take the forms‡
\[
\begin{align*}
\frac{\partial \xi}{\partial u} du - \eta_1 r dv + \xi du + t q du - \eta_1 (r du + r_1 dv) &= 0, \\
\xi r_1 du + \frac{\partial \eta_1}{\partial v} dv + \eta_1 dv - t p_1 dv + \xi (r du + r_1 dv) &= 0.
\end{align*}
\]

The elimination of \( t \) between these two equations gives

* Eisenhart, p. 168, formulas (48), and p. 170, formulas (55).
† Eisenhart, p. 174, formulas (75).
‡ Eisenhart, p. 170.
the following equation of the curves on $S$ defining the developable surfaces of the congruence:

$$q\xi(r + r_1)du^2 + \left[p_1 \left(\frac{\partial \xi}{\partial t} + \xi - \eta_1 r\right) + q \left(\frac{\partial \eta_1}{\partial v} - \eta_1 + \xi r_1\right)\right]dudv - p_1 \eta_1 (r + r_1)dv^2 = 0.$$  

The elimination of the ratio $\frac{dv}{du}$ between equations (7) gives the following equation for the distances along $l$ to the focal points:

$$p_1 q t^2 + \left[p_1 \left(\frac{\partial \xi}{\partial t} + \xi - \eta_1 r\right) - q \left(\frac{\partial \eta_1}{\partial v} + \eta_1 + \xi r_1\right)\right] t - \left(\frac{\partial \xi}{\partial u} + \xi - \eta_1 r\right) \left(\frac{\partial \eta_1}{\partial v} + \eta_1 + \xi r_1\right) - \xi \eta_1 (r + r_1)^2 = 0.$$  

The condition that equation (8) define an orthogonal system is

$$\xi^2 p_1 \eta_1 (r + r_1) - \eta_1^2 q \xi (r + r_1) = 0,$$

which may be written

$$(q \eta_1 - p_1 \xi)(r + r_1) = 0,$$

since $\xi, \eta_1 \neq 0$. Since (10) is identical with (5) we have the following theorem.

**Theorem II.** If the congruence of lines $l$ be normal the developables are represented on $S$ by an orthogonal system.

We note from (8) that if the congruence be normal by virtue of the relation $r + r_1 = 0$ the curves defining the developables are the lines of curvature.

4. *Asymptotic Lines on the Normal Surfaces.* Let $C$ be any curve on $S$ through $M$, the vertex of the trihedral, and let $l$ and $l_1$ be the lines of the congruence corresponding to $M$ and $M_1$, a neighboring point on $C$. As $M_1$ approaches $M$ along $C$ the foot on $l$ of the common perpendicular to $l$ and $l_1$ approaches a certain limiting position called the central point of the generator $l$. The locus of the central points is the line of striction of the ruled surface.

* Eisenhart, p. 80.
defined by $C$. We wish to determine the distance along $l$

to the line of striction of this ruled surface.

To this end we consider a second trihedral $T_0^*$ with
vertex at some fixed point in space, whose $x_0$-, $y_0$-, and
$z_0$-axes are in every position parallel to the $x$-, $y$-, and
$z$-axes of the moving trihedral. Relative to the trihedral
$T_0$ the coordinates of the point on the unit sphere cor-
responding to $l$ are $(0, 0, 1)$. As the vertex of the moving
trihedral is displaced along $C$ the absolute displacements
of the point $(0, 0, 1)$ in the directions of the axes of the
trihedral $T_0$ will be the variations experienced by the
direction-cosines of $l$. If these variations are denoted by
$\delta \alpha$, $\delta \beta$, $\delta \gamma$, we have† $\delta \alpha = q du$, $\delta \beta = -p_1 dv$, $\delta \gamma = 0$,
since for the motion of the trihedral $T_0$ the translations $\xi$
and $\eta_1$ are zero. The direction-cosines of $l_1$ relative to
the trihedral at $M$ are therefore $q du$, $-p_1 dv$, $1$. The displace-
ment of the central point on $l$ must be orthogonal to both
$l$ and $l_1$. Hence for that point we must have $\delta z = 0,$
$q du \delta x - p_1 dv \delta y + \delta z = 0$. Combining these equations we
get $q du \delta x - p_1 dv \delta y = 0$, which becomes

$$
q du \left[ \frac{\partial \xi}{\partial u} du - \eta_1 r dv + \xi du + g t du - \eta_1 r_1 dv - \eta_1 r_1 dv \right]
$$

$$
- p_1 dv \left[ \xi r_1 du + \frac{\partial \eta_1}{\partial v} dv + \eta_1 dv + \xi r_1 dv - \xi r_1 dv - tp_1 dv \right] = 0.
$$

