THE FUNDAMENTAL REGION
FOR A FUCHSIAN GROUP*

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1. Introduction. The present paper is an attempt to lay the groundwork of the theory of Fuchsian groups by basing the treatment on concepts of a very simple sort. The fundamental region to which we are led is not new. It is given by the Fricke-Klein method† under certain circumstances and is identical with that given by Hutchinson‡ in an important paper. However, we make use neither of non-euclidean geometry nor of quadratic forms, and we are able to derive the major results of the theory of Fuchsian groups in an unexpectedly simple manner.

2. The Group. Given a group of linear transformations with an invariant circle or straight line $K'$, the interior of $K'$ (or the half-plane on one side of $K'$) being transformed into itself by each transformation of the group. We shall assume that there exists a point $A$, not on $K'$, such that there are no points congruent to $A$ in a sufficiently small neighborhood of $A$.

Let $G$ be a linear transformation carrying $K'$ into the unit circle $K$ with center at the origin and carrying $A$ to the origin. Let $S$ be any transformation of the group; then the set of transformations

$$T = GSG^{-1}$$

is a group with $K$ as principal circle.§ Configurations

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† Fricke-Klein, Vorlesungen über die Theorie der automorphen Funktionen, vol. I, Chap. II.
‡ J. I. Hutchinson, A method for constructing the fundamental region of a discontinuous group of linear transformations, Transactions of this Society, vol. 8 (1907), pp. 261-269.
§ We use this order to mean the transformation $G^{-1}$, followed by $S$, followed by $G$. That is, writing $z' = S(z), z'' = G(z)$, etc., as the combining transformations, the new transformation is $z'' = T(z) = G\{S[G^{-1}(z)]\}$.
which are congruent by transformations of the new group are carried by $G^{-1}$ into configurations which are congruent by transformations of the original group. It will suffice, then, to find a fundamental region for the new group.

The condition that

$$z' = T(z) = \frac{az + b}{cz + d}$$

leave the unit circle unchanged is that $b = \bar{c}$, $d = \bar{a}$, where bars indicate conjugate imaginaries. Then

$$T = \frac{az + \bar{c}}{cz + \bar{a}}.$$

Since the origin, 0, must transform into an interior point, and $T(0) = \bar{c}/\bar{a}$, we must have $|c| < |a|$. Hence the determinant $a\bar{a} - \bar{c}\bar{c}$, which is real, must be positive. We shall insert such a positive factor in numerator and denominator that

$$a\bar{a} - \bar{c}\bar{c} = 1.$$

There is no point congruent to 0 in a suitably small neighborhood of 0. In particular, 0 is not a fixed point for any transformation; hence $c \neq 0$, unless $T$ be the identical transformation.

3. Two Locus Problems. Excluding the identical transformation, we shall solve the following two locus problems for the transformation $T$.

I. Find the locus of a point in the neighborhood of which lengths and areas are unchanged in magnitude.

Infinitesimal lengths are multiplied by $|T'(z)|$ and infinitesimal areas are multiplied by $|T'(z)|^2$. Since

$$T'(z) = \frac{1}{(cz + \bar{a})^2},$$

we have as the required locus the circle

$$C: |cz + \bar{a}| = 1,$$

or

$$|z + \bar{a}/c| = 1/|c|.$$
This circle, which will be called the $C$-circle of the transformation $T$, has its center at the point $-\bar{a}/c$; its radius is $1/|c|$. Writing the relation $a\bar{a} - \bar{c}c = 1$ in the form

$$1 + \frac{1}{|c|^2} = \left| \frac{\bar{a}}{c} \right|^2,$$

we see that the sum of the squares of the radii of $K$ and $C$ is equal to the square of the distance between their centers. Hence $C$ is orthogonal to $K$. It follows that $0$ is outside $C$.

If $z$ is within $C$ then

$$|z + \bar{a}/c| < 1/|c|, \quad |cz + \bar{a}| < 1, \quad |T'(z)| > 1,$$

and lengths and areas in the neighborhood of $z$ are increased in magnitude when transformed by $T$. Similarly, if $z$ is outside $C$ lengths and areas are decreased in magnitude when transformed by $T$.

