CRITERIA THAT ANY NUMBER
OF REAL POINTS IN \(n\)-SPACE SHALL LIE
IN AN \((n-k)\)-SPACE

BY H. S. UHLER

The object of the present paper is to establish an algebraic
identity from which may be deduced necessary and suffi­
cient conditions that any large number of real points in
\(n\)-dimensional linear space shall lie in a linear \((n-k)\)-space.

Let the following matrix, in which the number of columns
is \(m\) and the number of rows is \(n+1\) \([m \geq (n+1)]\), be
compounded with its conjugate:

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_{1,1} & x_{2,1} & \cdots & x_{m,1} \\
x_{1,2} & x_{2,2} & \cdots & x_{m,2} \\
\vdots & \vdots & \ddots & \vdots \\
x_{1,n} & x_{2,n} & \cdots & x_{m,n}
\end{bmatrix}
\]

The determinant of the resulting symmetric square
array is

\[
\begin{vmatrix}
m & \sum x_{i,1} & \sum x_{i,2} & \cdots & \sum x_{i,n} \\
\sum x_{i,1} & \sum x_{i,1} x_{i,1} & \sum x_{i,1} x_{i,2} & \cdots & \sum x_{i,1} x_{i,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sum x_{i,n} & \sum x_{i,n} x_{i,1} & \sum x_{i,n} x_{i,2} & \cdots & \sum x_{i,n} x_{i,n}
\end{vmatrix} = \Delta;
\]

\((i = 1, 2, 3, \ldots, m)\).

Multiply all of the rows of \(\Delta\) except the top row by \(m\),
compensate by prefixing \(m^{-n}\), and remove the factor \(m\)
now common to the constituents of the first column to get

\[
\Delta = m^{1-n} \begin{vmatrix}
1 & \sum x_{i,1} & \sum x_{i,2} & \cdots & \sum x_{i,n} \\
\sum x_{i,1} & m \sum x_{i,1} x_{i,1} & m \sum x_{i,1} x_{i,2} & \cdots & m \sum x_{i,1} x_{i,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sum x_{i,n} & m \sum x_{i,n} x_{i,1} & m \sum x_{i,n} x_{i,2} & \cdots & m \sum x_{i,n} x_{i,n}
\end{vmatrix};
\]

\((i = 1, 2, 3, \ldots, m)\).
Next subtract \( \sum_{i=1}^{m} x_{i,k} \) times the first column from the \((k+1)\)th column, \((k = 1, 2, 3, \ldots, n)\), in order to reduce to zero all the constituents of the top row, except the leading constituent, and to find \( \Delta = U_n / m^{n-1} \), where

\[
U_n = \begin{vmatrix}
\sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,n} \\
\sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_{n,n}
\end{vmatrix}
\]

and

\[
\sigma_{p,q} = m \sum_{i=1}^{m} x_{i,p} x_{i,q} - \left( \sum_{i=1}^{m} x_{i,p} \right) \left( \sum_{i=1}^{m} x_{i,q} \right) = m \sum_{i=j+1}^{m} \sum_{j=1}^{m-1} \left[ (x_{i,p} - x_{j,p}) (x_{i,q} - x_{j,q}) \right] = \sigma_{q,p}.
\]

Now the determinant \( \Delta \) produced by compounding the matrices specified above is known to equal the sum of the squares of all the \( \nu \) determinants of order \( n+1 \) that can be formed from the columns of the original matrix, where

\[
\nu = \binom{m}{n+1}.
\]

Let any one of these determinants be denoted by \( D_r \); then the required algebraic identity is

\[
(1) \quad U_n = m^{n-1} \sum_{r=1}^{\nu} (D_r^2).
\]

Thus far no special meaning has been assigned to the \( x \)'s; they may represent complex quantities, etc.