This may be written in the form

$$
(11) \quad q \left( \frac{\partial \xi}{\partial u} + \xi + g t - \eta_1 r \right) du^2 - (q \eta_1 + p_1 \xi) (r + r_1) du dv
$$

$$
- p_1 \left( \frac{\partial \eta_1}{\partial v} + \eta_1 + \xi r_1 - tp_1 \right) dv^2 = 0.
$$

When the value of $dv/du$ which determines the curve $C$ is put
in this equation we have an equation in $t$ which determines
the distance along $l$ to the line of striction of the ruled
surface defined by $C$. Conversely, given $t$ a function of $u$

---

* Eisenhart, p. 168.
† Eisenhart, p. 170.
and \(v\), the equation (11) determines two curves on \(S\), though not necessarily real, which define two ruled surfaces of the congruence for which \(t\) is the distance to their lines of striction.

We suppose that the congruence is normal by virtue of the relation \(r + r_1 = 0\). Hence equation (11) is of the form
\[
L(u, v) du^2 - M(u, v) dv^2 = 0,
\]
and consequently represents a system of curves symmetrically placed relative to the lines of curvature. Now the normals to a surface along the asymptotic lines form ruled surfaces for which the asymptotic lines are the lines of striction.* Hence we have the following theorem.

**Theorem III.** *If the congruence of lines \(l\) be normal by virtue of the relation \(r + r_1 = 0\), the curves on \(S\) which represent the asymptotic lines on the normal surfaces form a system which is symmetrically placed relative to the lines of curvature.*

5. **Minimal Surfaces.** We now suppose that \(S\) is a minimal surface with the parameters of the lines of curvature so chosen that the linear element of the surface has the form†
\[
(12) \quad ds^2 = q(du^2 + dv^2),
\]
where \(q\) is the absolute value of each principal radius. Hence we have‡
\[
(13) \quad \begin{cases}
    \hat{s} = \eta_1 = \sqrt{q}, & q = \frac{D}{VE} = -\frac{1}{\sqrt{q}}, \\
    p_1 = \frac{D'}{VG} = -\frac{1}{\sqrt{q}}, \\
    r = -\frac{1}{VG} \frac{\partial}{\partial v} \sqrt{E} = -\frac{\partial q}{2q'}, \\
    r_1 = \frac{1}{\sqrt{E}} \frac{\partial}{\partial u} \sqrt{G} = \frac{\partial q}{2q'}.
\end{cases}
\]

† Eisenhart, p. 253.
‡ Eisenhart, p. 174.
When the values of $\xi, \eta_1, q, p_1, r, r_1$, as given in (13) are substituted in (9), it is readily seen that the coefficient of $t$ vanishes. Hence we have the following theorem.

**Theorem IV.** If $S$ be a minimal surface with the parameters of the lines of curvature so chosen that the linear element has the form (12), the congruence of lines $l$ is normal and has for its middle envelope the given minimal surface.

The minimal surface $S$ is therefore the mean evolute of each of the normal surfaces.

6. **Envelope of a Two-Parameter Family of Surfaces.** Let $S$ be any surface referred to any parametric system. With every point $M$ of $S$ we associate the trihedral of the surface, giving the $x$-axis its most general orientation relative to the curve $v = \text{const}$. Let

\[ F(x, y, z, u, v) = 0 \]

be the equation relative to the trihedral at $M$ of some surface $\Sigma$. We consider the envelope of such a two-parameter family of surfaces.

The characteristic is defined by (14) and the equations

\[
\begin{align*}
\frac{\partial F}{\partial x} \frac{dx}{du} + \frac{\partial F}{\partial y} \frac{dy}{du} + \frac{\partial F}{\partial z} \frac{dz}{du} + \frac{\partial F}{\partial u} &= 0, \\
\frac{\partial F}{\partial x} \frac{dx}{dv} + \frac{\partial F}{\partial y} \frac{dy}{dv} + \frac{\partial F}{\partial z} \frac{dz}{dv} + \frac{\partial F}{\partial v} &= 0.
\end{align*}
\]