II. Find the locus of a point whose distance from 0 is unchanged.

The required locus is

$$|z| = \left| \frac{az + \bar{c}}{cz + \bar{a}} \right|$$

or

$$|cz^2 + \bar{a}z| = |az + \bar{c}|,$$

or

$$(cz^2 + \bar{a}z)(\bar{c}z^2 + a\bar{z}) = (az + \bar{c})(\bar{a}z + c).$$

On expanding and making use of the relation $\bar{c}c = a\bar{a} - 1$ this factors into

(a) $$(z\bar{z} - 1) [(cz + \bar{a})(\bar{c}z + a) - 1] = 0,$$

whence

$$|z| = 1, \quad \text{or} \quad |cz + \bar{a}| = 1.$$

The complete locus, then, consists of the two circles $C$ and $K$.

If $z$ is inside both $C$ and $K$ or outside both, the first member of (a) is greater than 0, and we find, on retracing our steps, that

$$|z| > \left| \frac{az + \bar{c}}{cz + \bar{a}} \right|. $$
The transform of \( z \) is nearer 0 than \( z \) is. Similarly, if \( z \) is inside one circle and outside the other the transform is farther from 0 than \( z \) is.

4. Geometric Interpretation of \( T \). The inverse of \( T \),

\[
T^{-1} = \frac{-az + c}{cz - a},
\]

has the \( C \)-circle

\[
C': \quad |z - a/c| = 1/|c|.
\]

It is clear that \( T \) carries \( C \) into \( C' \). For, \( T \) carries \( C \) into a circle \( C_0 \) without alteration of lengths; then \( T^{-1} \) transforms \( C_0 \) without alteration of lengths; whence \( C_0 \) coincides with \( C' \).

The region exterior to both \( C \) and \( C' \) is transformed by \( T \) into the interior of \( C' \) and by \( T^{-1} \) into the interior of \( C \). Let \( z \) be a point of the region, and let \( z \) be carried by \( T \) and \( T^{-1} \) into \( z' \) and \( z'' \) respectively. In both cases there is diminution of lengths and areas near \( z \) since \( z \) is exterior to both \( C \)-circles. Then the inverses, \( T^{-1} \) and \( T \), carry \( z' \) and \( z'' \) respectively back to \( z \) with increase of lengths and areas. Hence \( z' \) is in \( C' \) and \( z'' \) is in \( C \).

Let \( A, B \) and \( A', B' \) be the intersections of \( C \) and \( C' \) with \( K \), the points being so designated that motion around \( C \) from \( A \) to \( B \) inside \( K \) is counter-clockwise about \( C \), and motion from \( B' \) to \( A' \) around \( C' \) in \( K \) is counter-clockwise about \( C' \). Now, on applying \( T \), the interior arc \( AB \) of \( C \) is carried without alteration of length into either \( A'B' \) or \( B'A' \). The latter is impossible, for it is equivalent to a suitable rotation with 0 as fixed point, which is contrary to hypothesis. Hence \( A \) is transformed into \( A' \) and \( B \) into \( B' \).

We can now give simple geometric interpretations of \( T \). Let \( L \) be the perpendicular bisector of the line joining the centers of \( C \) and \( C' \). \( L \) passes through 0. (In the special case that \( C \) and \( C' \) coincide let \( L \) be the line joining 0 to the center of \( C \).) Either of the following pairs of inversions transforms the points of \( C \) exactly as \( T \) does and hence is identical with it:
(1) A reflection in $L$ followed by an inversion in $C''$;
(2) An inversion in $C$ followed by a reflection in $L$.

We can show from these inversions that $T$ is hyperbolic, elliptic, or parabolic according as $C$ and $C''$ are exterior to one another, intersect, or are tangent. The fixed points of the transformation are easily found geometrically.

5. The Arrangement of the C-Circles. We shall show first that there is an upper bound of the radii of the C-circles defined by the transformations of the group. There exists, by hypothesis, a circle of radius $\epsilon < 1$ with 0 as center having no point congruent to 0 in its interior. We have then for any transformation of the group

\[ |T(0)| = \left| \frac{\bar{c}}{a} \right| \geq \epsilon, \text{ whence } \left| \frac{\bar{a}}{c} \right| \leq \frac{1}{\epsilon}; \]

that is, the distance from 0 to the center of $C$ is not greater than $1/\epsilon$. Since 0 is outside $C$ it follows that the radius of $C$ is less than $1/\epsilon$.