To obtain the criteria contemplated advantage will be taken of the fact that \( D_r \) is squared in identity (1) so that if the \( x \)'s are real numbers \( D_r^2 \) will be incapable of becoming negative. Accordingly let the rectangular co-ordinates of a system of real points in \( n \)-dimensional flat space be

\[
(x_{i,1}, x_{i,2}, \ldots, x_{i,n}); \ i = 1, 2, 3, \ldots, m; \ m \geq (n+1).
\]

Also let \( S_t \) symbolize a linear space of \( t \) dimensions, a \( t \)-flat.
Now the vanishing of $\sum (D_r^2)$ is a necessary and sufficient condition that the $m$ given real points shall lie in the same $S_{n-1}$, hence, by formula (1), a necessary and sufficient condition that any number $m \geq (n+1)$ of real points in $S_n$ shall lie in the same $S_{n-1}$ is the vanishing of $U_n$.

When $m = n+1$, $v = 1$ so that there is only one $D_r$ in $\sum (D_r^2)$. This $D_r$ represents $n!$ times the content of the hyper-figure or simplex having the $n+1$ given points as vertices.* Hence, for $m > (n+1)$, $\sum (D_r^2)$ is proportional to the sum of the squares of the contents of all the simplexes that can be formed from the $m$ points taken $n+1$ at a time as vertices of each geometric figure. Accordingly the above italicized statement may also be interpreted as meaning that the contents of all the simplexes involved vanish.

Keeping $m = n+1$, and giving $n$ successively the values $1, 2, 3, 4, \ldots, n$, we may derive from the identity (1) the following expressions for the respective magnitudes of the length of a segment in $S_1$, the area of a triangle in $S_2$, the volume of a tetrahedron in $S_3$, the hyper-volume of a penta-hedroid in $S_4$, \ldots, the content of a simplex in $S_n$:

$$\frac{\sigma_{1,1}^{1/2}}{2\sqrt{3}}, \frac{|\sigma_{1,1}, \sigma_{2,2}|^{1/2}}{24}, \frac{|\sigma_{1,1}, \sigma_{2,2}, \sigma_{3,3}|^{1/2}}{120\sqrt{5}}, \ldots, \frac{|\sigma_{1,1}, \sigma_{2,2}, \ldots, \sigma_{n,n}|^{1/2}}{n!(n+1)^{n-1/2}}.$$

The extension of the above italicized statement from $S_{n-1}$ to $S_{n-k}$ is an immediate consequence of the well known properties of orthogonal projections of linear spaces. The fundamental idea is that identity (1) holds for a smaller number of coordinates than $n$ and hence it may be applied to the orthogonal projections of the $m$ given points upon all of the

$$\binom{n}{n-k+1}$$

coordinate-\(S_{n-k+1}\)'s. In other words the original matrix is to be replaced by

\[
\begin{pmatrix}
  n \\
  n-k+1
\end{pmatrix}
\]

matrices having the same top row of \(m\) 1's while the remaining rows are composed of \(n-k+1\) of the original rows of \(x\)'s. There will now be

\[
\begin{pmatrix}
  n \\
  n-k+1
\end{pmatrix}
\]

new systems of points,—one in each coordinate-\(S_{n-k+1}\),—to all of which the above italicized test must be applied. The orders of the \(U_n\)'s and \(D_r\)'s of formula (1) will be \(n-k+1\) and \(n-k+2\) respectively. Without further comment it should be perfectly clear that necessary and sufficient conditions that any number of real points in \(n\)-dimensional flat space shall lie in an \((n-k)\)-dimensional flat space are that all the

\[
\begin{pmatrix}
  n \\
  n-k+1
\end{pmatrix}
\]

determinants \(U\) of order \(n-k+1\) in the \(\sigma\)'s shall vanish while one, at least, of the determinants \(U\) of order \(n-k\) shall be finite.

The last sentence may be stated in terms of the rank of the \(U\) of order \(n\).* Incidentally the writer has found it possible to express the general criteria analytically in terms of only two determinants involving polynomial constituents composed of the \(\sigma\)'s.

Yale University

* G. Kowalewski, Determinantentheorie, § 52.