Since the characteristic is fixed in space, we must have

\[
\begin{align*}
\delta x &= dx + \xi du + \xi_1 dv + (qdu + q_1 dv)z - (rdu + r_1 dv)y = 0, \\
\delta y &= dy + \eta du + \eta_1 dv + (rdu + r_1 dv)x - (pdu + p_1 dv)z = 0, \\
\delta z &= dz + (pdu + p_1 dv)y - (qdu + q_1 dv)x = 0,
\end{align*}
\]

for all values of $\frac{dv}{du}$. Hence

\[
\begin{align*}
\frac{dx}{du} &= ry - qz - \xi, & \frac{dy}{du} &= pz - rx - \eta, & \frac{dz}{du} &= qx - py, \\
\frac{dx}{dv} &= r_1 y - q_1 z - \xi_1, & \frac{dy}{dv} &= p_1 z - r_1 x - \eta_1, & \frac{dz}{dv} &= q_1 x - p_1 y.
\end{align*}
\]
By (16), equations (15) become

\[
\begin{aligned}
\frac{\partial F}{\partial x}(ry - qz - \xi) + \frac{\partial F}{\partial y}(p\xi - rx - \eta) \\
+ \frac{\partial F}{\partial z}(qi - r_1x - p_1y) + \frac{\partial F}{\partial u} = 0,
\end{aligned}
\]

\[
\begin{aligned}
\frac{\partial F}{\partial x}(r_1y - q_1z - \xi_1) + \frac{\partial F}{\partial y}(p_1z - r_1x - \eta_1) \\
+ \frac{\partial F}{\partial z}(q_1x - p_1y) + \frac{\partial F}{\partial v} = 0.
\end{aligned}
\]

The coordinates \((x, y, z,\xi, \eta)\) of the characteristic are therefore given by equations (14) and (17).

7. Applications. As an application, consider the envelope of certain two-parameter families of planes. We choose \(S\) as any surface referred to its lines of curvature, and choose the \(x\)-axis of the trihedral tangent to the curve \(v = \text{const}\). Consider the two two-parameter families of planes

\[
x = \xi,
\]

and

\[
y = \eta_1.
\]

For the family of planes (18), equations (17) become

\[
ry - qz - \xi - \frac{\partial \xi}{\partial u} = 0, \quad r_1y + \eta_1 = 0.
\]

Hence solving equations (18) and (20) the coordinates of the characteristic of the planes (18) are

\[
x_1 = \xi, \quad y_1 = -\frac{\eta_1}{r_1}, \quad z_1 = -\frac{r_1\left(\xi + \frac{\partial \xi}{\partial u}\right) + \eta_1}{qr_1^2}.
\]

For the family of planes (19), equations (17) become

\[
rx + \xi r_1 = 0, \quad p_1z - r_1x - \eta_1 - \frac{\partial \eta_1}{\partial v} = 0.
\]

The solution of equations (19) and (22) gives the following for the coordinates of the characteristic of the planes (19):

\[
x_2 = -\frac{\xi r_1}{r}, \quad y_2 = \eta_1, \quad z_2 = \frac{r\left(\eta_1 + \frac{\partial \eta_1}{\partial v}\right) - \xi}{r_1^2}.
\]
Let us now assume the relation

\[(24) \quad r + r_1 = 0.\]

Then, from (21) and (23), we have

\[
\begin{align*}
(25) & \quad \left\{\begin{array}{ll}
x_1 = x_2 = \xi, & y_1 = y_2 = \eta_1, \\
z_1 = \frac{\eta_1 r - \xi - \frac{\partial \xi}{\partial u}}{q}, & z_2 = \frac{\xi r_1 + \eta_1 + \frac{\partial \eta_1}{\partial v}}{p_1}.
\end{array}\right.
\]

Hence when the relation (24) holds, the characteristics of both families of planes lie on the line \(l\). Moreover, when the relation (24) holds the roots of equation (9) are

\[
(26) \quad t_1 = \frac{\eta_1 r - \xi - \frac{\partial \xi}{\partial u}}{q}, \quad t_2 = \frac{\xi r_1 + \eta_1 + \frac{\partial \eta_1}{\partial v}}{p_1}.
\]

Hence, since \(t_1\) and \(t_2\) are identical with \(z_1\) and \(z_2\) in (25) we have the following theorem.

**Theorem V.** If \(S\) be a surface referred to its lines of curvature for which the relation \(r + r_1 = 0\) holds, the envelopes of the two families of planes \(x = \xi\) and \(y = \eta_1\) are the two focal sheets of the normal congruence of lines \(l\).

The planes (18) and (19) are therefore the focal planes of the congruence. From (8), the curves defining the developables are the lines of curvature. Hence from (7),

\[
t = t_1 = \frac{\eta_1 r - \xi - \frac{\partial \xi}{\partial u}}{q},
\]

for \(v = \text{const.},\) and

\[
t = t_2 = \frac{\xi r_1 + \eta_1 + \frac{\partial \eta_1}{\partial v}}{p_1},
\]

for \(u = \text{const.}\) Consequently the planes \(x = \xi\) and \(y = \eta_1\) envelope the focal sheets determined by the developables \(v = \text{const.}\) and \(u = \text{const.},\) respectively.

Yale University