Let

\[ T_1 = \frac{ax + \bar{y}}{y^2 + \bar{a}} \]

be a second transformation of the group. Designate by $C_1$, $C_1'$ the C-circles of $T_1$ and $T_1^{-1}$. Thus $C_1$ is $|z + \bar{a}/\gamma| = 1/|\gamma|$. Let us now make the transformation

\[ TT_1^{-1} = \frac{(-a\bar{a} + \bar{c}\gamma)z + a\bar{\gamma} - \bar{c}a}{(-\bar{c} \bar{a} + \bar{c}\gamma)z + \bar{c}\gamma - \bar{c}a}, \]

which, since $T_1 \neq T$, is not the identical transformation. Designating the radii of $C$, $C_1$, and the C-circle of $TT_1^{-1}$ by $r$, $r_1$, $r_2$, respectively, we have

\[ r_2 = \frac{1}{\left| -c\bar{a} + \bar{a}\gamma \right|} = \frac{1}{\left| \gamma \right| \left| -\frac{\bar{a}}{c} - \left( \frac{-\bar{a}}{c} \right) \right|} = \frac{rr_1}{d}, \]

where $d$ is the distance between the centers of $C$ and $C_1$. Since $r_2 < 1/\epsilon$ we have

\[ d = \frac{rr_1}{r_2} > \epsilon r_1. \]
Consider now all $C$-circles whose radii equal or exceed some positive number $k$. The distance between the centers of any two of the circles satisfies the inequality
\[ d > \varepsilon k^2. \]
Since the centers lie in the finite region bounded by $K$ and by the circle of radius $1/\varepsilon$ with center at 0 it follows that the number of such circles is finite.

It follows from the result just found that any closed region lying entirely within $K$, for example a circle with 0 as center and radius less than 1, is exterior to all but a finite number of $C$-circles.

Another consequence is that the transformations of the group are denumerable, since the radii of the corresponding $C$-circles are denumerable.

6. The Fundamental region. Let $R$ be the region within $K$ which lies outside all $C$-circles formed for the transformations of the group. The region is connected, since any of its points can be joined to 0 by a straight line segment which does not cross the boundary. We shall show that $R$ is a fundamental region for the group; that is, (1) that no two interior points of $R$ are congruent, and (2) that no region adjacent to $R$ and lying in $K$ can be added to $R$ without the inclusion of points congruent to points of $R$.

The proof of the first property is immediate. The transform of any interior point of $R$ by any transformation of the group, the identical transformation excepted, lies within some $C$-circle and hence is exterior to $R$.

To establish the second property we shall show first that if $P$, a point of $C$, the $C$-circle of some transformation $T$ of the group, lies on the boundary of $R$, then $P'$, the transform of $P$ by $T$, also lies on the boundary of $R$. $P'$ lies on $C'$, the $C$-circle of $T^{-1}$.

Suppose $P'$ does not lie on the boundary of $R$. Then $P'$ lies within the $C$-circle of some transformation $T_1$ of the group. Consider the transformation $T_1T$. By the trans-
formations $T$ lengths in the neighborhood of $P$ are carried without alteration of magnitude into the neighborhood of $P'$. By $T_1$ lengths in the neighborhood of $P'$ are magnified. Hence $T_1T$ magnifies lengths in the neighborhood of $P$; consequently $P$ is within the $C$-circle of $T_1T$. This is contrary to hypothesis; hence $P'$ is on the boundary of $R$.

It follows from the preceding that if an arc $ab$ of $C$ forms part of the boundary of $R$ the congruent arc $a'b'$ of $C'$ is a part of the boundary. The transformation $T^{-1}$ carries $R$ into a region abutting $R$ along $ab$. Any region adjacent to $R$ abuts along some $C$-circle and contains points congruent by a suitable transformation to points of $R$. The second property is thus established.

The following properties of the region $R$ are consequences of the preceding analysis:

1. $R$ is bounded by arcs of circles orthogonal to $K$. The number of bounding arcs in a circle $|z| = r < 1$ is finite.

2. The bounding arcs are arranged in congruent pairs. Two congruent arcs of the boundary are equal in length, and congruent points thereof are equidistant from the center of $K$.

3. The vertices of a cycle (congruent vertices) lie on a circle concentric with $K$, since all are equidistant from the center of $K$. If the vertices of a cycle lie within $K$ their number is finite.

4. $R$ is the fundamental region of maximum area. For, a different fundamental region must contain points congruent to all points of $R$, and a shift of any part of $R$ to a congruent position effects a diminution of area.

We shall now prove that $R$ and the regions congruent to it fill up without overlapping the whole interior of $K$.

Suppose $R_i$ and $R_j$, the transforms of $R$ by $T_i$ and $T_j$, overlap. Let $z_1, z_2, z_3$ be three points common to $R_i$ and $R_j$. These are the transforms by $T_i$ of three points $z_1', z_2', z_3'$ of $R$, and the transforms by $T_j$ of three points $z_1'', z_2'', z_3''$ of $R$. If $z_1 = z_1'$, $z_2 = z_2'$, $z_3 = z_3'$ then $T_i$ and $T_j$ are the same transformation, since they transform three points in the
same way, and $R_i$ and $R_j$ coincide. Otherwise the points of one pair, say $z'_1$ and $z''_1$, are unequal. Being both congruent to $z_1$ they are congruent, which is impossible.

Let $K_r$ be the circle $|z| = r < 1$. Let $a_1b_1, a'_1b'_1, \ldots, a_nb_n, a'_nb'_n$ be the sides of $R$ lying wholly or in part within $K_r$; and let $T_i, i = 1, 2, \ldots, n$, be the transformation carrying $a_ib_i$ into $a'_ib'_i$. The transforms of all remaining sides of $R$ are exterior to $K_r$, since the distance from 0 of any point on such a side is not decreased by any transformation of the group.

By applying $T_1, \ldots, T_n$ and their inverses we get regions congruent to $R$ abutting on $R$ along the sides $a_1b_1, \ldots, a'_nb_n$. The sides of the new regions which lie in $K_r$ are congruent to the sides just mentioned. By combinations of $T_1, \ldots, T_n$ and their inverses we can adjoin further regions along sides of these new regions, provided the sides lie in $K_r$; and the process can be continued as long as there are any free sides in $K_r$.

This process will end in a finite number of steps; for, each transformation carries $R$ into the interior of a particular $C$-circle, and there is but a finite number of $C$-circles intersecting $K_r$. Hence $K_r$ is covered by a finite number of regions. Since $r$ may be chosen as near 1 as we like, it follows that the whole interior of $K$ is covered.

We note from the preceding that all transformations of the group are formed by combinations of the transformations connecting congruent sides of $R$. These are therefore called generating transformations of the group. A further interesting fact is that a transformation which carries $R$ into a region lying wholly or in part in a circle $K_r$ concentric with $K$ is a combination of those generating transformations only whose $C$-circles intersect $K_r$.

7. An Important Special Case. The sides of $R$ may be finite or infinite in number. There are certain groups in which a fundamental region not extending to the principal circle is known to exist; for example, some of the groups
arising in connection with the uniformization problem.* For this case we have the following proposition.

If there exists a fundamental region $F$ lying within $K_r(|z| = r < 1)$, then $R$ lies within $K_r$, and the number of sides and of generating transformations is finite.

$R$ and a finite number of its transforms, $R_1, \ldots, R_m$, will cover $F$ completely. Carry the portion of $F$ lying in each $R_i$ into $R$ by means of the transformation which carries $R_i$ into $R$. The totality of these transforms of parts of $F$, together with the portion of $F$ originally in $R$, fill up $R$ completely. If this were not so we could construct a region $D$ adjacent to one of these transformed regions and containing no points congruent to points of $F$. On carrying $D$ back to the boundary of $F$ we should have a region abutting on $F$ and containing no points congruent to points of $F$, which is contrary to hypothesis.

Finally $R$ is in $K_r$, for on transforming the parts of $F$ into $R$ the distance of no point from 0 is increased.

Since only a finite number of $C$-circles intersect $K_r$, it follows that $R$ has a finite number of sides and that the number of generating transformations is finite.